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A VARIABLE ARC-LENGTH METHOD

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ABSTRACT

Methods of nonlinear analysis of structures are discussed. The paper will concentrate on the geometric nonlinear analysis . Developments of these methods in recent years are reviewed, and their advantages and disadvantages will be mentioned. Finally, a method for nonlinear analysis of structures is presented. The formulation of the method is written and its ability will be shown by solving an example.

Key words : Variable arc-length, geomenric nonlinear analysis, cylndrical arc- length, incremental-iterative scheme, load factor, limit points.

INTRODUCTION

Strength of any structure against the applied forces is dependant on its geometric shape. In general methods of structural analysis, the stiffness matrix is written according to initial geometry of structure and is assumed to be constant during loading stages and deflection of structure. This assumption is valid for small displacements. Nevertheless, displacements of different points of structure are possible to become so large that the above assumption will be inbalid. In this case, the geometry of structure and its stiffness vary. This results in geometric nonlinear behavior of structure.

Linear analysis of structures is easy and cheap. On the other hand, the nonlinear behavior is not only difficult to formulate but also makes the analysis expensive. It should be emphasized that the long time spent for analysis is another difficulty. Nevertheless, advances in engineering sciences and computer technology has focused the attention to nonlinear analyses and valuable investigations are eing performed at the present time. These investigations follow two important aims. In the first aim, the load-bearing capacity of structure beyond the small displacements is determined. Using this capacity will be more economical than the one obtained by linear analysis. In the second aim, the failure behavior of structures is numerically simulated by these investigations and costly expenses of experiments are reduced. In fact, numerical methods can be performed by computer and need no material and testing equipments.

However , the nonlinear analysis techniques have some difficulties in formulation and utilization. On the other hand, the complex duty of structures requires exact study of their behavior.

There have been much advances in the methods of nonlinear structural analysis up to now. Different methods with special properties have been proposed. Nevertheless, no perfect and general method has ever been suggested. A suitable method for nonlinear analysis must have some specifications. It must possess a simple sturcture and be compatible with nonlinearity roots. It must result in exact solution with fast convergence. At last, the method should be general and expandable and facilate more than one solution procedure for structural analysis.

Developments of nonlinear analysis of structures will be discussed in the following sections. At first, history of well-known methods with their specifications is overeviewed. Then , the properties of incremental-iterative methods is described. Following that, some points on the Arc-length methods will be given. At the end, a new method for nonlinear analysis of structures with its formulation are introduced. The method possesses good computational ability. Its merit is discussed by giving numerical example.

Shor review

Different methods of nonlinear analysis were established after the computer was used in structural analysis. Turner, Dill, Martin and Melosh were among the first persons who started analyzing complex structures with large displacements under temperature and external loads in 1960. The method proposed by these researchers is known as direct stiffness method. This method was used for analyzing structures with geometric nonlinear behavior at that time. One of the disadvantages of the direct stiffness method is excluding some limit point. The researchers used Newtonian schemes, Which are utilized as correcting methods, for analyzing structures with geometric nonlinear behavior up to 1970. The Newtonian methods were used initially by Oden in 1967, Mallet and Marcel in 1968 and Muray and Wilson in 1969. Stricklin et al started evaluating self-correcting, virtual force, energy and Newtonian methods in 1970. Performance of Newtonian methods were noticed in this year and these schemes were used in ninlinear analysis of structures extensively. Reinbold and Ortega suggested Newton- Raphson methods for nonlinear analysis of structures in this year [R3]. Nonlinear analysis by these methods is done by linear analysis and successive correction of equation governing the bahavior of structure and attempting to obtain suitable answer. One of the disadvantages of these methods is their inability to pass through some limit points. It should be noted that Ortega proposed high-order methods in addition to Newton-Raphson ones in 1970.

Among the updating methods the DFP and BFGS methods have been changed more. These methods were first proposed by Davidon in 1959 and Flecher and Powell in 1963. These schemes were known as quasi-Newton ones at that time. In the quasi-Newton methods , the researcher is approximating stiffness matrix at each step of loading. This is done by gathering ingormations from the previous steps. After some years, this process changed into Broydon's, PSB, BFGS and DFP methods [R3]. In PSB, DFP and Broydon's, the stiffness matrix is updated at each step of loading. This is done by the relation given by the resercher. In BFGS method, the inverse of stiffness matrix is updated at each step of the loading. These methods have suitable convergence rate. Among the quasi-Newton methods, the BFGS method is more applicable because of fewer computations and input data. One of the disadvantages of these methods is storing some matrices for updating the stiffness matrix during analysis process. In addation, the stiffness matrix obtained at each step is not sparse which increases the time spent for solving system of equation. This property reduces the applicability of the method for large and complex structures. The aforementioned methods have been much changed in the reent year.

Argyris suggested the displacement- increment method with the aim of passing through the load limit points in 1965. According to the method, a displacement of a degree-of-freedom of structure is assumed to be known and is applied to the structure incrementally. Then , load factor and displacements of other degrees-of-freedom are calculated by governing equations. It should be noted that the stiffness matrix loses its symmetry and banded properties during the analysis. Pian (1970) and Zienkiewicz (1971) continued work on the method and gave a simpler form for it [A1]. They eliminated the need for inversing stiffness matrix from the available equations. This procedure is known as modified displacement-increment method. Hisler, Stricklin and Key proposed the self-correcting methods [H1]. This scheme is similar to Argyris , Pian, and Zienkiewicz's method. The unknown displacements are divided into two parts in this method. The first part is concerned with base load and the secong relates to unbalanced load. On the other hand, the governing system of equations is divided into three parts. The first part consists of euations with unknown displacements relating to the base load. The unknown displacements concerened with unbalanced loads are included in the second part. The third part is the equations determining load factors. Batoz and Dhatt proposed onother method of displacement increment methods [B2]. Three methods of analysis are discussed in their paper. The first two ones are the Argyris and Zienkiewicz and Pian's method and the third one is proposed by the authors. Their method is famed as displacement-control method. The method is recommended as a supper scheme and it is the modified form of previous procedures. There is no need for some variables and matrices to be stored in this method. One of the most importand specifications of this method is passing the load limit points smoothly. On the other hand, the stiffness matrix keeps its symmetry and banded form. It is noticable that all of the displacement-incremental methods are not able to pass limit points of displacement. In addition, determining a value for a particular degree-of=freedom of the structure is a difficult task.

Reinbold classified the available methods in 1974 [R3]. He divided them into four groups. They were linearizing, updating, minimizing and higer-order methods. He investigated the specifications of each group separately and compared them with each other. Monddar and Powell evaluated some of the existing methods in a paper in 1977 [M1]. The first scheme studied in this paper is the unbalanced sterp-by-step method. It is belong to incremental methods. The stiffness matix is not corrected at each step of the loading in this mehod. Step-by-step method with correcting stiffness matrix, Newton-Raphson, and iterative method with constant stiffness are the other procedures explained by these researchers. They compared them by each others and evaluated their advantages and disadvantages. Finally, a method was suggested with previous advantages and more speed. It is known as the Newton-Raphson method with modifying stiffness matrix after several iterations. In this method, by controlling a proposed relation, the stiffness matrix is corrected after some iterations if the convergance speed decreases.

Crisfield (1979) suggested a fast method of the modified Newton-Raphson ones [C2]. It is known as the fast modified Newton-Raphson method. This researcher had given a relation which increases the speed of finding displacements in iterative analyses. The disadvantage of the method is storing some of the matrices for use in next iterations. On the other hand, it is not able to pass some of the limit points. Bergan, Holand and Soride suggested equations for automatizing the nonlinear analysis in the same year [G1]. The proposed relation gave the current stiffness factor or CSP. The relaton is written between the energy of present step of the loading and energy of initial step of the loading. In this way, increasing or decreasing of structure stiffness will increase or decrease the factor of structure stiffness. These authors used the stiffness factor for determining the convenient coefficient of incremental load at the beginning of each step of loading and also determining the location of limit point and passing it. For the first time, the researcher could analyze the structure without controlling sequentially.

