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A FAMILY OF 13-NODE PLATE BENDING TRIANGULAR ELEMENTS

M. REZAIEE-PAJAND* AND M. R. AKHTARY
Department of Civil Engineering, Ferdowsi University of Mashhad, Mashhad, Iran

SUMMARY

This paper investigates the importance and accuracy of a family of high-order triangular elements. After a brief review of characteristics of high-order triangular elements, six new incompatible 13-node triangular elements are presented. The accuracy of the proposed elements is illustrated by comparing their numerical results with the other investigators' solutions, and the best element is introduced. © 1998 John Wiley & Sons, Ltd.

KEY WORDS finite element; plate bending; triangular element; displacement method; polynomial function

INTRODUCTION

Kirchhoff's theory is used for finite element analysis of thin plate bending. Based on this theory, the normals to the middle plane of a plate remain normal and unstrained after deformation.[1] Using this assumption, the main variable of the problem is the normal deflection of the structure. In the finite element displacement method, the field function and its first derivatives have to be continuous. However, there are incompatible elements with discontinuous derivatives, which yield acceptable results.[2]

In this paper, several high-order elements are presented. Some of these elements have been introduced and evaluated by other investigators. After presenting a family of 13-node plate bending elements, six new elements of this type are introduced. These new elements are used to solve several plate bending problems. The capabilities and the accuracy of the derived elements are illustrated by comparing the results with the other solutions of the problems.

HIGH-ORDER TRIANGULAR ELEMENTS

Plate bending elements are classified in two major groups: compatible and incompatible elements. In compatible elements, two adjacent elements have identical deflections and slopes. A major source of incompatibility is the difference in slopes of two adjacent elements. There are two slopes in the longitudinal and normal directions to the border. Usually, the longitudinal slope uniquely exists due to continuity of the displacement at the sides of the elements. Normal slope, in some cases, is not defined uniquely. The absence of some node parameters yields incompatible

* Correspondence to: M. Rezaiee-Pajand, Dept. of Civil Engineering, Ferdowsi University, PO Box 1111-91775, Mashhad, Iran

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elements with different normal slopes. These incompatible elements can, however, converge to a correct solution.[3,4]

In order to establish the compatibility in triangular elements, the number of nodes and/or degrees of freedom in elements can be increased. To do so, high-order polynomials are used. As a result, more complex calculations are needed when high-order polynomials are used. In most cases, it is impossible, especially for calculating the stiffness matrix which needs the integration, to find the explicit formulation for these elements. Numerical integration is used for this purpose. In the following sections, some of these elements are discussed.

T-18 element

This is a high-order triangular compatible element with three nodes at the corners. Each node has six degrees of freedom: w, $w_{,x}$, $w_{,y}$, $w_{,xx}$, $w_{,yy}$, $w_{,xy}$. Displacement methods and Kirchhoff's assumption are used to derive this element.[5–7]

T-21 element

This compatible element is a six-node triangular element. Here, w, $w_{,x}$, $w_{,y}$, $w_{,xx}$, $w_{,yy}$, $w_{,xy}$ are six degrees of freedom associated with each corner node. $w_{,n}$ is the only degree of freedom of the mid-side nodes. A fifth-order complete polynomial is used for this element. Bell,[5] Argyris *et al.*[7] and Irons[8] presented this element.

TUBA triangular elements

This set of high-order triangular elements were presented by Argyris *et al.*[7] Kirchhoff's assumption and displacement method were utilized to obtain these elements. One of these elements is TUBA-6, which has six nodes and 21 degrees of freedom. This element is the same as the T-21 element. Another element is TUBA-13, which has 13 nodes and 28 degrees of freedom. A sixth-order complete polynomial is used for this element. Figure 1 shows the TUBA-13 element. Finally, the last TUBA element is TUBA-15. This element has 15 nodes and 36 degrees of freedom. A seventh-order complete polynomial is used for the formulation of the TUBA-15 element. Figure 2 shows this element.

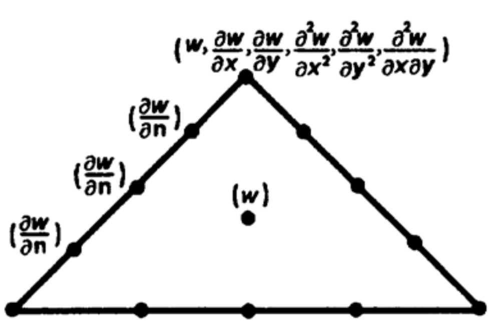

Figure 1. TUBA-13 element

T-33 element

This triangular element has ten nodes. There are six degrees of freedom associated with each corner node. Each side node has two degrees of freedom and the centroid node has three degrees of freedom. This element was presented by Svek and Gladwel.[6] Figure 3 shows the T-33 element. A seventh-order complete polynomial was utilized for this element. Three degrees of freedom associated with the centroid node were omitted.

