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THREE-DIMENSIONAL SENSITIVITY ANALYSIS USING A FACTORING TECHNIQUE

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Abstract—A new formulation for the sensitivity analysis of three-dimensional finite elements is presented. The method is based upon the implicit differentiation and calculates all parts of deformed element stiffness matrix. These parts are found by a rather simple factoring technique. The design element procedure along with the isoparametric concept are used in this presentation. Computer coding of the method is compatible with the finite element programming technique and can be easily used in structural optimization. Numerical examples, which show the validity as well as application of the formulation, are presented. According to the results, the new formulation calculates the sensitivity analysis quickly.

NOTATION

$[\mathbf{B}]$	strain matrix of initial finite element
$[\hat{\mathbf{B}}]$	strain matrix of deformed finite element
$[\mathbf{D}]$	elasticity matrix
$\{\mathbf{F}\}$	equivalent nodal forces
$\{\hat{\mathbf{F}}\}$	vector of pseudo-loads
f	objective function
f_i	design element shape function
g_j	constraint function
$[\mathbf{J}]$	Jacobian matrix
$\|\mathbf{J}\|$	determinant of Jacobian matrix
$\|\mathbf{J}\|_x, \|\mathbf{J}\|_y, \|\mathbf{J}\|_z$	derivatives of $\|J\|$ with respect to master node coordinates
$[\mathbf{K}]$	global stiffness matrix
$[\mathbf{k}]_e$	element stiffness matrix
N_i	finite element shape function
$\{\mathbf{Q}_j\}$	vector of virtual loads
$\{\mathbf{q}_j\}$	vector of virtual displacements
r, s, t	local coordinates of design element
$\{S\}$	vector of shape variables
$\{S\}_l, \{S\}_u$	lower and upper bounds of $\{S\}$
s_k	a particular shape variable
$\{U\}$	nodal displacements
$\mathbf{u}_j$	a particular displacement component
X, Y, Z	global coordinates of design element nodes
x, y, z	global coordinates of finite element nodes
$\{\lambda_j\}$	vector of adjoint variables
$\delta X_k, \delta Y_k, \delta Z_k$	design changes

1. INTRODUCTION

In most structural optimization studies, sizing design variables such as the cross-sectional area of bars, the moment of inertia of beams and thickness of plates are optimized while the shape of the structure is considered unchanged. However, it is known that for many problems taking the boundary shape of structures as variables may yield a further reduction of weight and cost. This is the case of shape optimization problems which have received much attention in the last 15 years. A typical shape optimization problem can be mathematically defined in terms of shape variables $\{S\}$ and behavior variables $\{U\}$ as

$$\begin{aligned} &\text{minimize} \quad f(\{S\}, \{U\}) \\ &\text{subject to} \quad g_j(\{S\}, \{U\}) \leqslant 0, \quad j = 1, \ldots, J \\ &\qquad \{S\}_l \leqslant \{S\} \leqslant \{S\}_u, \end{aligned} \tag{1}$$

where f is the objective function which is a criterion for selection of the optimal design. The weight or volume of structures are widely used as objective functions in the field of optimization. Furthermore, g_j presents constraint functions that are to be satisfied at the optimum design. Stress and displacement constraints under various load cases are such functions. Finally, $\{S\}_l$ and $\{S\}_u$ show some technological limitations that are imposed on the shape design variables $\{S\}$.

Structural optimization with shape design variables is more complicated than with sizing design variables. Since the shape of the structure is continuously changing, it is difficult to maintain an adequate finite element mesh and accurate analysis throughout the design process. Another difficulty of shape optimization arises from complex implicit relations between response of structures and shape design variables.

Most efficient optimization techniques use derivatives of structural responses with respect to design variables to obtain a new improved feasible design. However, calculation of these derivatives with respect to shape variables, which is called *shape design sensitivity analysis*, is more expensive and time consuming due to the implicit relations of structural responses and shape variables.

Sensitivity derivatives in structural optimization problems are widely calculated by using the well-known finite difference techniques [1–3]. A disadvantage of these techniques is that a proper step size change should be chosen for the design variables. Furthermore, for a problem with k design variables, finite difference calculations of the displacement derivatives with respect to design variables requires analysis of $k+1$ different stiffness matrices. However, the large number of analyses associated with finite difference calculations can be avoided by analytical computation of sensitivity derivatives.

The first analytical formulation for design sensitivity analysis of continuum structures was presented by Zienkiewicz and Campbell in 1973 [4] and Ramakrishnan and Francavilla in 1974 [5]. It should be noted that these formulations were only introduced for two-dimensional problems.