The arc-length methods were initially suggested by Wempner in 1971 and Riks in 1972 [D1]. Another relation is proposed to solve the system of equations besides the governing equations of structure in this group of methods. This relation is known as condition equation. The arc-length methods are of the incremental-iterative ones. The coefficient of incremental load at each step is determined by condition equation when assuming a value for arc-length, which is the locus of distance between the results of iterative analyses and the previous equilibrium point on the load-displacement curve. The incremental displacement of this load is then calculated by incremental analysis and the attemption is made to minimize the unbalanced load resulting from the iterative analyses. Finally, a new equilibrium point is obtained on the load-displacement curve. It should be noted that all of the specifications of aforementioned analysis depend on the kind of condition equation. Therefore, the researchers tried to find a convenient form of the condition equation. Riks suggested a special condition equation in 1979 [R2]. The relation is based on the displacement and incremental load factor of the loading step. To reach the arc-length, Riks assumed that all of the equilibrium points resulting from the iterative analyses in a step of loading are located on a straigth line. By this way, a relation for determining the load factor at each iterative analyses was obtained. Riks's procedure is known as the normal plane method. One of its disadvantages is the incapability of passing through the displacement limit points. Crisfield investigated Riks's method in a paper in 1981 [C3]. He suggested the spherical plane and cylinderical plane scheme to overcome the disadvantages of normal plane method. These methods are modified form of Riks's one. In the spherical plane method, crisfield proposed a special condition equation by assuming that the locus of equlibrium points of iterative analyses in a loading step are located on a spherical surface. On the other hand, in the cylindrical arc-length method, he assumed the projection of the curve of equilibrium points on the coordinates plane of structure displacements, is a circle. Crisfield's methods introduces the creation of powerul methods in nonlinear analysis of structures at that time.

Ramm defined incremental-iterative methods and investigated the existing powerful methds in 1981. He explained displacement control method simultaneously. Ramm neglected some of the matrices related to the load factor to obtain an easier form of the technique. Although this assumption increased the speed of computations, passing some of the limit points of displacement was still a problem. Ramm corrected Crisfield's cylindrical arc-length method and the normal plane scheme proposed by Riks and wimpner. A more convenient form of normal plane method was obtained in this way. Ramm compared the methods by solving examples and explained the abilities of modified normal plane method. Park suggested a new method named as "Parabolic arc-length method", in 1982 [B1]. The projection of the curve of the equilibrium points obrained from iterative analyses on the coordinates plane relating to two particular degrees of freedom of structure is a parabola in this method. It should be noted that speed of computations is not very high and procedure is the same as Crisfield's arc-length method.

Dhatt and Touzot investigated the fundamental methods of nonlinear analysis in 1985 [D1]. At first, they explained the substitution method. This method is one of the nonlinear analysis procedures which is not of great use. The secant stiffness matrix is used to analyze the structure in substitution method. The load is applied to structure completely and unbalanced forces are calculated in this method. Then, the stiffness matrix of structure is updated and another analysis is done with the whole load. This process is continued till the unbalanced forces reach a small value. One of the disadvantages of this method is disconvergence in analysis of some structures. Dhatt and Touzot also explained the Newton-Raphson and incremental load methods and discussed the advantages of them. The aforementioned methods are not able to pass that part of the load-displacemtn curve after the first load limit point. According to Dhatt and Touzot's suggestion, analysis must be done by displacement increment method in this region. Ryu and Arora wrote a paper in nonlinear analysis methods with substructuring process at the same year and investigated and calssified nonlinear analysis methods [R3]. All nonlinear analysis methods have been classified and their charactristics have been investigated in the paper. Attention has been paid to Reinbold's classification in 1974. Finally, the best group of nonlinear analysis methods has been introduced by comparing them.

Kolar and Kamel proposed a special kind of arc-length methods in 1986 [K1]. As it is seen in the condition equation of arc-length, scheme the relation is dependant on two vectors, namely load and displacement. The authors changed the influence of each vector in the condition equation by multiplying these vectors by a variable. They suggested a special method by choosing particular values of these multipliers. On the other hand, these researchers updated the arc-length relating to new step of loading by determining stiffness factor and multiplying it by the arc-length of previous step. The iterative procedure used at each step of loading is of the modified Newton-Raphson kind. One of the disadvantages of the method is time- consuming computations. In other words, much computations and time are needed to obtain the condition equation at each step of loading. The method has good speed for some of structures with particular initial load factor. If the initial load factor varies, the speed of method will decrease comparing to similar ones in the group of arc-length methods. In other words, the speed of the method is much dependant on structure type and initial load factors.

Forde and Stimer suggested other methods of arc-length family [F2]. The locus of obtained points at each iterative analysis are segmented lines in all of the methods proposed by these researchers. In other words, the aforementioned surface contains sequential broken lines. The proposed methods perform similarly and their computations are limited. Neverthelss, the methods resemble much to Crisfied's cylindrical arc-length process. These authors have compared the methods to cylindrical arc-length process by solving some examples and at the end, it had been concluded that the cylindrical arc-length method is more capable. In this procedure, the locus of equilibrium points obtained from iterative analysis, which is formed of broken lines, is almost on the circular surface of the cylindrical arc-length method. Belliny and Chulya investigated and compared spherical arc-length, cylindrical arc-length and parabolic arc-length method in the same year [B1]. They proposed some relations for updating the size of the arc-length at the beginning of the step. These suthors compared the methods by solving four examples. Belliny and Chulya concluded that the cylindrical arc-length method is more efficient than the two other techniques, because there is a good compatibility between condition equation and relation updating the arc-lenght at each step of loading. Time of computations and number of steps for updating the arc-length is less than other methods. Chan and lau investigated another group of iterative incremental methods at the same year [C1]. As it was mentioned before nonlinear analysis of large structures with numerous degrees of freedom is time-consuming and expensive. Therefore, special methods shlould be used to reduce the computations in these sturctures. At first, Chan and Lau explained the cylindrical arc-length methods for nonlinear analysis of structures. They chose a special matrix and multiplied it by stiffness matrix to reduce its dimensions. In this way, the dimensions of stiffness matrix were reduced and less time is needed for solving governing equations. The special matrix is made of several displacement vectors. In addition, Chan and Lau suggested a new method for analysis of structures by mixing the explained technique and cylindrical arc-longth process. It should be noted that the capabilities of the method is reduced for small structures because of excessive number of matrix coefficient.

Chan investigated arc-length and other incremental methods in a paper in 1988 [C5]. He believes that all the incremental-iterative methods have time-consuming computations which are not essential. He proposed an incremental-iterative scheme named as minimum residual displacement method. The author's aim was minimizing the residual displacement obtained at each step of iterative analysis. This method possesses good convergence speed with less computation. Safjan investigated reduced basis technique in the same year [S1]. He evaluated some of these of structure is written as linear composition of some vectors. In other words, the unknown displacement vector is decomposed into the matrix of known base vectors and the general vector of degrees-of-freedom. The base vectors matrix is N*N and vector of degrees-of-freedom is N*1. Entering these two vectors into the governing equations makes them more simple. The vectore of edegrees-of-freedom is obtained by solving the new equations. This task is not expensive, but enters some errors in the calculations. On the other hand, the precision of solution and ability of the process depends on the selection of the base vectors. These vectors should be chosen to reduce the expense of the computation time and they shuold be independant from member type and nonlinear factor. It should be noted that entering the effect of base vectors will increase the computations time in some cases. The aformentioned method has some disadvantages and the researchers are trying to overcome them.