T21-CSW element

This element was presented by Carmanlian, Selby and Will.[6] There are six nodes in this element. The degrees of freedom in corners and mid-sides of the element are three and four, respectively. A fifth-order complete polynomial with 21 terms is used for the T21-CSW element. Figure 4 shows this element.

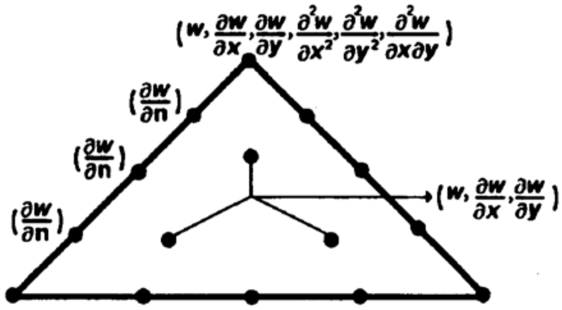

Figure 2. TUBA-15 element

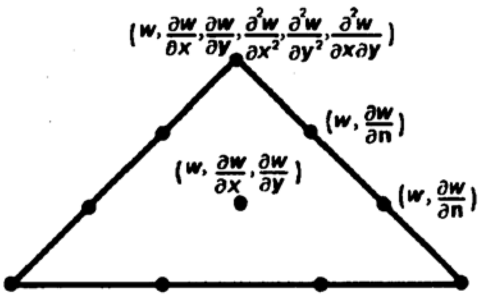

Figure 3. T-33 element

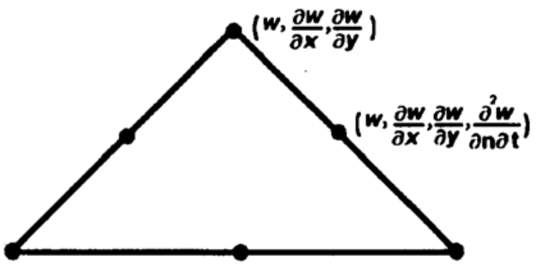

Figure 4. T21-CSW element

A TRIANGULAR ELEMENT WITH 13 NODES

In this study, a fifth-order polynomial is used for the displacement field of the problem. This function has the following form:

$$\begin{aligned} w(\xi_i, \xi_j, \xi_k) = {} & \alpha_1 \xi_i^5 + \alpha_2 \xi_i^4 \xi_j + \alpha_3 \xi_i^4 \xi_k + \alpha_4 \xi_i^3 \xi_j^2 + \alpha_5 \xi_i^3 \xi_j \xi_k + \alpha_6 \xi_i^3 \xi_k^2 + \alpha_7 \xi_i^2 \xi_j^3 + \alpha_8 \xi_i^2 \xi_j^2 \xi_k \\ & + \alpha_9 \xi_i^2 \xi_j \xi_k^2 + \alpha_{10} \xi_i^2 \xi_k^3 + \alpha_{11} \xi_i \xi_j^4 + \alpha_{12} \xi_i \xi_j^3 \xi_k + \alpha_{13} \xi_i \xi_j^2 \xi_k^2 + \alpha_{14} \xi_i \xi_j \xi_k^3 \\ & + \alpha_{15} \xi_i \xi_k^4 + \alpha_{16} \xi_j^5 + \alpha_{17} \xi_j^4 \xi_k + \alpha_{18} \xi_j^3 \xi_k^2 + \alpha_{19} \xi_j^2 \xi_k^3 + \alpha_{20} \xi_j \xi_k^4 + \alpha_{21} \xi_k^5 \end{aligned} \tag{1}$$

Figure 5 shows the location of element nodes discussed in this paper. Deflection of the plate, w, and its derivatives are the degrees of freedom used in this study. There are two types of co-ordinates for the element: Cartesian co-ordinates (x, y), which are used for any nodes, and (t, n) co-ordinates, which are utilized for mid-side nodes. Figure 6 shows the aforementioned co-ordinates.