An implicit differentiation approach for sensitivity analysis in shape optimization of three-dimensional continuum structures was developed by Wang *et al.* in 1985 [6]. They used a limited number of master nodes to charaterize the surface of a set of isoparametric finite elements and their coordinates were adopted as design variables for shape optimization. A similar approach was used for sensitivity analysis and shape optimization of axisymmetric structures by Cheu in 1989 [7].

The purpose of this paper is to present an efficient formulation for the sensitivity analysis of three-dimensional continuum structures. A two-dimensional formulation has been used by the authors. In the formulations that follow, the technique of isoparametric mapping is used to generate the finite element mesh from a master element, and nodal coordinates of this master element are selected as design variables. Numerical examples are solved and some related discussions are presented.

2. SENSITIVITY ANALYSIS

This section deals with the calculation of derivatives of static structural response with respect to general design variables $\{X\}$. Naturally, such derivatives can be calculated when the structure is modeled by finite elements. By using a finite element analysis, nodal displacements of structures are obtained from the solution of the following equilibrium equation

$$[\mathbf{K}]\{\mathbf{U}\}=\{\mathbf{F}\}, \tag{2}$$

where $[\mathbf{K}]$ is the symmetric stiffness matrix of the structure formed by assembling all of the element stiffness matrices $[\mathbf{k}]_e$, $\{\mathbf{U}\}$ is the vector of unknown nodal displacements, and $\{\mathbf{F}\}$ the vector of applied forces. Generally, both $[\mathbf{K}]$ and $\{\mathbf{F}\}$ are functions of design variables. It is believed that over 80% of the computational effort is spent in analysis and design sensitivity analysis during the entire design process [8]. Hence, the efficiency of the process can be improved by adoption of efficient analysis and sensitivity analysis techniques.

The objective of sensitivity analysis is to compute the derivatives of nodal displacements, and a typical function of them such as a behavior constraint function, with respect to design variables. A clear description of analytical methods for calculation of such derivatives has been presented by Arora and Haug [9–11]. They distinguished three methods of sensitivity analysis: the direct or design space method, the adjoint variable or state space method, and the virtual load method. It has been shown that the virtual load method can be derived from both the direct and adjoint methods.

In spite of the fact that the three aforementioned methods of sensitivity analysis differ in some manner, all of them are based on the direct differentiation of eqn (2) as

$$[\mathbf{K}]\left\{\frac{\partial U}{\partial X}\right\}=\left\{\frac{\partial F}{\partial X}\right\}-\left[\frac{\partial K}{\partial X}\right]\{\mathbf{U}\}=\{\bar{\mathbf{F}}\}, \tag{3}$$

where $\{\bar{\mathbf{F}}\}$ is referred to as a pseudo-load. It is clear that by applying the pseudo-load to the structure and obtaining the solution of eqn (3), the desired displacement derivatives can be achieved.

2.1. *Design space method*

In this method, sensitivity derivatives of nodal displacements (i.e. $\{\partial U/\partial X\}$) are directly computed by solving eqn (3). Having these in hand, the derivatives of a general constraint function, $g_j(\{X\},\{U\})$, with respect to design variables, can be obtained from the following equation

$$\left\{\frac{\mathrm{d}g_j}{\mathrm{d}X}\right\}=\left\{\frac{\partial g_j}{\partial X}\right\}+\left\{\frac{\partial g_j}{\partial U}\right\}^T\left\{\frac{\partial U}{\partial X}\right\}. \tag{4}$$

It should be added that for N_l distinct loading conditions, eqn (3) must be solved N_l times for each design variable. Therefore, this process is costly when the number of design variables and loading conditions is large.

2.2. *State space method*

If the displacements, $\{U\}$, and design variables, $\{X\}$, are assumed as independent variables, the first-order variations of the constraint function, $g_j(\{X\},\{U\})$, can be computed as

$$\delta g_j=\left\{\frac{\partial g_j}{\partial X}\right\}^T\{\delta X\}+\left\{\frac{\partial g_j}{\partial U}\right\}^T\{\delta U\}. \tag{5}$$

Now an adjoint equation should be defined to express the effect of variation of design variables $\{X\}$ on the displacements $\{U\}$ as follows:

$$[\mathbf{K}]\{\lambda_j\}=\left\{\frac{\partial g_j}{\partial U}\right\}, \tag{6}$$

where $\{\lambda_j\}$ is the vector of adjoint variables associated with the constraint function g_j. Having the adjoint variables $\{\lambda_j\}$, the derivatives of g_j with respect to design variables can be obtained with

$$\left\{\frac{\mathrm{d}g_j}{\mathrm{d}X}\right\}=\left\{\frac{\partial g_j}{\partial X}\right\}+\{\lambda_j\}^T\{\mathbf{F}\}. \tag{7}$$

At first, the adjoint equations have been looked upon only as a numerical tool for obtaining sensitivity derivatives. However, Belegundu gave them a physical interpretation and showed that the equations offer a new method for obtaining influence coefficients [12]. In fact, the adjoint vector $\{\lambda_j\}$ associated with the constraint function g_j indicates how sensitive the function g_j is with respect to applied forces $\{\mathbf{F}\}$.