Lee studied quasi-newton methods in 1989 [L2]. He believes that the application of these methods increase the speed of convergence. Lee explained al of the quasi-Newton methods at first. He investigated the BFGS and DFP methods and concluded that DFGS has more ability than the DFP. Lee explained the cases in which the BFGS method is not efficient. One of these cases is determining stiffness matrix or inversing of it near the limit points. Lee suggested another relation for updating stiffness matrix to eliminate the disadvantages of BFGS method. There is a variable named as convergence ratio in the equation. The situation of analysis is determined by the value of this ratio. In other words, the convergence or divergence of the analysis is known from this ratio. One of the disadvantages of the method is the time spent for computations and also difficulty of programming. It shlould be noted that lee's method is not able to distinguish a curve with negative slope in the load diagram. Sheu et al proposed the automatic Newton-Raphson method to obtain higher speed of convergence in the modified Newton-Raphson procedure [S2]. They suggested a relation based on the determinant of siffness matrix in each iterative analysis. They updated the stiffness matrix at a special time by choosing limits for aforementioned relation. in other words, if the value obtained by the relation is in the suggested range, the stiffness matrix must be updated. This will increase the speed of analysis. This method has the same disadvantage as Newton-Raphson method. The difference is in hte speed of analysis which is varying automatically in this scheme. Jeusette, Lasche and Idelsohn investigated some of the arc-ength method in this year [J1]. They initially explained the nonlinear relation governing the behavior of structure and then introduced the condition equation during the arc-length procedure. Ageneral form of condition for this technique is given. Relations were written for determining in the incremental load factor at the beginning of each step of the loading and he most convenient of them was chosen. One of the specifications of these two methods is obtaining load factors during each iterative analysis. Aspecial procedure should be used to obtain the correct load factor. Jeusette et al explained the procedure to obtain the convenient load factor with a relation. They Suggested the normal plane and external work with constant increment ara-length methods to prevent the calculations of the oad factor. The condition equations in these two procedures is so that only a single value is obtained for the load factor in each iterative analysis.

Al-Rabsy investigated some of the arc-length methods used for norlinear analysis of structures in 1989 [A1] and 1991 [A2]. At first, he explained the Newton-Raphson methods and discussed their characterstics. He explained the needs of powerful methods by investigating the advantages and disadvantages of Newton-Raphson methods. In the following, he expressed the requirement of the stiffness factor for passing the limit points and also determining the incremental load at the beginning of each step of loading. The characteristics of a convenient convergence and the criteria for obtaining the convergence were evaluated then. Al-Rasby investigated four schemes of the arc-length methods group and compared them by solving examples. He concluded that crisfield's spherical arc-length method is more efficient than the other ones.

Wong and Tin-Loi used the arc-length and Newton-Raphson methods simultaneously to increase the speed of analysis in 1990 [W1]. The aforementioned computational technique has been suggested according to the special capaibities of each method. A part of the load-displacement curve with low variations of slope is traced by Newton-Raphson method and the other part is followed by arc-length methods. The time for using the arc-length methods is determined by a relation suggested by Tin-Loi. In other words, a relation had been given according to variations of determinant of the stiffness matrix so that if the value obtained by it is limited to and analyst's suggestive number. the arc-length method can be used intead of Newton-Raphson ones. The disadvantage of the method is that it can be used just once and in the part of curve with positive slope. In other words, the method is not applicable near the limit points of displacement. In addition, the two methods can not be converted to each other near these points. Clarke and Hancock investigated and compared some of the nonlinear analysis methods in this year [C6]. The authors classified the methods into some parts for simplicity and explained each part seperately. The iterative procedures were explained in the first part and iterative-incremental and load increment methods were discussed in the second and third par, respectively. Relations for controlling the existing errors during each norlinear analysis were given then. They finally compared the methods by solving some examples. Zienkiewicz investigated some of the nonlinear analysis methods in his book in 1991. All ot the procedures discussed in this book are fundamental methods of nonlinear analysis [Z1]. Criteria for convergence and relations for increasing the speed of convergence were given there. The characteristics of nonlinear analysis methods have been studied too.

Crisfield accumulated some of the arc-length methods and classified them in his book in 1991 [C4]. The charactersitics of nonlinear behavior of structures and fundamental methods of nonlinear analysis have been reviewed at the beginning. The nonlinear analysis of plate and shell structures have been explained in other chapters. The arc-length method and some similar ones have been explained in the book. Reasons for utilization of these methods have also been given. In the following, the spherical arc-length, normal plane and updating normal plane methods have been explained. Stiffness factors and reasons to use them in nonlinear analysis have been discussed.

Rothart and Gebbeken evaluated some of the powerful methods in nonlinear analysis of structures in 1992 [R1]. The arc-length method is one of them. A method has also been suggested after explaningstiffness factor which uses the simultaneous effect of incremental load and incremental displacement methods for passing the limit points. In other words, a part of the load-displacement curve is specified by incremental load method and the other part is signified by incremental displacement displacement procedure and vice versa. Lam and Morley suggested the modified constant arc-length method in a paper [L1]. They found a new locus for the equilibrium points of iterative analysis by assuming multipliers for the condition equation of Riks and Wempner's method. They proposed relations for neglecting the complex roots. The capability of the method in passing the limit points has been also presented. It should be noted that this method has a simple form and can be used in nonlinear analysis of many problems.

Fafard and Massicotie investigated Crisfield's spherical arc-length and Ramm's normal plane method and proposed a new technique [F1]. The method is known as the modified Criesfield-Ramm procedure. The locus of equilibrum points of each iterative analysis in this method is on segmented lines like Ramm's method and these lines are limited to the locus of equilibrium points of iterative analysis obtained from Crisfields methld. In other words, the end of the lines intersect with spherical surface of Crisfield's method at different points. The modiefied Crisfield-Ramm's method has the advantages of spherical surface and normal plane method, and in addition, it is fast in passing the limit points.

In spite of different methods which have ever been proposed for nonlinear analysis of structures, there is not a method capable of analyzing structures precisely, fast and perfectly. So the researchers are always searching for a method with explained characteristics by vast investigations. It should be noted that there are other methods for nonlinear analysis of structures rather than the discussed schemes. The methods with more applicability were only discussed in this paper.

incremental-interative methods

An incremental-iterative analysis is performed to determine the displacement of structure udner the applied loads. Figure (1) demonstrates the result of such an analysis for a single degree-of-freedom structure. Point m is an equilibrium point of structure in the figure. The displacement of structure at this point under the load $\lambda_m P$ is U_m. Here, λ_m is the factor of applied load at this point, and P is the total load applied on the structure. The distance between m and next equilibrium point, M+1, is the step of loading. This distance is traced by performing iterative analysis and a new equilibrium point is obtained [H4]. The incremental load $\Delta\lambda^{(1)}$ P is initially applied on the structure during iterative analyses. The result is point (1) on the diagram. This point is not on the load-displacement curve because the analysis is performed by assuming linear behavior for structure and it should be transformed vertically to the value of $R^{(1)}$ until it is located on the curve. The vertical distance $R^{(1)}$ is known as unbalanced load. This load is obtained by difference between applied load on structure ($\Delta\lambda^{(1)}P$) and internal forces ($F^{(1)}$). In the next iteration, the load $R^{(1)}$ is applied on the structure, and after calculating the new stiffness matrix at ercent point, analysis is continued. The result of this analysis is point (2). The unbalanced load $R^{(2)}$ is calculated then. This procedure is kept on until the unbalance load R(m+1) is less than its allowable value. Finally, point (m+1) is obtained.