Several types of degrees of freedom are used in this study. Both first and second derivatives of w, with respect to x, y, n and t, are utilized. All degrees of freedom for these elements are as follows: w, $w_{,x}$, $w_{,y}$, $w_{,n}$, $w_{,nt}$.

The following relationships exist between x, y, n and t co-ordinates:

$$\begin{cases} n = Cx + Sy \\ t = -Sx + Cy \end{cases}, \quad \begin{cases} S = \sin\theta \\ C = \cos\theta \end{cases} \tag{2}$$

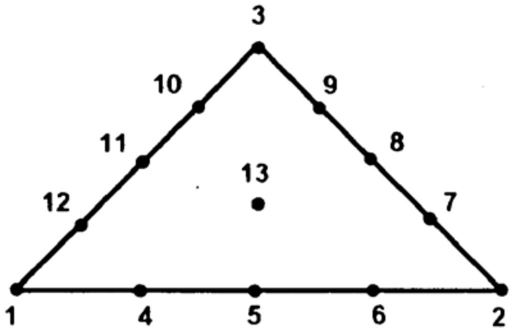

Figure 5. Triangular element with 13 nodes

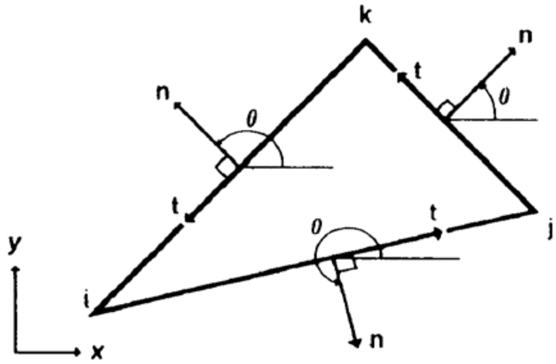

Figure 6. Element co-ordinates

In order to investigate the effect of elemental degrees of freedom, several elements are suggested. All of them have 13 nodes, with different degrees of freedom, and they are not C^1 compatible elements. Six such elements are explained in the following lines:

1. T13-1 element. The degrees of freedom for the nodes 1, 2, 3 and 13 are w, $w_{,x}$ and $w_{,y}$. For nodes 4–12, $w_{,nt}$ is the only degree of freedom.
2. T13-2 element. This element has w, $w_{,x}$ and $w_{,y}$ as degrees of freedom for nodes 1, 2, 3 and 13. In addition to these, $w_{,n}$ is used for nodes 4–12.
3. T13-3 element. For nodes 1, 2, 3 and 13, the degrees of freedom w, $w_{,x}$ and $w_{,y}$ are utilized. Also, w is the only degree of freedom for all other nodes.
4. T13-4 element. The nodes 1, 2, 3 and 13 have three degrees of freedom, w, $w_{,x}$ and $w_{,y}$. $w_{,nt}$ is used for the nodes 4, 6, 7, 9, 10 and 12. In addition to these, nodes 5, 8 and 11 have w as a degree of freedom.
5. T13-5 element. This element uses w, $w_{,x}$ and $w_{,y}$ as degrees of freedom for nodes 1, 2, 3 and 13, and $w_{,n}$ for nodes 4, 6, 7, 9, 10 and 12. Also, nodal deflection, w, is utilized for nodes 5, 8 and 11 as a degree of freedom.
6. T13-6 element. The degrees of freedom for nodes 1, 2, 3 and 13 are w, $w_{,x}$ and $w_{,y}$. For nodes 4, 6, 7, 9, 10 and 12 w is the only degree of freedom. In addition to these, $w_{,n}$ is utilized for nodes 5, 8 and 11.

It should be noted that T13-6 is a C^0 compatible element. Also, in T13-1 and T13-2, the adjacent elements have a unique normal slope.

NUMERICAL EXAMPLES

In this paper, many problems were solved by using the proposed elements. They had different shape, mesh, loading and boundary conditions. These numerical tests showed the convergence requirements. It is not possible to give all of the numerical results here. However, a few plate problems will be solved. These problems are analysed by a computer program which has been written by the authors. Some of the results obtained in this paper are compared with others. These numerical results will illustrate the ability of the element. It appears that only T13-5 gives good results.

Square plate with four columns at the corners

In this example, a square plate which is supported by four columns at its corners is analysed. The structure is under uniformly distributed load of intensity q. The nodal loads are obtained from the related integrations formula. Each side of the plate is equal to L. The supports are assumed to be simple. Therefore, there is zero deflection at the corners of the plate ($w = 0$). This plate is shown in Figure 7.

