چهلمین کنفرانس ریاضی کشور
40TH ANNUAL IRANIAN MATHEMATICS CONFERENCE
Accepted for presentation
Available at http://aimc40.ir/node/622

انجمن ریاضی ایران
Iranian Mathematicsl Society

دانشگاه صنعتی شریف
Sharif University of Technology

A New Approach for estimation of attraction region and asymptotic stability a system of nonlinear ODE

S. Effati
email: effati911@.yahoo.com

O. Baghani

H. Baghani

Tarbiat Moallem University of Sabzevar

Abstract

In this paper, we use the measure theory method for estimation of attraction region and asymptotic stability of an autonomous system of first order of the non-linear ordinary differential equations (ODE's). Corresponding with attraction region, two optimization problems are defined. We prove that if solutions of two problems be the same, the attraction region is determined completely. The solutions of these optimization problems are used to find an estimation of the region of attraction and then a piecewise-constant control function. Using the approximated control signals, the approximate trajectories and the error functional related to asymptotic stability problem are obtained.Mathematics Subject Classification:

Keywords: ODE's, asymptotic stable, region of attraction, infinite-horizon problem, measure theory, nonlinear optimal control, linear programming.

AMS Mathematics Subject Classification (2000): 93D20, 34B40, 34B15, 49J15

1 Introduction and formulation of problem

The offered methods for determining region of attraction, about a equilibrium point, is divided to two groups. The first group, consist of the methods which Lyapunov's function is used in them structure, for example, Michel method [2], and the second group, consist of the methods that don't use Lyapunov's function, for example, the methods that is offered by Infante [3] and Loprio [6]. Generally, the offered methods, are used for limited problems and we don't use them for all nonlinear systems. Now, we want to use a new method for estimation of attraction region and asymptotic stability of the autonomous system by using measure theory. The measure theory has been used to solve optimal shape design by Fakharzadeh and Rubio [1], and to solve linear and nonlinear ODE' s and infinite- horizon optimal control problems by Effati et al. [7].

An autonomous system of ordinary nonlinear differential equations is in the following form

$$x'(t) = G(x(t)) \tag{1}$$

where $G : A \to I\!R^m$ is Lipschitz function and A is a compact set. G is Lipschitz function, if G satisfy in the following relation

$$\forall y, z \in A \ \ , \ \|G(y) - G(z)\|_p \leq L\|y - z\|_p \ ; \quad (L > 0)$$

$\|x\|_p$ for $x = (x_1, x_2, \cdots, x_m) \in I\!R^m$ is defined by

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_m|^p)^{\frac{1}{p}} \ \ ; \quad (1 \leq p < \infty).$$

Definition 1.1 *Let $x(t)$ is a solution of system (1), and there exists $t^* \geq 0$, such that $x(t^*) = x^* \in I\!R^m$. x^* point is said stable point if $x(t^*) = x(t^* + t)$ for each $t > 0$. We always assume that $x^* = 0$ is stable point, since with $y = x - x^*$ the stable point for new system $y' = G(y + x^*)$ is zero point.*

Definition 1.2 *Let $x(t)$ is a solution of the system (1) with starting point $x(0)$. The set of all points $x(0)$ in A that $\lim_{t\to\infty} x(t) = 0$, is said region of attraction, denoted by RA.*

Determining of RA is difficult in generally, but we can obtain an estimate of this region.

Definition 1.3 *Upper radius $\overline{r}_p$ and lower radius $\underline{r}_p$ of system (1) are defined in the following form*

$$\overline{r}_p = \sup\{\|x(0)\|_p \mid \lim_{t\to\infty} x(t) = 0\}, \tag{2}$$

$$\underline{r}_p = \inf\{\|x(0)\|_p \mid \lim_{t\to\infty} x(t) \ \textit{not exis or} \ \lim_{t\to\infty} x(t) \neq 0\}. \tag{3}$$

Since A is compact, $\overline{r}_p$ and $\underline{r}_p$ are well define.

Theorem 1.1 *Assume that upper radius and lower radius attraction are $\overline{r}_p$ and $\underline{r}_p$ respectively, then*
i) $\overline{r}_p = \sup\{r \mid RA \subset N_p(r)\}$,
ii) $\underline{r}_p = \inf\{r \mid RA \subset N_p(r)\}$.
where $N_p(r) = \{x \mid \|x\|_p \leq r\}$.

Proof. It is clear.

Corollary 1.1 *Let that upper radius and lower radius attraction are $\overline{r}_p$ and $\underline{r}_p$ respectively, then*
i) $\underline{r}_p \ \leq \ \overline{r}_p$.
ii) If $\underline{r}_p = \overline{r}_p$ *then* $RA = N_p(\overline{r}_p)$.
iii) Let $x^* = 0$ be a stable point for system (1), then $x^* = 0$ is asymptotically stable point if only if $\underline{r}_p > 0$.