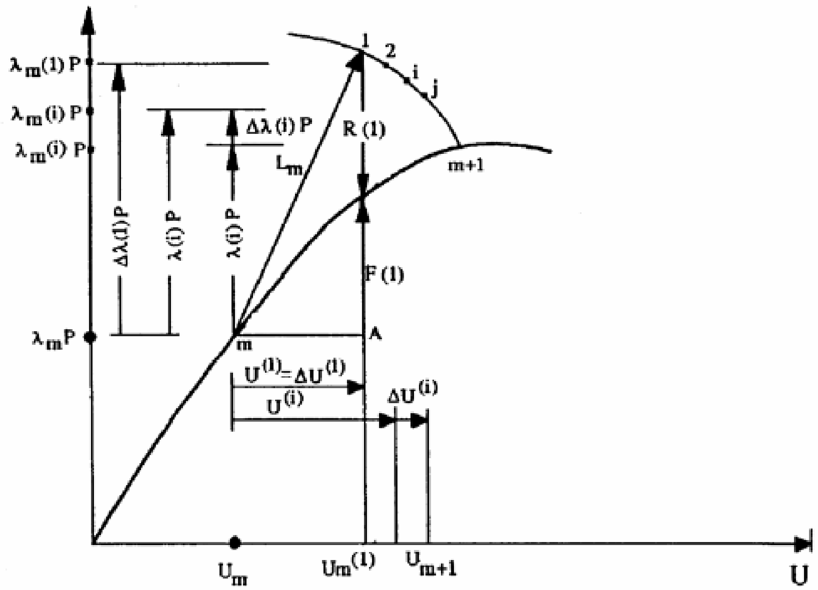

Fig (1) : Inerementai-iterative methods

The mathematical relations of the method aer presented in the following. If $\{U\}^{(i)}$, $\{P\}^{(i)}$ and $\lambda_m(i)$ are displacement, load and load factor up to (j) th iteration in the 9m) th step of loading, respectively, the following relations can be written for every point on the diagram of structure behavior :

$$\{P\}_m = \lambda_m \{P\} \tag{1}$$

$$\{P\}^{(j)}_m = \{P\}_m + \{P\}^{(j)} = \{P\}_m + \{P\}^{(i)} + \{\Delta P\}^{(j)} \tag{2}$$

$$\lambda^{(j)}_m = \lambda_m + \lambda^{j)} = \lambda_m + \lambda^{i)} + \Delta\lambda^{(j)} \tag{3}$$

$$\{U\}^{(j)}_m = \{U\}_m + \{U\}^{(j)} = \{U\}_m + \{U\}^{(i)} + \{\Delta U\}^{(j)} \tag{4}$$

$\{U\}_m$, $\{P\}m$ and λ_m in the above equations are displacement, load and load factor of the structure at point m, respectively. Points I and j are the result of iterative analysis. If the internal forces in structure are shown by $\{F\}^{(i)}$, the equation of the structure behavior with tangential sfiffness matrix $[K]_T^{(i)}$ can written as :

$$[K]_T^{(i)}\{\Delta U\}^{(j)}=\{\Delta P\}^{(j)}+\{P\}_m^{(i)}-\{F\}^{(i)} \quad (5)$$

$$\{R\}^{(i)}=\{P\}_m^{(i)}-\{F\}^{(i)} \quad (6)$$

$$[K]_T^{(i)}\{\Delta U\}^{(j)}=\Delta\lambda^{(j)}\{P\}+\{R\}^{(i)} \quad (7)$$

In order to find more simple relations, the displacement of structure under applied loads $\Delta\lambda\{P\}^{(i)}+\{R\}^{(i)}$ is divided into two parts. First part ($\{\Delta U\}_p^{(j)}$) is the displacement related to base load, $\{P\}$, and the second part ($\{\Delta U\}_r^{(j)}$) is displacement depending on unbalance load, $\{R\}^{(i)}$. Therefore, the previous equations can be written as [D1]:

$$\{\Delta U\}^{(j)}=\Delta\lambda^{(i)}\{\Delta U\}_p^{(j)}+\{\Delta U\}_r^{(j)} \quad (8)$$

$$[K]_T^{(i)}\{\Delta U\}_p^{(j)}=\{P\} \quad (9)$$

$$[K]_T^{(i)}\{\Delta U\}_r^{(j)}=\{R\}^{(i)} \quad (10)$$

If incremental-iterative analysis is only preformed by these equation, some of equilibrium points are not reached and the respoinse does not converge. This problem usually cccurs near limit points of the load and displacement. Other relations and assumptions are suggested to overcome this problem in the most of incremental-iterative methods. These relations make it possible to pas the limit points and obtain convergence. The additional relations are known as condition equations. In an incremental-iterative method, an equilibrium point on the load-displacement curve can be obtained by several iterative analysis. The arc-length methods suggest a simple procedure for nonlinear analysis by modifying the governing equations and giving a condition equation [A2,F1,F2]. It should be known that two surfaces in the load-displacement diagram are important in arc-length methods. The first surface is the locus of the results of iterative analyses. These values specify the range of results of all iterative analyses during a step of the loading. This surface passes the points (1),(2),(i),(j),.. and (m+1) in Fig (1). The second one is the distance between equilibrium point and the curve of iterative analyses and is known as the arc-length. The distance between point (m) and point (1) is this surface in Fig (1). A special method can be obtained by noticing on relative position of these two surfaces. In other words, a particular method of the arc-length methods can be concluded from how these surface intersect.

The condition equation is always written between load and displacement in arclength methods. This equation displays the distance between locus of the results of iterative analyses and previous equilibrium point. The distance between this surface and previous equilibrium point. The distance between this surface and previous equilibrium point (m) is assumed to be L_m. The following equation can be written for triangle (mA1) in Fig (1):

$$\{\Delta U\}^{(1)T}\ \{\Delta U\}^{(1)}+(\Delta\lambda^{(1)})^2+\{P\}^T\ \{P\} = L_m^2 \tag{11}$$

The given condition equation is used in a special way in each of the arc-length methods. The most poweful one of these procedures is discussed in the following section.

Cylindrical arc-length method

In cylindrical arc-length method it is assumed that distance between locus of the results of iterative analyses and previous equilibrium point on the curve obtained from projection of load-displacement diagram on the plane of coordinates with exes representing displacements of degrees-of-freedom, is constant [C3,C4,F1,M2]. Figure (2) represents the load-displacement diaram of the degrees-of-freedom U_1 and U_2. As it is seen in the figure, the distance between all points of iterative analyses and equilibrium point (m) must be the same for the assumption to be valid. It should be known that the advantage of cylindrical arc-length method is its simplicity of equations. If L_m is the distance from circular surface to equlibrium point (m), deflection $\{\delta U\}^{(j)}$ will represent the displacements of degrees-of-freedom relative to equilibrium point (m) in (j) th iteration. According to Fig (2) the following expression can be written:

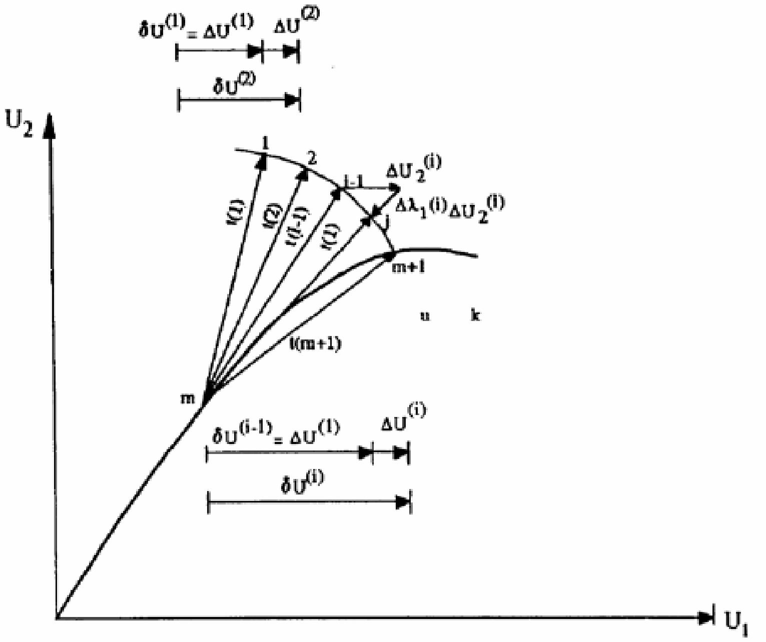

Fig (2): Cylindrical arc-length method

$$\{\delta U\}^{(j)T}\{\delta U\}^{(j)} = L_m^2 \tag{12}$$

Where $\{\delta U\}^{(j)}$ is equal to:

$$\{\delta U\}^{(j)} = \{\delta U\}^{(j-1)} + \{\Delta U\}^{(j)} \tag{13}$$

$\{\Delta U\}^{(j)}$ in above equition is the vector of displacements of the degrees-of-freedom after each iterative analysis and is defined as follows:

$$\{\Delta U\}^{(j)} = \Delta\lambda^{(j)}\{\Delta U\}_p^{(j)} + \{\Delta U\}_r^{(j)} \tag{14}$$

Inserting equations (13) and (14) into equation (12) will result in the following quadratic eqution:

$$a(\Delta\lambda^{(j)})^2 + b(\Delta\lambda^{(j)}) + c = 0 \tag{15}$$

$$a = \{\Delta U\}_p^{(j)T}\{\Delta U\}_p^{(j)} \tag{16}$$

$$b = 2\{\{\delta U\}^{(j-1)} + \{\Delta U\}_r^{(j)}\}^T\{\Delta U\}_p^{(j)} \tag{17}$$

$$c = \{\{\delta U\}^{(j-1)} + \{\Delta U\}_r^{(j)}\}^T\{\{\delta U\}^{(j-1)} + \{\Delta U\}_r^{(j)}\} - L_m^2 \tag{18}$$

The following situations can be discussed according to the sign of discriminant of the equation.