It is obvious that the adjoint variable method requires solving eqn (6) once for each constraint function g_j. Therefore, in cases where a limited number of constraints should be considered at a current design point, the method is preferred to the design space method. This is the case where an active set strategy is used to solve the optimal design problem.

2.3. *Virutal load method*

In order to calculate the derivatives of a particular displacement component $\mathbf{u}_j$ with respect to design variables $\{X\}$, it can be expressed as follows:

$$\mathbf{u}_j=\{\mathbf{Q}_j\}^T\{U\}, \tag{8}$$

where $\{\mathbf{Q}_j\}$ is a virtual load vector and has a unit value at the jth component and all other terms are equal to zero. Corresponding to the virtual load $\{\mathbf{Q}_j\}$, a virtual displacement field $\{\mathbf{q}_j\}$ can be obtained which satisfies the following equation

$$[\mathbf{K}]\{\mathbf{q}_j\}=\{\mathbf{Q}_j\}. \tag{9}$$

After the virtual displacements $\{\mathbf{q}_j\}$ are computed, the derivatives of desired displacement component $\mathbf{u}_j$ can be easily obtained as follows:

$$\left\{\frac{\mathrm{d}U_j}{\mathrm{d}X}\right\}^T=\{\mathbf{q}_j\}^T\{\mathbf{F}\}. \tag{10}$$

The virtual load method requires solving eqn (9) once for each displacement component $\mathbf{u}_j$ whose derivatives are needed.

3. FORMULATIONS

As the first step of sensitivity analysis based on the implicit differentiation of eqn (2), the derivatives of global stiffness matrix $[\partial\mathbf{K}/\partial\mathbf{X}]$ and global nodal forces $\{\partial\mathbf{F}/\partial\mathbf{X}\}$ should be calculated. These derivatives can be obtained by assembling all derivatives of element stiffness matrices and element nodal forces. In the case of shape optimization, such derivatives should be calculated with respect to shape variables $\{\mathbf{S}\}$. A treatment for the formulation of $\{\partial\mathbf{F}/\partial\mathbf{S}\}$ for isoparametric elements is presented in detail by Wang *et al.* [6]. Therefore, only a new formulation for finding $[\partial\mathbf{K}/\partial\mathbf{S}]$ is presented here.

The stiffness matrix of an isoparametric finite element can be formulated as follows [13]:

$$[\mathbf{k}]_e=\iiint_v[\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}[\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]|\mathbf{J}|\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{11}$$

where $[\mathbf{B}]$ is the strain matrix which operates on nodal displacements to produce element strains, $[\mathbf{D}]$ is the elasticity matrix which relates the element stresses and strains and $|\mathbf{J}|$ is the determinant of the Jacobian matrix $[\mathbf{J}]$ which is the multiplier that yields area $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in global coordinates from $\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$ in curvilinear local coordinates.

In most optimal design problems, the mechanical properties of material are prescribed and do not change during the optimization process. Hence, the derivative of the element stiffness matrix with respect to a shape variable s_k can be written in the usual manner as follows:

$$\left[\frac{\partial K}{\partial s_k}\right]_e=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}\left\{\left(\left[\frac{\partial B}{\partial s_k}\right]^T[\mathbf{D}][\mathbf{B}]+[\mathbf{B}]^T[\mathbf{D}]\left[\frac{\partial B}{\partial S_k}\right]\right)|\mathbf{J}|+[\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]\times\frac{\partial|\mathbf{J}|}{\partial s_k}\right\}\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{12}$$

It is known that a typical coefficient in $[\mathbf{B}]$ depends on local coordinates and has ξ,η,ζ polynomials in both numerator and denominator. Therefore, the parametric intergration of the stiffness matrix and its derivatives are complex and it must be done numerically. It is obvious that evaluation of eqn (12) requires the derivatives of $|\mathbf{J}|$ and $[\mathbf{B}]$ with respect to shape variables. Here a new technique for calculation of such derivatives is introduced. In order to do this, an m-node isoparametric design element is selected with natural coordinates r, s, and t, such as the one shown in Fig. 1. The design element consists of several finite elements.