According to (2), URA problem is

$$\text{Sup} \ \|x(0)\|_p$$

$$\text{subject to} \begin{cases} x'(t) = G(x(t)) \\ \lim_{t\to\infty} x(t) = 0, \end{cases}$$

and by (3), LRA problem is

$$\text{Inf } \|x(0)\|_p$$

$$\text{subject to} \begin{cases} x'(t) = G(x(t)) \\ \lim_{t\to\infty} x(t) \neq 0. \end{cases}$$

Here assume $x'(t) = u(t)$ (we may call $u(\cdot)$ as artificial control function). define $\varphi(t, x(t), u(t)) = \|u(t) - G(x(t))\|_2^2$ and the error functional E into $E[x(\cdot), u(\cdot)] = \int_0^\infty \varphi(t, x(t), u(t))dt$, for all $t \in [0, \infty)$, where $\|\cdot\|$ is the l^2- norm in $I\!R^m$. Now by the change of variable $\theta = \frac{2}{\pi}\tan^{-1}(t)$, we transform the interval $[0, \infty)$ to $[0, 1)$. Assume $y(\theta) = x\left(\tan(\frac{\pi}{2}\theta)\right)$ and $v(\theta) = u\left(\tan(\frac{\pi}{2}\theta)\right)$, since $x(0) = y(0)$, we get the following problems. For URA problem we have

$$\text{Sup} \quad I[y(\cdot), v(\cdot)] = \|y(0)\|_p - \lambda \int_{[0,1)} \varphi\left(\tan(\frac{\pi}{2}\theta), y(\theta), v(\theta)\right) \frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta)d\theta \tag{4}$$

$$\text{subject to} \quad \left.\begin{array}{l} y'(\theta) = g(\theta, y(\theta), v(\theta)) \\ v(\theta) \in U \subseteq I\!R^m \\ \lim_{\theta\to 1^-} y(\theta) = y^1 = 0 \\ y(\theta) \in A \subseteq I\!R^m. \end{array}\right\} \tag{5}$$

and for LRA problem we have

$$\text{Inf} \quad I[y(\cdot), v(\cdot)] = \|y(0)\|_p + \lambda \int_{[0,1)} \varphi\left(\tan(\frac{\pi}{2}\theta), y(\theta), v(\theta)\right) \frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta)d\theta \tag{6}$$

$$\text{subject to} \quad \left.\begin{array}{l} y'(\theta) = g(\theta, y(\theta), v(\theta)) \\ v(\theta) \in U \subseteq I\!R^m \\ \lim_{\theta\to 1^-} y(\theta) \neq 0 \\ y(\theta) \in A \subseteq I\!R^m. \end{array}\right\} \tag{7}$$

where $g(\theta, y(\theta)) = \frac{\pi}{2}v(\theta)\sec^2(\frac{\pi}{2}\theta)$ and $\lambda > 0$ is a big number.

Definition 1.4 *A pair $w = [x(\cdot), u(\cdot)]$, where $x(\cdot), u(\cdot) : [0, \infty) \to I\!R^m$ are said to be admissible for URA problem if $u(\cdot)$ is measurable and bounded, the trajectory function $x(.)$ is differential and the constrains of the (5) are satisfied. We assume that the set of all admissible pairs, is non-empty. Similarly the set of all admissible pairs for LRA is definable.*

Theorem 1.2 *If there exists an optimal solution $w^* = [x^*(\cdot), u^*(\cdot)]$ for the infinite-horizon optimal control problem (4)-(5) such that the optimal value E^* be zero, then the optimal solution w^* is exact.*

Proof. Suppose w^* is the optimal solution for problem (4)-(5). It is obvious that w^* is an admissible solution. Furthermore, since the function φ is continuous function and non-negative, then $\varphi \equiv 0$, thus, the first order system (1) will be hold for all $t \in [0, \infty)$, and the exact solution of (1) is obtained. The above theorem is true for (6)-(7) problem.

Remark 1.1 *If $w^* = [x^*(\cdot), u^*(\cdot)]$, be an optimal solution for the problem (4)-(5) such that the optimal value E^* be a.e. zero, then the optimal solution w^* is the approximate solution for (1).*