1-when $b^2-4ac \geq 0$: The equation has two real roots in this case. If the roots are $\Delta\lambda_1^{(j)}$ and $\Delta\lambda_2^{(j)}$, respectively, they can be obtained by considering the same direction for vectors $\{\delta U\}^{(j)}$ and $\{\delta U\}^{(j-1)}$. At first, displacement vectors $\{\delta U\}_1^{(j)}$ and $\{\delta U\}_2^{(j)}$ must be calculated by the roots. This is done by equations (13) and (14). The angle between these vectors and $\{\delta U\}^{(j-1)}$ is obtained by dot product. As a results, the angles θ_1 and θ_2 are calculated by the following equation:

$$\text{Cos}\theta_1 = \frac{\{\delta U\}_1^{(j)T}\{\delta U\}^{(j-1)}}{\left|\{\delta U\}_1^{(j)}\right|\left|\{\delta U\}^{(j-1)}\right|} \tag{19}$$

$$\text{Cos}\theta_2 = \frac{\{\delta U\}_2^{(j)T}\{\delta U\}^{(j-1)}}{\left|\{\delta U\}_2^{(j)}\right|\left|\{\delta U\}^{(j-1)}\right|} \tag{20}$$

The load factor depending on a angle with positive sign, is a covenient root of equation (15). If θ_1 and θ_2 have the same sign, the root which is closer to following value, will be acceptable:

$$\Delta\lambda = -\frac{c}{b} \tag{21}$$

Equation (21) represents linear solution of equation (15). In order to find a simple solution for existance of the roots of equation (15), the values a,b and c are inserted into the discriminant relation and existance of roots are investigated:

$$\{V\}^{(j)} = \{\delta U\}^{(j-1)} + \{\Delta U\}_r^{(j)} \tag{22}$$

$$\begin{aligned} b^2\text{-}4ac &= 4(\{V\}^{(j)T}\{\Delta U\}_p^{(j)})^2 - 4(\{\Delta U\}_p^{(j)T}\{\Delta U\}_p^{(j)})(\{V\}^{(j)T}\{V\}^{(j)} - L_m^2) \\ &= 4(\{V\}^{(j)T}\{\Delta U\}_p^{(j)})^2 - 4\left|(\{\Delta U\}_p^{(j)T}\right|^2(\left|\{V\}^{(j)}\right|^2 - L_m^2) \end{aligned} \tag{23}$$

If $\beta^{(j)}$ is the angle between vectors and $\{U\}_p^{(i)}, \{V\}^{(i)}$ the following equation can be written:

$$\{V\}^{(j)T}\{\Delta U\}_p^{(j)} = \left|\{V\}^{(j)}\right| \left|(\{\Delta U\}_p^{(j)}\right| \text{Cos } \beta^{(j)} \tag{24}$$

The following relation is obtained by substituting equation (24) in (23)

$$b^2\text{-}4ac = 4\left|\{\Delta U\}_p^{(j)}\right|^2 [\left|\{V\}^{(j)}\right|^2 (\text{Cos } \beta^{(j)})^2 - \left|\{V\}^{(j)}\right|^2 + L_m^2] \tag{25}$$

If the condition for real roots of equation (25) is written, the following relation will be valid:

$b^2 - 4ac \geq 0$ (26)

$$(\text{Cos } \beta^{(j)})^2 \geq 1 - \frac{L_m^2}{\left|\{V\}^{(j)}\right|^2} \tag{27}$$

In the above equation, when $\text{Cos } \beta^{(j)} = 0$, the vector $\{U\}_p^{(j)}$ is perpendicular to $\{V\}^{(j)}$. In this case, equation (27) can be written as follows:

$$\left|\{\delta U\}^{(j-1)} + \{\Delta U\}_r^{(j)}\right| = \left|\{V\}^{(j)}\right| \leq L_m \tag{28}$$

If the length of $\{V\}^{(j)}$ is less than the arc-length L_m, equation (15) will always have real roots. If $\{V\}^{(j)}.\ \{U\}_p^{(j)}$ is equal to zero, then, the roots of equation (15) will be:

$$b = 0\ ,\ \Delta\lambda^{(1)} = 0 \tag{29}$$

If $\{V\}^{(j)}.\ \{U\}_p^{(j)}$ is not equal to zero, the following relations will be valis:

$$\Delta\lambda_1^{(j)} = -\frac{b}{a} = -\frac{2\{V\}^{(j)T}\{\Delta U\}_p^{(j)}}{\{\Delta U\}_p^{(j)T}\{\Delta U\}_p^{(j)}} \quad (30)$$

If the length of $\{V\}^{(j)}$ is greater than L_m, then, the existance of real roots for equation (15) will be dubious.

2-when b^2-4ac < 0 : The equation has complex roots in this case. This situation is occured when the structure becomes unstable and it usually happens in analysis of space structures, end-hinged arches and, ... In this case, the value of initial incremental load had been large and the amount of the arc-length or external load should be reduced. Therefore, if complex roots occur in any step of the loading, the analysis must restart from previous step. By assuming a value for L_m, at the beginning of each step of loading, the following equations are used to determine $\Delta\lambda^{(1)}$.

$$\{\Delta U\}^{(1)} = [K]_T^{(-1)} \Delta\lambda^{(1)} \{P\} \quad (31)$$

$$\{\Delta U\}^{(1)T} \{\Delta U\}^{(1)} = L_m^2 \quad (32)$$

Values of $\Delta\lambda^{(1)}$ and vector $\{\Delta U\}^{(1)}$ are unknown in the above equation. Combining these reletions and eliminating one of the unknowns leads to following equation:

$$\Delta\lambda^{(1)} = \pm \sqrt{\frac{L_m^2}{([K]^{-1T}[K]^{-1})\{P\}^T\{P\}}} \quad (33)$$

It should be noted that equation (33) specifies the incremental load factor at the beginning of each step of the loading. The correct sign of $\Delta\lambda^{(1)}$ is obtained by determinant of tangential stiffness matrix at the beginning of each step of the loading. It should be mentioned that change in sign of the determinant represents change in the sign of incremental load factor $\Delta\lambda^{(1)}$.

Calculating the Arc-Length

The arc-length can be assumed to be constant during an incremental-iterative analysis. The speed of analysis does not vary very much in this way. However, one can update these values and speed up the convergence. Most of the researchers suggest relations for updating the arc-length or load factor at the beginning of each step of the loading. Some of these relations are presented in the following which are of great use in the arc-length methods. Here, L_m and L_{m+1} are the amount of the arc-length at (m) th and (m+1)th step of the loading, respectively, J_d represents number of arbitry iterative analysis at each step of the loading, and J is number of performed iterative analysis at (m) th step of loading. Ramm and Crisfield suggested the following relations for updating the arc- length at (m+1)th step [C6]:

$$L_{m+1} = L_m \sqrt{\frac{J_d}{J}} \tag{34}$$

$$L_{m+1} = L_m \frac{J_d}{J} \tag{35}$$

Belline and chulya suggested the following relations for updating the arc-length at the beginning of each step of the loading [B1]:

$$L_{m+1} = L_m \sqrt[4]{\frac{J_d}{J}} \tag{36}$$

$$L_{m+1} = L_m \sqrt{\frac{J}{J_d}} \tag{37}$$

When the curvature of load-displacement curve is large, the value obtained by eauation (35) is small. On the other hand, if this curve is linear of its cuvature is small, this value will be lare. Therefore, when increasing or decreasing the amount of the arc-length, the need for this task should be determined by controlling the calculated value. To prevent this problem, most of the researchers substitute special numbers for J_d. Value of J_d is determined according to the numbers of degrees-of-freedom of structure (DOF):

1- If DOF < 25 , $J_d = 5$.