This problem is solved by six different meshes. These meshes have 4, 8, 16, 32, 64 and 72 elements, and are shown in Figure 8.

The deflection of the plate centre, which is obtained by each mesh, is shown in Table I. All six elements are used to solve this problem. In addition to these, the solution for elements T13-2 and T13-5 are compared with the result of other investigators in Figure 9. This Figure shows the

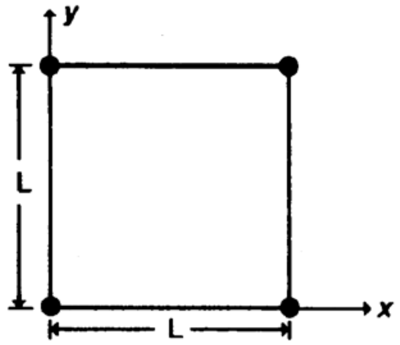

Figure 7. Square plate with four columns at the corners

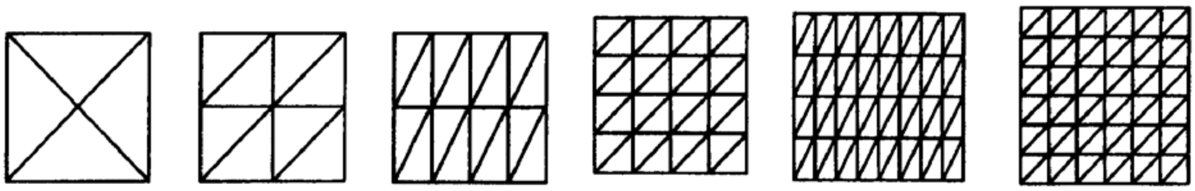

Figure 8. Meshes for square plate with four columns at the corners

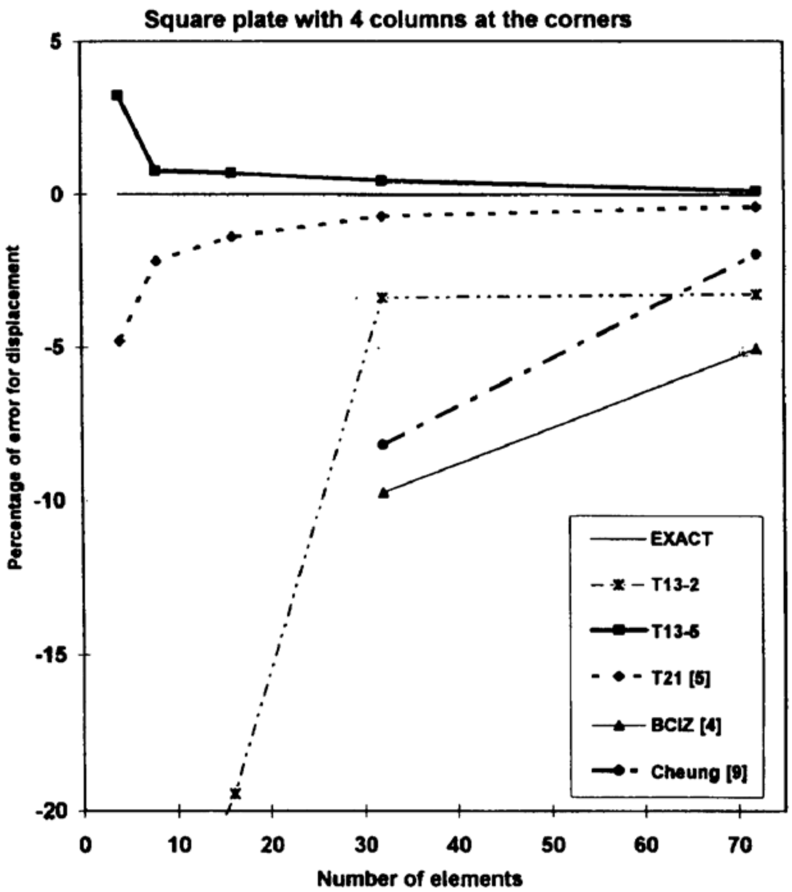

Figure 9. Convergence of the square plate deflection

Table I. Deflection of the square plate centre with four columns at the corners

No.	Element	Number of elements 4	8	16	32	64	72
1	T13-1	0·01774	0·01124	0·00286	0·00088	0·00019	0·00018
2	T13-2	0·02567	0·02908	0·01865	0·02483	0·00345	0·02229
3	T13-3	−0·16434	0·03079	0·00371	0·00139	0·00014	0·00020
4	T13-4	0·03155	0·05170	0·06039	0·05326	0·06025	0·05338
5	T13-5	0·02653	0·02590	0·02588	0·02582	0·02580	0·02573
6	T13-6	0·02937	0·03175	0·03219	0·03077	0·03084	0·03050
7	Exact			0·0257			