2 Transformation of URA into Infinite-dimensional optimization

Consider $\Omega = J \times A \times U$, where $J = [0, 1)$. Assume $J_\epsilon = [0, 1-\epsilon]$ and $\Omega_\epsilon = J_\epsilon \times A \times U$. Since J_ϵ, A and U are compact subsets of $I\!R$, $I\!R^m$ and $I\!R^m$ respectively, then Ω_ϵ is a compact subset of $I\!R^{2m+1}$ and $\Omega_\epsilon \to \Omega$ as $\epsilon \to 0$. Let $w = [y(\cdot), v(\cdot)]$ be an admissible pair and B an open ball in $I\!R^{m+1}$ containing $J \times A$, that $C'(B)$ be the space of real-valued continuously differentiable functions on B such that the first derivative is also bounded. Also $D(J^\circ)$ be the space of all infinitely differentiable real-valued functions with compact support in J° (see [4]), where $J^\circ = (0, 1)$ and $C_1(\Omega)$ is subspace of the space $C(\Omega)$ of all continuous functions on Ω depending only on the variable θ. The mapping $\Lambda_w : F \to \int_J F(\theta, y(\theta), v(\theta))d\theta \quad (F \in C_c(\Omega))$, defines a positive linear functional on $C_c(\Omega)$, the space of all bounded continuous functions with compact support. Similarly, we define the mapping $\Lambda_\epsilon : F \to \int_{J_\epsilon} F(\theta, y(\theta), v(\theta))d\theta \quad (F \in C(\Omega))$. By the Riesz representation theorem (see [8]) there exist two unique positive Radon μ and μ_ϵ on Ω and Ω_ϵ respectively, such that $\Lambda_w = \int_J F(\theta, y(\theta), v(\theta))d\theta = \int_\Omega F d\mu \equiv \mu(F) \quad (F \in C_c(\Omega))$ and $\Lambda_\epsilon = \int_{J_\epsilon} F(\theta, y(\theta), v(\theta))d\theta = \int_{\Omega_\epsilon} F d\mu_\epsilon \equiv \mu_\epsilon(F) \quad (F \in C(\Omega_\epsilon))$. If $\phi^g(\theta, y(\theta), v(\theta)) = \frac{d}{d\theta}(\phi(\theta, y))$ and $\psi^j(\theta, y(\theta), v(\theta)) = \frac{d}{d\theta}(\psi.y_j)$ then we have

$$\left.\begin{array}{l} \mu(\phi^g) = \Delta\phi \quad (\phi \in C'(B)) \\ \mu(\psi^j) = 0 \quad (j = 1, 2, ..., m; \quad \psi \in D(J^\circ)) \\ \mu(\dot{f}) = a_f \quad (f \in C_1(\Omega)). \end{array}\right\} \tag{8}$$

Proposition 2.1 *Let $w = [y(\cdot), v(\cdot)]$ be an admissible pair, for the URA problem, then*
i) $y_i(0) = -\mu(g_i) \quad (i = 1, 2, \ldots, m)$
ii) $\|y(0)\|_p = (\sum_{i=1}^m |\mu(g_i)|^p)^{\frac{1}{p}}$
iii) $\Delta\phi = \phi(1, 0) - \phi(0, -\mu(g_1), -\mu(g_2), \ldots, -\mu(g_m))$

Proof. If $\phi(\theta, y) = y_j$, then $\phi^g(\theta, y) = g_j$, therefore $\mu(\phi^g) = \mu(g_j) = \int_{[0,1)} g_j d\theta = -y_j(0)$, for $j = 1, 2, \ldots, m$. The proof of (ii) and (iii) by (i) is clear.

Proposition 2.2 $\lim_{\epsilon\to 0} \mu_\epsilon(F) = \mu(F), \qquad (F \in C(\Omega_\epsilon))$.

Proof. We have

$$\lim_{\epsilon\to 0} \mu_\epsilon(F) = \lim_{\epsilon\to 0} \int_{\Omega_\epsilon} F d\mu_\epsilon = \lim_{\epsilon\to 0} \int_{J_\epsilon} F(\theta, y(\theta), v(\theta))d\theta = \lim_{\epsilon\to 0} \int_J \chi_{J_\epsilon} F(\theta, y(\theta), v(\theta))d\theta$$

where, χ is characteristic function. Since $|\chi_{J_\epsilon} F| \le |F|$, therefore

$$\lim_{\epsilon\to 0} \mu_\epsilon(F) = \int_J \lim_{\epsilon\to 0} \chi_{J_\epsilon} F(\theta, y(\theta), v(\theta))d\theta = \int_\Omega F(\theta, y(\theta), v(\theta))d\theta = \mu(F).$$

Theorem 2.1 *Let $M^+(\Omega)$ be the set of all positive Radon measure on Ω and $Q(\Omega)$ be the set of all positive Radon measure on Ω satisfying* (8). *Similarly define $M^+(\Omega_\epsilon)$ and $Q(\Omega_\epsilon)$. Then*

$$\lim_{\epsilon\to 0} \; Sup_{\;Q(\Omega_\epsilon)} I(\mu_\epsilon) = \; Sup_{\;Q(\Omega)} I(\mu).$$

Proof. See [4].