2- If 25 < DOF < 250 , $J_d = 6$.

3- If DOF > 250 , $J_d > 6$.

It should be noted that the incremental load factor can be updatd by similar equations at each step of the loading. Other relations have also been suggested by some researchers. It should be added that all of available relations are not able to determine convenient value for the arc-length and reduce speed of analysis in some structures. Authors of this paper suggest the following general form of updating equation. Different value can be substituted for η:

$$L_{m+1} = L_m \left(\frac{J}{J_d}\right)^{\eta} \tag{38}$$

Convergence Criterion

To find out when convergence and equilibrium points are obtained, during an iterative process, the researchers have suggested relations depending on incremental displacement variables [C6]. If N is the number of total degrees-of=freedom, $\Delta U^{(j)}$ is the last incremental displacement in present analysis, $U^{(j)}$ is displacement up to the present iteration and k represents the number of degree-of-freedom, some of these equations will be as follows:

1- Modified Absolute Norm:

$$\varepsilon_1 = \frac{1}{N}\sum_{k=1}^{N}\left|\frac{\Delta U_k^{(j)}}{U_k^{(j)}}\right| \tag{39}$$

2- Modified Euclideam Norm:

$$\varepsilon_2 = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left|\frac{\Delta U_k^{(j)}}{U_k^{(j)}}\right|^2} \tag{40}$$

3- Maximum Norm:

$$\varepsilon_3 = \mathrm{Max}\left|\frac{\Delta U_k^{(j)}}{U_k^{(j)}}\right| \tag{41}$$

The third one is of the great use among the above equations. Variable β_c is the tolerance of convergence and it is usually between 0.01 and 0.00001, according to the precision of problem. The convergence is obtained when the following equation is valid:

$$\varepsilon_3 < \beta c \tag{42}$$

A Variable Arc-Length Method

The aim of presenting a variable arc-length method is easily passing limit points, specially displacemtn limit points. The attempt is made to reduce the number of iterative analyses and increase steps of the loading in vicinity of these points. Speed of reaching equilibrium points is increased in this way. Figure (3) shows the load-displacement diagram of the structure with a single degree-of-freedom. The figure shows the way to reach the equilibrium point(m+1) at (m)th loading step. Point (1) is obtained by performing an incremental analysis in the beginning. Position of point (A) on the load-displacement curve is obtained after determining the unbalanced load $R^{(1)}$. Points (2),(3), ... (m+1) are obtained by iterative analyses in the step of the loading. It should be noted that the distance between points (1),(2), and (m) equals to $L_m^{(1)}$. In other words, point (1), (2) represent an arc of circle with radius $L_m^{(1)}$. Curvature of this arc changes after point (2). Radius of the arc and position of its center must change for doing this task. The new radius of the arc is $L_m^{(1)}$ and its center is (A). Point (3) is next obtained by an itrative analysis.

The same procedure is done for this point and a new arc-length is obtained. Therefore, curvature of the curve varies at each point. Each two sequential points on the locus of answers, display two ends of an arc of the eircle with specified center and radius. Therefore, the curve passing through points (1),(2),...(m+1) intersects with the load-displacement curve in a shorter length rather than other methods. The equilibrium points are accessed very rapidly near the limit points in this technique. The reason is the close relation between the number of iterative analyses in the step of the loading and value of the load increment in the next step. On the other hand, number of iterative analyses in each step of loading is less and therefore, size of loading step and speed of analysis inerease.

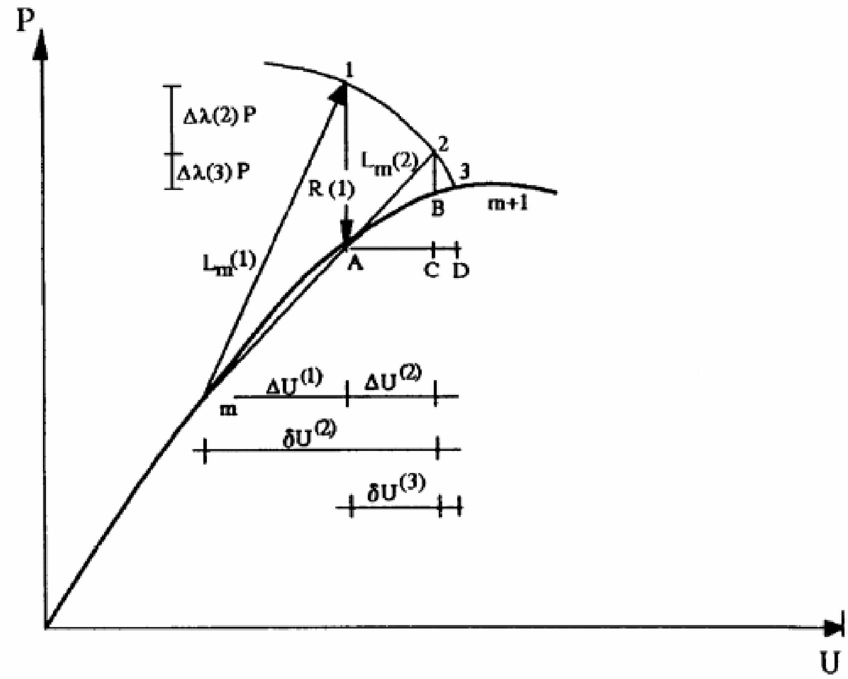

Fig 3: Variable arc- length ,method

Formulation of proposed method

The following relation can be written in triangle (m1A') of Fig (3):

$$\Delta U^{(1)}\, \Delta U^{(1)} + \Delta\lambda^{(1)} P\, \Delta\lambda^{(1)} P = (L_m^{(1)})^{(2)} \quad \textbf{(43)}$$

The same equation can be written in triangle (m2B'):

$$\delta U^{(2)}\, \delta U^{(2)} + (\Delta\lambda^{(1)} - \Delta\lambda^{(2)})^2 PP = (L_m^{(1)})^{(2)} \quad \textbf{(44)}$$

$\delta U^{(2)}$ in equation (44) is the displacement up to the present iteration from the equilibrium point at the beginning of loading step (m) and is ealculated by following relation:

$$\delta U^{(2)} = \Delta U^{(1)} + \Delta U^{(2)} \quad \textbf{(45)}$$

$\Delta U^{(2)}$ is obtained from the following equation in each iterative analysis:

$$\Delta U^{(2)} = \Delta\lambda^{(2)}\, \Delta U_p^{(2)} + \Delta U_r^{(2)} \quad \textbf{(46)}$$

$U_r^{(2)}$ and $\Delta U_p^{(2)}$ in equation (46) are displacements of unbalanced load and base load in second iteration of the step, respecitively. Substiuting $\Delta U^{(2)}$ in equation (45) and combining the result with equation (44), and substituting $L_m^{(1)}$ from equation (43) in equation (44), will result in the following quardratic eqution for finding $\Delta\lambda^{(2)}$:

$$a_1 = (\Delta\lambda^{(2)})^2 + b_1(\Delta\lambda^{(2)}) + c_1 = 0 \quad \textbf{(47)}$$

$$a_1 = \Delta U_p^{(2)} \Delta U_p^{(2)} + P\,P \quad \textbf{(48)}$$

$$b_1 = 2\,[\Delta U_p^{(2)} (\Delta U_r^{(2)} + \Delta U^{(1)}) - \Delta\lambda^{(1)} P\,P] \quad \textbf{(49)}$$

$$c_1 = (\Delta U_r^{(2)} + \Delta U^{(1)})(\Delta U_r^{(2)} + \Delta U^{(1)}) - \Delta U^{(1)}\ \Delta U^{(1)}] \quad \textbf{(50)}$$