errors of the plate deflection in terms of number of elements. It is observed that the T13-5 element has excellent convergence. The exact deflection of the plate centre is given as[10]

$$w_c = 0{\cdot}0257\frac{qL^4}{D} \tag{3}$$

In this relationship, D is the rigidity of the plate and can be written in terms of plate thickness, t, E and ν as follows:

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{4}$$

Torsional cantilever plate

Figure 10 shows a cantilever plate with sides a and b, which is divided into two elements. Two equal forces, P, are applied at C and B, in opposite directions. The structure is twisted under the influence of the couple. The boundary conditions at the fixed end are as follows:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0 \tag{5}$$

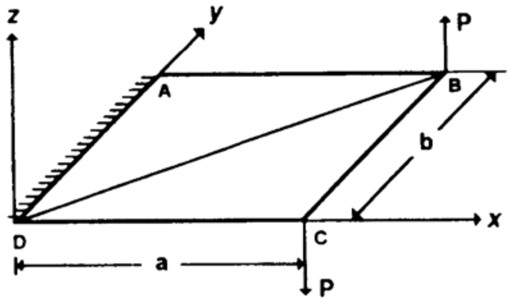

Figure 10. Torsional cantilever plate

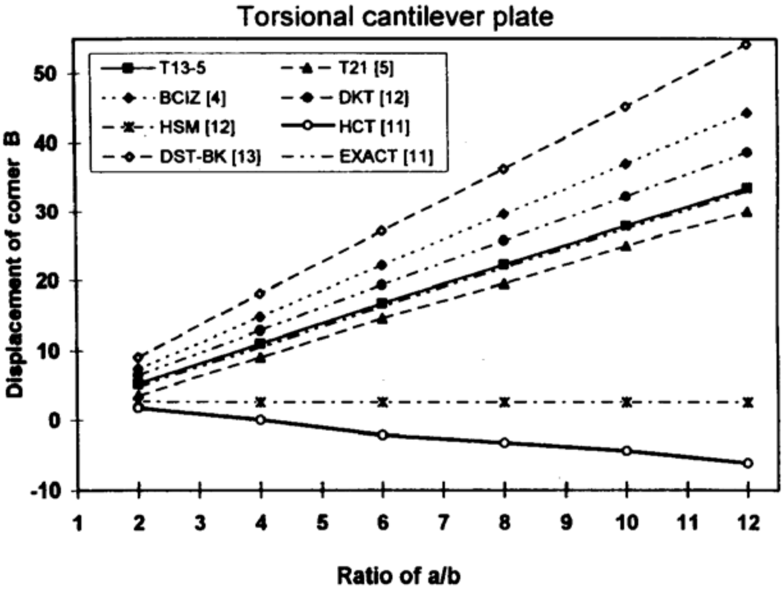

Figure 11. Displacement of corner B for different ratios of a/b

This plate is analysed with different ratios of a/b. The a/b ratios are: 2, 4, 6, 8, 10 and 12. In all cases, only two elements are used to model the structure. The deflection of the corner B is calculated with the different elements.

The displacement of the corner B for different aspect ratios (a/b) are shown in Figure 11. It is clear that the T13-5 element gives the closest results to the exact one.

CONCLUSIONS

It is generally believed that using a higher-order displacement function in the finite element method gives better answers. Also, some analysts may feel that the selection of number of nodes or degrees of freedom in any node does not alter the results very much. In this paper, a family of incompatible plate bending triangular elements is presented. The performances of the proposed elements are examined by solving different example problems. The results illustrated the ability of convergence to the exact answers. It is concluded that the T13-5 element is the best one. This element has 13 nodes, which are located at the corners and centroid, and also there are three nodes in each side. The degrees of freedom for this element are: w, $w_{,x}$ and $w_{,y}$ in the corners and centroid, w in the mid-side and $w_{,n}$ in other nodes. It should be added that, although the other proposed elements have 13 nodes and the same displacement function, their degrees of freedom are not very effective.

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