By above theorem also the Riesz representation theorem, the URA problem is estimated to following form

$$\text{Sup}_{\ \mu_\epsilon \in Q(\Omega_\epsilon)} \quad I(\mu_\epsilon) = (\sum_{i=1}^{n} |\mu_\epsilon(g_i)|^p)^{\frac{1}{p}} - \lambda \mu_\epsilon(f_0) \tag{9}$$

3 Approximation of URA problem

Let $s_1 = \{\phi_i \ ; \ i = 1, 2, \cdots\}$, such that the linear combinations of the functions $\phi_i \in C'(B)$ are uniformly dense in $C'(B)$. For instance, these functions can be taken to be monomials in the components of the $m-$ vectors x. Also assume $s_2 = \{\sin(2\pi r\theta),\ 1-\cos(2\pi r\theta)\ ;\ r = 1, 2, \cdots\}$ and $s_3 = \{f_s \ ; \ s = 1, 2, \ldots, L\}$. If $Q_\epsilon(M_1, M_2)$ be the set of all positive Radon measure on Ω_ϵ satisfying in (8), then by theorem III.3 of [4], we have $\lim_{(M_1,M_2)\to(\infty,\infty)} \text{Sup}_{\ \mu_\epsilon \in Q_\epsilon(M_1,M_2)}\ I(\mu_\epsilon) = \text{Sup}_{\ \mu_\epsilon \in Q_\epsilon(\Omega_\epsilon)}\ I(\mu_\epsilon)$. Therefore the following problem is an approximate of the problem (9).

$$\text{Sup}_{\ \mu_\epsilon \in Q_\epsilon(M_1,M_2)} \qquad I(\mu_\epsilon) = (\sum_{i=1}^{m} |\mu_\epsilon(g_i)|^p)^{\frac{1}{p}} - \lambda \mu_\epsilon(f_0) \tag{10}$$

By Proposition III.2 of [4] we obtain an approximation for the optimal measure μ_ϵ^* by a finite combination of atomic measures, such that $\mu_\epsilon^* = \sum_{i=1}^{M1+M2} \alpha_i^* \delta(z_i^*)$ $(\alpha_i^* \geq 0, z_i^* \in \Omega_\epsilon)$. Here $\delta(z)$ is the unitary atomic measure characterized by $\delta(z)(F) = F(z)$ where $F \in C(\Omega_\epsilon)$ and $z \in \Omega_\epsilon$. This structural result points the way towards a nonlinear problem in which the unknowns are the coefficients α_i^* and supports $\{z_i^*\}$ $(i = 1, 2, \cdots, M1 + M2)$. To transform this problem to a linear programming, we use the another approximation. If w_ϵ^N is a countable dense subset of Ω_ϵ, we can approximate μ_ϵ^* by a measure $\nu_\epsilon \in M^+(\Omega_\epsilon)$ such that $\nu_\epsilon = \sum_{i=1}^{M1+M2} \alpha_i^* \delta(z_i)$. where $z_i \in w_\epsilon^N = \{z_1, z_2 \ldots, z_N\}$ (Proposition III.3 of [4]). Now we shall consider only a finite mnuber M_1 of the functions ϕ of the type $\phi_1 = y_1,\ \phi_2 = y_2, \ldots,\ \phi_m = y_m,\ \phi_{m+1} = y_1^2,\ \phi_{m+2} = y_2^2, \ldots$. Since every continuous function can be written as a linear combination of monomials of the type $1, y, y^2, \cdots$. Also assume

$$\psi_r(\theta) = \begin{cases} \sin(2\pi r\theta)\ (r = 1, 2, \cdots, M_{21}) \\ 1 - \cos(2\pi r\theta)\ (r = M_{21} + 1, M_{21} + 1, \cdots, 2M_{21}) \end{cases}$$

then we have $M_2 = 2mM_{21}$. Finally, let $\xi_r \in C_1(\Omega)$, $\xi_r(\theta, y(\theta), v(\theta)) = \theta^r$ $(r = 0, 1, \cdots)$ Then $\{\xi_r,\ r = 0, 1, \cdots\}$ is dense in $C_1(\Omega)$. Assume that there are L of them in the set $\{\phi_i^g\}_{i=1}^{M_1}$. It is necessary to choose L number of function of the time only, to replace the functions ξ_r $(r = 1, 2, \cdots)$ which were not found suitable, so we have chosen some suitable functions, to be denoted by f_s $(s = 1, 2, \cdots L)$, as follows

$$f_s(\theta) = \begin{cases} 1 & \text{if } \theta \in J_s \\ 0 & \text{otherwise,} \end{cases}$$

will be considered, where $J_s = (\frac{(s-1)}{L}, \frac{s}{L})$, $(s = 1, \cdots, L)$. The set $\Omega_\epsilon = J_\epsilon \times A \times U$ will be covered with a grid, where the grid will be defined by taking all points in Ω_ϵ as $z_j = (\theta_j, y_{1j}, y_{2j}, \ldots, y_{mj}, v_{1j}, v_{2j}, \ldots, v_{mj})$, The points in the grid will be numbered sequentially from

1 to N. So URA problem is approximated by the following nonlinear programming problem which z_i for $i = 1, \cdots, N$ belongs to a dense subset of Ω_ϵ.