The following relation is written in the third iterative analysis:

$$\Delta U^{(2)}\, \Delta U^{(2)} + (R^{(1)} - \Delta\lambda^{(2)} P)(R^{(1)} - \Delta\lambda^{(2)} P) = (L_m^{(2)})^2 \quad \textbf{(51)}$$

Similar equation can be written in triangle (A3D):

$$\delta U^{(3)}\, \delta U^{(3)} + [R^{(1)} - (\Delta\lambda^{(2)} + \Delta\lambda^{(3)})\, P]\, [R^{(1)} - (\Delta\lambda^{(2)} + \Delta\lambda^{(3)})\, P] = (L_m^{(2)})^2 \quad \textbf{(52)}$$

$\delta U^{(3)}$ in equation (52) is the incremental displacement between points (A) and (C), and is calculated from the following equation:

$$\delta U^{(3)} = \Delta U^{(2)} + \Delta U^{(3)} \quad \textbf{(53)}$$

On the other hand, $\Delta U^{(3)}$ is calculated from displacements $\Delta U_r^{(3)}$ and $\Delta U_p^{(3)}$:

$$\Delta U^{(3)} = \Delta U^{(3)}\, \Delta U_p^{(3)} + \Delta U_r^{(3)} \quad \textbf{(54)}$$

Combination of relations (51) , (52) , (53) , (54) results in equations similar to (47) , (48) , (49) , (50) :

$$a_2 = (\Delta\lambda^{(3)})^2 + b_2(\Delta\lambda^{(3)}) + c_2 = 0 \quad \textbf{(55)}$$

$$a_2 = \Delta U_p^{(3)} \Delta U_p^{(3)} + P\,P \quad \textbf{(56)}$$

$$b_2 = 2\,[\Delta U_p^{(3)} (\Delta U_r^{(3)} + \Delta U^{(2)}) - (R^{(1)} - \Delta\lambda^{(2)} P)\, P] \quad \textbf{(57)}$$

$$c_2 = (\Delta U_r^{(3)} + \Delta U^{(2)})(\Delta U_r^{(3)} + \Delta U^{(2)}) - \Delta U^{(2)} \ \Delta U^{(2)}] \quad \textbf{(58)}$$

Value of $\Delta\lambda^{(3)}$ is obtained by solving quadratic eauation in relation (55). Relations similar to (55),(56),(57),(58) are found during each iteration by continuing the procedure. Authors have written a program according to the presented relations. As a result, speed of the analysis is the same as similar schemes. However, discriminant of equations (47) and (55) becomes negative in some cases. The reason is the influence of the load factor in the relations. In this case, a new step of the loading is selected. To reduce the effect of the load factor in analysis, one can initialize it with small values at each step of the loading. This task needs precise control of the program during running. Another method to reduce the effect of the load factor is eliminating this variable from the relations. Doing this, similar relations to the cylindrical arc-length method will be obtained. Eliminating the load factor from relations will result equations (59)-(62) for the second analysis, and (63-(66) for third analysis and after that:

$$a_1 = (\Delta\lambda^{(2)})^2 + b_1(\Delta\lambda^{(2)}) + c_1 = 0 \quad \textbf{(59)}$$

$$a_1 = \{\Delta U\}_p^{(2)T} \{\Delta U\}_p^{(2)} \quad \textbf{(60)}$$

$$b_1 = 2\,[\{\Delta U\}^{(1)} + \{\Delta U\}_r^{(2)}]^T \{\Delta U\}_p^{(2)} \quad \textbf{(61)}$$

$$c_1 = \{\{\Delta U\}^{(1)} + \{\Delta U\}_r^{(2)}\}^T \{\Delta U\}^{(1)} + \{\Delta U\}_r^{(2)}\} - \{\Delta U\}^{(1)T} \{\Delta U\}^{(1)} \quad \textbf{(62)}$$

$$a_{(j-1)} = (\Delta\lambda^{(j)})^2 + b_{(j-1)}(\Delta\lambda^{(j)}) + c_{(j-1)} = 0 \quad \textbf{(63)}$$

$$a_{(j-1)} = \{\Delta U\}_p^{(j)T} \{\Delta U\}_p^{(j)} \quad \textbf{(64)}$$

$$b_{(j-1)} = 2\,[\{\Delta U\}^{(j-1)} + \{\Delta U\}_r^{(j)}]^T \{\Delta U\}_p^{(j)} \quad \textbf{(65)}$$

$$c_{(j-1)} = [\{\Delta U\}^{(j-1)} + \{\Delta U\}_r^{(j)}]^T [\{\Delta U\}^{(j-1)} + \{\Delta U\}_r^{(j)}] - \{\Delta U\}^{(j-1)T} \{\Delta U\}^{(j-1)} \quad \textbf{(66)}$$

Now, incremental load factors in iterative analyses of any step can be determined by equations (59)-(66). Authors have written another program according to these relations. The problem discussed in previous program is almost overcome in the recent one. However, in are case discriminant of quadratic equations (59) and (63) becomes negative in some iterative analyses. Solving different examples declared that the phenomenum occurs in a part of diagram which has large curvature. To overcome this problem one must increase the radius of the circular arcs. In other words, whenever one of the discriminant of the equations becomes negative, radius of the arc will be increased at that iteration and analysis is performed again. It should be noted that if discriminant of new equation becomes negative, radius of the circular arc will be increased again. This is done only twice. If discriminant becomes begative for the third time, analysis will be stopped and step of loading will be continued from the beginning by choosing new arc-length. It should be remembered that a new center for the arcs should be chosen to increase their radius. For example, if discriminant becomes negative during second iteration, center of the arc (B) will be replaced by point (A) in Fig (3). The new radius equals to A3.

Numerical Example

One example is solved to show the capabilities of the presented method. An arc-structure of Fig (4) is selected [C6]. Behavior of this structure is so that its load-displacement curve has different limit points. This structure was initially analyzed by Harrison by discrete element method [H2]. This method is usable for structures restricted by two supports [H3]. Arc structures and portal frame are of this type. It should be mentioned that material nonlinearity efect is neglected in this structure. Only geometric nonlinearity effect is considered here. In the following, analysis of circular arc-structure in figure (4) will be discussed. Two ends of structure are hinge supports. A concentruted load, P=1 is applied on structure at 200 mm from its symmetry axis. Cross section area, monent of inertia and modulus of elasticity are as follows:

$A = 100$, $I = 100000$, $E = 20000$

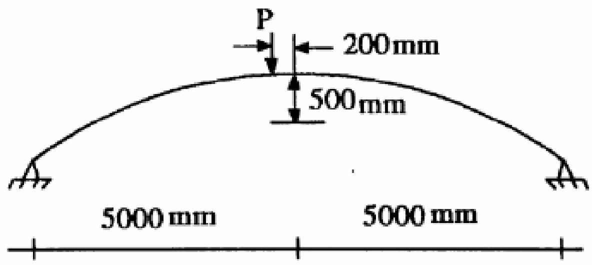

Fig (4) : Arc-structure

Analysis is done to obtain the diagram of vertical deflection of the point under the applied concentrated load. The structure is divided into smaller parts and each part is assumed to be a straight member. Nonlinear analysis is begun by stiffness matrix of plane frame member. Constants which are used during analysis have the following values.

$\Delta\lambda^{(1)} = 400$ $J_d = 6$ $J_{max} = 15$ $\beta_c = 0.0001$ $\eta = 0.5$

Figures (5) and (6) display the load-displacement diagrams obtained by the cylindrical arc-length and suggested method, respectively. The full lines display the curve obtained by

Harrison. Equilibrium points are shown with o on the diagrams. The sign shows number of iterative analyses to reach the equilibrium point. In other words, there is a need for another analysis at the points specified by this sign because of divergence of the procedure. Therefore, the number of+sign equals to number of iterative analyses required for convergence. Limit points have been specified on diagram of Fig (5). As it is seen, there are four load limit points and two displacement limit points.

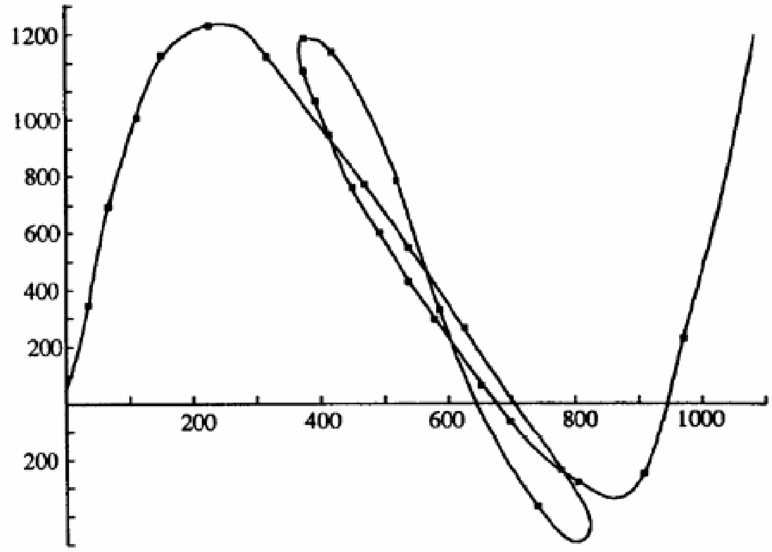

Fig 5: Solution by cylindrical arc-length method

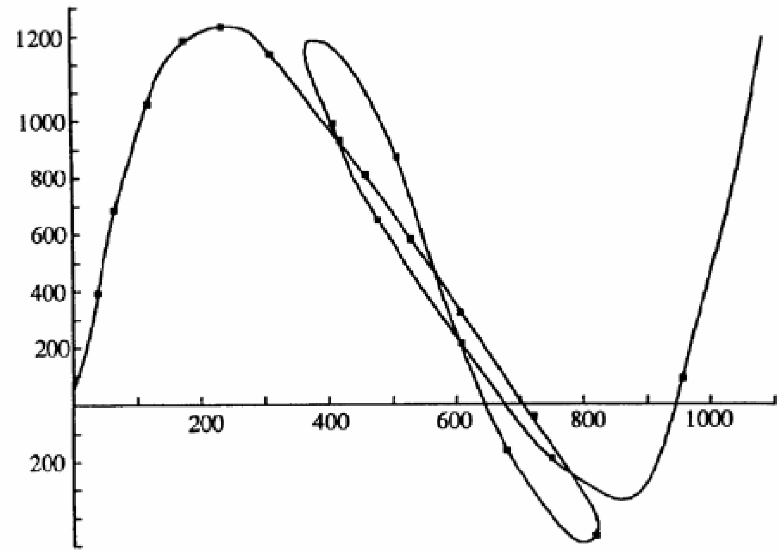

Fig 6: Solution by suggest method

REFERENCES

A1. S.N.AL-Rasby, Finite element analysis of space frames considering large displacements and finite rotations, Ph.D Dissertation, university of washington, seattle, WA (1989.

A2. S.N.AL-Rasby, "Solution techniques in nonlinear structural analysis", Comput. Struct., 40, 985-993 (1991)

B1. P.X.Belliny and A.Chulya, "An improved automatic incremental algorithm for the efficient solution of nonlinear finite element equation", Comput. Struct., 26, 99-110 (1987).

B2. Q.L.Batoz and G.Dhatt, "Incremental displacement algorithms for nonlinear problems", Int. J. Numer. Meth. Engng., 14, 1262-1267 (1979).

C1. A.S.L.Chan and T.B.Lau, "Further development of the reduced basis method for geometric nonlinear analysis", Comput. Methods. Appl. Mech. Engrg., 62, 127-144 (1987).

C2. M.A.Crisfield, "A faster modified Newton-Raphson iteration", Comput. Methods. Appl. Mech. Engng., 20, 267-278 (1979).

C3. M.R.Crisfield, "A fast incremental/iterative solution procedure that handles snap-through", Comput. Struct., 13, 55-62 (1981).

C4. M.A.Crisfield, Nonlinear finite element analysis of solids and structures, John Willey & sons Ltd. (1991).

C5. S.L.Chan, "Geometric and material non-linear analysis of beam-columns and frames using the minimum residual displacement method", Int. J. Numer. Meth. Engng., 26. 2657-2669 (1988).

C6. M.J.Clarke and G.J.Hancock, "A study of incremental iterative strategies for non-linear analysis", Int. J. Numer. Meth . Engng., 29 , 1365-1391 (1990).

D1. G.Dhatt and G.Touzut, The finite element method displayed, John Willey & sons Ltd. (1985).

F1. M.Fafard and B.Massicotie, "Geometrical interpretation of the are-length method", Comput. Struct., 46, 603-615 (1993).

F2. B.W.R.Forde and S.F.Stimer, "Improved arc-length orthogonlity methods for nonlinear finite element analysis", Comput. Struct., 27, 625-630 (1987).

G1. Glowinski, Rodin and Zienkiewicz, Energy method in finite element analysis, John Willy & sons Ltd. (1979).

H1. W.E.Haisler and J.A.Stricklin, "Displacement incrementation in non-linear structural analysis by the self-correcting method", Int. J. Numer. meth. Engng., 11, 3-10 (1977).

H2. H.B.Harrison, "Elastic post-buckling response of plane frames", pp56-65, in Instability and plastic collapse of steel structures, (L.J.Morris ed), Granada. (1983).

H3. H.B.Harrison, Structural Analysis and Design: some Minicomputer Methods, Pergamon Press, Oxford (1980)

H4. E.Hinton, D.R.J.Owen and C.Taylor, Recent advances in non-linear computational mechanics, Pineridge Press Limited (1982).

J1. J.P.Jeusette, G.Laschet and S.Idelsohn, "An effective automatic incremental iterative method for static nonlinear structural analysis", Comput. Struct., 32, 125-135 (1989).

K1. R.Kolar and H.A.Kamel, "On some effecient solution algorithms for non-linear finite element analysis", pp425-445, in finite element method for nonlinear problems (Ed.P.G.Bergan et al.) proc. Us-Europe symposium, Springer verleg Trondheim, (1985).

L1. W.F.Lam and C.T.Morley, "Arc-length method for passing limit point in structural calculation", J.Struct. Engng., 118, 169-185 (1992).

L2. S.H.Lee, "Rudimentary considerations for effective quasi-Newton updates in nonlinear finite element analysis", Comput. Struct., 33, 463-476 (1989).

M1. D.P.Mondkar and G.H.Powell, "Evaluation of solution schemes for nonlinear structures", Comput. Struct., 9, 223-236 (1978).

M2. J.L.Meek and S.Loganathan, "Large displacement analysis of space-frame structures, Comput. Meth. Appl. Mech. Engng., 72, 57-75 (1989).

R1. H.Rothert and N.Gebbeken, "On numerical results of reticulated shell bucking", Int. J.Space. Struct., 7, 310-316 (1992).

R2. E.Riks, "An incremental approach to the solution of snapping and buckling problems", Int. J.Solids. Struct., 15, 524-551 (1979).

R3. Y.S.Ryu and J.S.Arora, "Review of nonlinear FE methods with substructures", J.Engng. Mech., 111, 1361-1379 (1985).

S1. A.Safjan, "Nonlinear structural analysis via reduced basis technique", Comput. Struct., 29, 1055-1061 (1988).

S2. C.H.Sheu, G.DE Roeck, M.Van laethem and .Geyskens, "Multi-level substructuring and on exprimental self-adaptive Newton-Raphson method for two dimensional nonlinear analysis", Comput. Struct., 33, 489-497 (1989).

W1. M.B.Wong and F.Tin-Loi, "Geometric nonlinear analysis of elastic framed structures", Comput. Struct. 34, 633-640 (1990).

Z1. O.C.Zienkiewicz, The finite element method, Mc Graw-Hill, London, (1991).