$$\text{maximize} \qquad (\sum_{j=1}^{m} |c_i|^p)^{\frac{1}{p}} - \lambda \sum_{j=1}^{N} \alpha_j f_0(z_j) \tag{11}$$

subject to

$$\left.\begin{array}{l} \sum_{j=1}^{N} \alpha_j \phi_i^g(z_j) + \phi_i(0, c_1, c_2 \ldots, c_m) = \phi_i(1, 0) \quad (i = 1, ..., M_1) \\ \sum_{j=1}^{N} \alpha_j \psi_r^j(z_j) = 0 \quad (r = 1, ..., \frac{M_2}{m}) \ (j = 1, 2, ..., m) \\ \sum_{j=1}^{N} \alpha_j f_s(z_j) = a_{f_s} \quad (s = 1, ..., L) \\ \alpha_j \geq 0, \quad (j = 1, 2, \cdots, N) \end{array}\right\} \tag{12}$$

where $c_i = -\mu(g_i) \quad (i = 1, 2, \ldots, m)$. Therefore, the upper region of attraction by solving this problem is obtained.

4 Approximation of LRA problem

In this section, a nonlinear programming problem to estimate the optimal solution of LRA is obtained. Put $y(0) = [y_1(0), y_2(0), \ldots, y_m(0)]^T$ and $y(\theta_\epsilon) = [y_1(\theta_\epsilon), y_2(\theta_\epsilon), \ldots, y_m(\theta_\epsilon)]^T$, where $\lim_{\epsilon \to 0} \theta_\epsilon = 1$, $\theta_\epsilon < 1$.

Proposition 4.1 *Let $w = [y(\cdot), v(\cdot)]$ be an admissible pair, for the LRA problem satisfying in (7), then*
i) $y_i(0) = \frac{1}{\theta_\epsilon} \mu_\epsilon(y_i + (\theta - \theta_\epsilon) g_i) \quad (i = 1, 2, \ldots, m)$
ii) $y_i(\theta_\epsilon) = \frac{1}{\theta_\epsilon} \mu_\epsilon(y_i + \theta g_i) \quad (i = 1, 2, \ldots, m)$
iii) $\Delta_\epsilon \phi = \phi(\theta_\epsilon, y_1(\theta_\epsilon), y_2(\theta_\epsilon), \ldots, y_m(\theta_\epsilon)) - \phi(0, y_1(0), y_2(0), \ldots, y_m(0))$

Proof. The proof is clear, if wee choose $\phi(\theta, y) = \theta y_j$ and $\phi(\theta, y) = (\theta - \theta_\epsilon) y_j$ for $j = 1, 2, \ldots, m$.

In the other hand, if $y(\theta)$ be a admissible solution and $\lim_{\theta \to 1^-} y(\theta) \neq 0$ we have

$$\exists k > 0 \ \ \forall s \in I\!N \ \ \exists \theta_s \ s.t \ \ 1 - \frac{1}{s} \leq \theta_s \leq 1 \text{ and } \|y(\theta_s)\| \geq \frac{1}{k}.$$

By the similar process used for URA, we obtain the following optimization problem for LRA.

$$\text{minimize} \qquad (\sum_{j=1}^{m} |c_i|^p)^{\frac{1}{p}} + \lambda (\sum_{j=1}^{N} \alpha_j f_0(z_j)) \tag{13}$$

subject to

$$\left\{\begin{array}{l} \sum_{j=1}^{N} \alpha_j \phi_i^g(z_j) - \phi_i(1, d_1, d_2, \ldots, d_m) + \phi_i(0, c_1, c_2 \ldots, c_m) = 0 \quad (i = 1, \ldots, M_1) \\ \sum_{j=1}^{N} \alpha_j \psi_r^j(z_j) = 0 \quad (r = 1, \ldots, \frac{M_2}{m}) \ (j = 1, 2, \ldots, m) \\ \sum_{j=1}^{N} \alpha_j (y_i + (\theta - \theta_\epsilon) g_i)(z_j) + \theta_\epsilon c_i = 0 \quad (i = 1, 2, \ldots, m) \\ \sum_{j=1}^{N} \alpha_j (y_i + \theta g_i)(z_j) + \theta_\epsilon d_i = 0 \quad (i = 1, 2, \ldots, m) \\ (\sum_{i=1}^{m} |d_i|^p)^{\frac{1}{p}} \geq \frac{1}{k} \\ \sum_{j=1}^{N} \alpha_j f_s(z_j) = a_{f_s} \quad (s = 1, \ldots, L) \\ \alpha_j \geq 0, \quad (j = 1, 2, \ldots, N) \end{array}\right. \tag{14}$$

where $c_i = -\frac{1}{\theta_\epsilon} \mu_\epsilon(y_i + (\theta - \theta_\epsilon) g_i)$ and $d_i = -\frac{1}{\theta_\epsilon} \mu_\epsilon(y_i + \theta g_i) \quad (i = 1, 2, \ldots, m)$.

5 Numerical example

Consider the following nonlinear differential equation

$$\begin{cases} x_1'(t) = -3x_1^3(t) - x_2(t) \\ x_2'(t) = x_1^5(t) - 2x_2^3(t) \end{cases} \tag{15}$$

First we define function

$$\varphi(t, x(t), u(t)) = (u_1(t) + 3x_1^3(t) + x_2(t))^2 + (u_2(t) - x_1^5(t) + 2x_2^3(t))^2,$$

Now by suitable change of variable, for URA problem we have

$$\text{Sup} \quad I[y(\cdot), v(\cdot)] = \|y(0)\|_p - \lambda \int_{[0,1)} \frac{\pi}{2}\varphi(\theta, y(\theta), v(\theta)) \sec^2(\frac{\pi}{2}\theta) d\theta$$

subject to

$$\left.\begin{array}{l} y_1'(\theta) = \frac{\pi}{2} v_1(\theta) \sec^2(\frac{\pi}{2}\theta) \\ y_2'(\theta) = \frac{\pi}{2} v_2(\theta) \sec^2(\frac{\pi}{2}\theta) \\ \lim_{\theta\to 1^-} y_1(\theta) = 0 \\ \lim_{\theta\to 1^-} y_2(\theta) = 0. \end{array}\right\}$$

and for LRA problem we have

$$\text{Inf} \quad I[y(\cdot), v(\cdot)] = \|y(0)\|_p + \lambda \int_{[0,1)} \frac{\pi}{2}\varphi(\theta, y(\theta), v(\theta)) \sec^2(\frac{\pi}{2}\theta) d\theta$$

subject to

$$\left.\begin{array}{l} y_1'(\theta) = \frac{\pi}{2} v_1(\theta) \sec^2(\frac{\pi}{2}\theta) \\ y_2'(\theta) = \frac{\pi}{2} v_2(\theta) \sec^2(\frac{\pi}{2}\theta) \\ \lim_{\theta\to 1^-} y_1(\theta) \neq 0 \\ \lim_{\theta\to 1^-} y_2(\theta) \neq 0. \end{array}\right\}$$

Let $\theta \in J_\epsilon = [0, 1-\epsilon]$, $\epsilon = \frac{1}{100}$, so $\theta_\epsilon = 0.99$. Also $y(\theta) = [y_1(\theta), y_2(\theta)] \in A = A_1 \times A_2$, where $A_1 = [0, 0.1]$, $A_2 = [-0.1, 0.122]$ and $v(\theta) = [v_1(\theta), v_2(\theta)] \in U = U_1 \times U_2$ where $U_1 = U_2 = [-1, 1]$. And let the set $J_\epsilon = [0, 1-\epsilon]$ be divided into 15 equal subintervals, the sets A_1, A_2, U_1 and U_2 be divided into 10 equal subintervals, so that $\Omega_\epsilon = J_\epsilon \times A \times U$ is divided into 150000 equal subsets. We assumed $z_p = (\theta_p, y_{1p}, y_{2p}, v_{1p}, v_{2p})$, $(p = 1, ..., 150000)$ is obtained by the following

```
for l=1:15
    for k=1:10
        for j=1:10
            for i=1:10
                for w=1:10
                    p=w+10*(i-1)+100*(j-1)+1000*(k-1)+10000*(l-1);
                    y1(p)=0.01*i;
                    y2(p)=-0.1+0.0222*j;
                    v1(p)=-0.99+0.2*k-0.0004;
                    v2(p)=-0.99+0.2*w-0.0004;
                    if l<=10
                        teta(p)=0.08*l-0.0776;
                    elseif l<=14
```

```
                    teta(p)=0.9+0.02*(l-10)-0.0799;
                else
                end
            end
        end
    end
  end
end
```

Now for URA and LRA we chose $\phi = y_1^p \ , \ y_2^p$. In the other hand, put $M_2 = 8$ and $L = 15$, also $\beta_1 = |y_1(0)|^p$, $\beta_2 = |y_2(0)|^p$, $\beta_3 = |y_1(\theta_\epsilon)|^p$ and $\beta_4 = |y_2(\theta_\epsilon)|^p$. Then we have a nonlinear programming problem for URA and LRA as follows

$$\text{maximize } (\beta_1+\beta_2)^{\frac{1}{p}} - \lambda \sum_{j=1}^{150000} \frac{\pi}{2}\{(v_{1_j}+3y_{1j}^3+y_{2_j})^2+(v_{2_j}-y_{1_j}^5+2y_{2_j}^3)^2\}\sec^2(\frac{\pi}{2}\theta_j)\alpha_j$$

subject to

$$\begin{cases} \sum_{j=1}^{150000} p(y_{1_j})^{p-1}(v_{1_j}\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j))\alpha_j = -\beta_1 \\ \sum_{j=1}^{150000} p(y_{2_j})^{p-1}(v_{2_j}\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j))\alpha_j = -\beta_2 \\ \sum_{j=1}^{150000}\{2\pi h y_{l_j}\cos(2\pi h\theta_j)+v_{l_j}\sin(2\pi h\theta_j)\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j)\}\alpha_j = 0 \\ \sum_{j=1}^{150000}\{2\pi h y_{l_j}\sin(2\pi h\theta_j)+v_{l_j}(1-\cos(2\pi h\theta_j))\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j)\}\alpha_j = 0 \\ \alpha_{1+10000(i-1)}+\cdots+\alpha_{10000+10000(i-1)} = 0.066, \quad i=1,\cdots,15 \\ \beta_1,\beta_2 \geq 0 \\ \alpha_j \geq 0, \ j=1,2,\cdots,150000 \quad (\ l=1,2, \ h=1,2). \end{cases} \tag{16}$$

If we put $k = 10$, the LRA problem is equivalent to the following problem

$$\text{minimize } (\beta_1+\beta_2)^{\frac{1}{p}} + \lambda \sum_{j=1}^{150000} \frac{\pi}{2}\{(v_{1_j}+3y_{1j}^3+y_{2_j})^2+(v_{2_j}-y_{1_j}^5+2y_{2_j}^3)^2\}\sec^2(\frac{\pi}{2}\theta_j)\alpha_j$$

subject to

$$\begin{cases} \sum_{j=1}^{150000} p(y_{1_j})^{p-1}(v_{1_j}\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j))\alpha_j = \beta_3-\beta_1 \\ \sum_{j=1}^{150000} p(y_{2_j})^{p-1}(v_{2_j}\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j))\alpha_j = \beta_4-\beta_2 \\ \sum_{j=1}^{150000}\{y_{1_j}+(\theta_j-\theta_\epsilon)(v_{1_j}\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j))\}\alpha_j = \theta_\epsilon(\beta_1)^{\frac{1}{p}} \\ \sum_{j=1}^{150000}\{y_{12_j}+(\theta_j-\theta_\epsilon)(v_{2_j}\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j))\}\alpha_j = \theta_\epsilon(\beta_2)^{\frac{1}{p}} \\ \sum_{j=1}^{150000}\{y_{1_j}+\theta_j(v_{1_j}\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j))\}\alpha_j = \theta_\epsilon(\beta_3)^{\frac{1}{p}} \\ \sum_{j=1}^{150000}\{y_{12_j}+\theta_j(v_{2_j}\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j))\}\alpha_j = \theta_\epsilon(\beta_4)^{\frac{1}{p}} \\ \sum_{j=1}^{150000}\{2\pi h y_{l_j}\cos(2\pi h\theta_j)+v_{l_j}\sin(2\pi h\theta_j)\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j)\}\alpha_j = 0 \\ \sum_{j=1}^{150000}\{2\pi h y_{l_j}\sin(2\pi h\theta_j)+v_{l_j}(1-\cos(2\pi h\theta_j))\frac{\pi}{2}\sec^2(\frac{\pi}{2}\theta_j)\}\alpha_j = 0 \\ \alpha_{1+10000(i-1)}+\cdots+\alpha_{10000+10000(i-1)} = 0.066, \quad i=1,\cdots,15 \\ \beta_3+\beta_4 \geq (\frac{1}{10})^p \\ \alpha_j \geq 0, \ j=1,2,\cdots,150000 \quad (\ l=1,2, \ h=1,2). \\ \beta_1,\beta_2,\beta_3,\beta_4 \geq 0 \end{cases} \tag{17}$$

The problems (16)-(17) are nonlinear generally. But for $p = 1$ two problems are linear. So by solving them for $p = 1$, we obtain an initial estimation for β_1, β_2, β_3 and β_4. Now for $p = 2$ and $p = 4$, by using Taylor expansion function about these points, the nonlinear problems are transformed into linear optimization problems. Therefore we have

λ	$\underline{r}_1$	$\bar{r}_1$	*Lower error*	*Upper error*
1000	0.0748	0.1035	0.0012	0.0012

Table 1. p=1.

λ	$\underline{r}_2$	$\bar{r}_2$	*Lower error*	*Upper error*
1000	0.07408	0.1240	0.00093	0.0012

Table 2. p=2.

λ	$\underline{r}_4$	$\bar{r}_4$	*Lower error*	*Upper error*
1000	0.1036	0.3162	0.0012	0.2558

Table 3. p=4.

The attraction region is shown in Figs. 1-2. Now we solve the problem (16) with $p = \lambda = 1$ and initial point $y_1(0) = y_2(0) = 0.1$. Therefore we obtain the trajectory from the initial point $y(0) = (0.1, 0.1)$ to end point $\lim_{\theta\to 1^-} y(\theta) = (3.3117 \times 10^{-4}, 0.0043)$. The error functional E is 0.0484. Thus the autonomous system (15) is asymptotically stable. The graphs of the piecewise constant control function and the trajectory functions are shown in Fig. 3.

6 Conclusion

In this paper, corresponding with the attraction region problem, two optimization problems are defined. Using these problems, $\overline{r}_p$ and $\underline{r}_p$, $1 \le p < \infty$, are obtained. In the other hand, with obtained optimal α_j, $j = 1, 2, \dots, N$, we can calculate the approximate control function then the trajectory for an autonomous system $x'(t) = G(x(t))$.

References

[1] A. Fakharzadeh and J. E. Rubio, *Shapes and Measures*, Ima. J. Math. Control & information, **16** (1999), 207-220.

[2] A. N. Michel, B. H. Nam and V. Vittal, *Computer generated Lyapunov functions of iterconnected systems, improve results with application to power systems*, IEEE Trans., Circuits Syst., CAS-31, Feb (1984), 189-198.

[3] E. F. Infante and L.G. Clark, *A method for the determination to power system*, J. Appl. Mech., vol 86, June (1964).

[4] J. E. Rubio, *Control and Optimization; the Linear Treatment of Non-Linear Problems*, Manchester, U. K., Manchester University Press, (1986).

[5] J. Xiao, X. Zhu, *On linerly topological structure and property of fuzzy normed linear space*, Fuzzy set and System 125 (2002) 153161.

[6] K. A. Loparo and G. L. Blankenship, *Estimating the domain of attraction of nonlinear feedback systems*, IEEE Trans., Automat. Contr., AC-23, Aug (1978), 602-608.

[7] S. Effati, A. V. Kamyad and R. A. Kamyabi- Gol, *On Infinite-Horizon Optimal Control Problems*, J. Z. Anal. Anw. **19** (2000), 269-278.

[8] W. Rudin, *Real and complex analysis,* third ed., McGraw-Hill, New York, (1987).

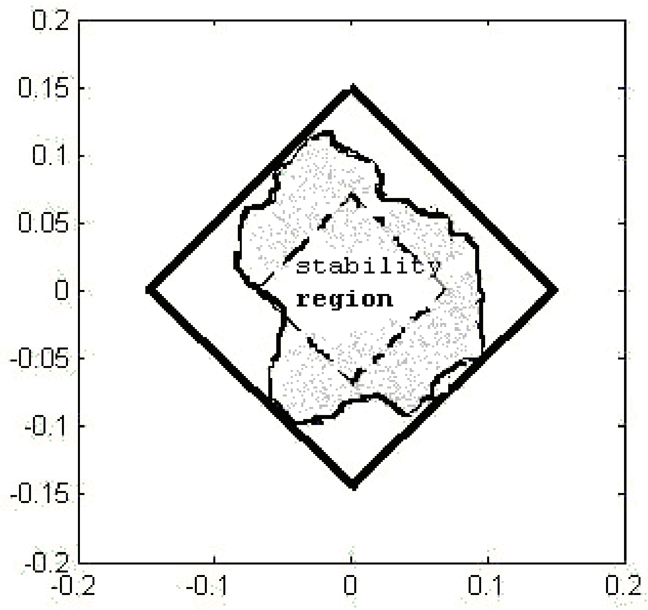

Fig.1. $p = 1$

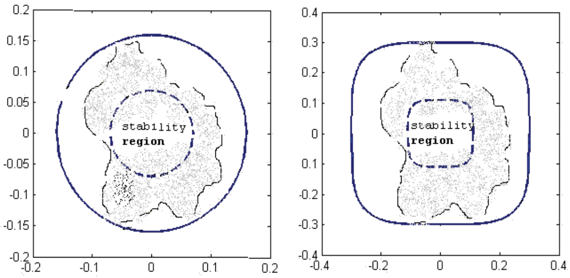

Fig.2. $p = 2$ $p = 4$

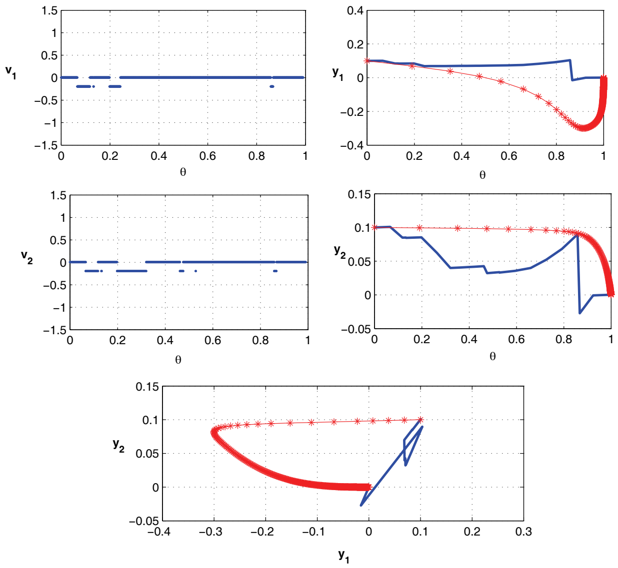

Fig.3. The Control Function of v_1 , v_2 and the Optimal Trajectories of y_1 , y_2.
" $-$ " Measure Theory and " $-*$" 4-nd Runge-Kouta