Once the nodal coordinates of the design element are determined, the coordinates of its internal nodes, such as finite elements nodal points, can be computed by using the technique of isoparametric mapping. It should be noted that a comprehensive description of this approach for shape representation of structures is presented by Wang *et al.* [6] and readers are referred to it for more information. Using this

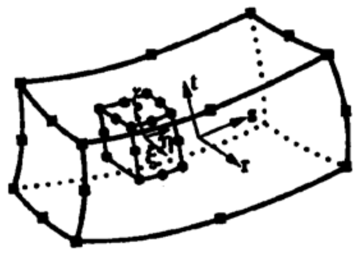

Fig. 1. Design element and its associated finite element. ■ Design element nodes; ● finite element nodes.

technique, the coordinates of finite element nodal points are generated in the following form

$$x = \sum_{i=1}^{m} f_i(r, s, t) X_i$$

$$y = \sum_{i=1}^{m} f_i(r, s, t) Y_i$$

$$z = \sum_{i=1}^{m} f_i(r, s, t) Z_i, \tag{13}$$

where X_i, Y_i, Z_i and x, y, z are global coordinates of the design element and its associated finite element, respectively. Furthermore, f_i shows the isoparametric shape function corresponding to the ith node of design element. However, instead of the isoparametric shape function, a spline blending function typical of computer graphics can also be chosen for the definition of design element, such as the one used by Braibant and Fleury [14].

If the coordinates X_k, Y_k and Z_k, related to the Kth master node of design element, take small changes as δX_k, δY_k and δZ_k respectively, nodal coordinates of each deformed finite element can be computed as

$$\hat{x} = x + f_k(r, s, t)\delta X_k$$

$$\hat{y} = y + f_k(r, s, t)\delta Y_k$$

$$\hat{z} = z + f_k(r, s, t)\delta Z_k. \tag{14}$$

Here the quantities associated with the deformed finite element are distinguished by the hat. Having the nodal coordinates of deformed finite elements, the components of their stiffness matrices and their derivatives can be calculated.

3.1. *Derivatives of the Jacobian matrix*

The mapping between global and natural coordinates of finite element is defined by means of the determinant of the Jacobian matrix. Since numerical integration of the isoparametric element stiffness matrix is done in natural coordinates, the determinant of the Jacobian matrix appears in the formulation of the stiffness matrix. Hence, its derivatives with respect to design variables are needed in sensitivity calculations.

If N_i and $N_{i,\xi}, N_{i,\eta}, N_{i,\zeta}$ represent finite element shape function and its local derivatives respectively, the Jacobian matrix of deformed element can be presented in the following form

$$[\hat{J}] = \begin{bmatrix} N_{1,\xi} \cdots N_{i,\xi} \cdots N_{n,\xi} \\ N_{1,\eta} \cdots N_{i,\eta} \cdots N_{n,\eta} \\ N_{1,\zeta} \cdots N_{i,\zeta} \cdots N_{n,\zeta} \end{bmatrix} \begin{bmatrix} \hat{x}_1 & \hat{y}_1 & \hat{z}_1 \\ \vdots & \vdots & \vdots \\ \hat{x}_i & \hat{y}_i & \hat{z}_i \\ \vdots & \vdots & \vdots \\ \hat{x}_n & \hat{y}_n & \hat{z}_n \end{bmatrix}$$

$$= \begin{bmatrix} N_{1,\xi} \cdots N_{i,\eta} \cdots N_{n,\xi} \\ N_{1,\eta} \cdots N_{i,\eta} \cdots N_{n,\eta} \\ N_{1,\zeta} \cdots N_{i,\zeta} \cdots N_{n,\zeta} \end{bmatrix} \left\{ \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_i & y_i & z_i \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} + \begin{bmatrix} f_k(r_1, s_1, t_1) \\ \vdots \\ f_k(r_i, s_i, t_i) \\ \vdots \\ f_k(r_n, s_n, t_n) \end{bmatrix} \begin{bmatrix} \delta X_k \\ \delta Y_k \\ \delta Z_k \end{bmatrix}^T \right\} \tag{15}$$

It should be noted that $N_{i,\xi}$, $N_{i,\eta}$ and $N_{i,\zeta}$ are only functions of the natural coordinates of Gauss sampling points and are independent of global coordinates if the number of Gauss sampling points is held fixed during the design process. Obviously the Jacobian matrix of deformed finite element can be presented in a linear form in terms of design changes δX_k, δY_k and δZ_k as

$$[\hat{J}] = [J] + [J]_x \delta X_K + [J]_y \delta Y_K + [J]_z \delta Z_K, \tag{16}$$

where

$$[J] = \begin{bmatrix} x_{,\xi} & y_{,\xi} & z_{,\xi} \\ x_{,\eta} & y_{,\eta} & z_{,\eta} \\ x_{,\zeta} & y_{,\zeta} & z_{,\zeta} \end{bmatrix}, \quad [J]_x = \begin{bmatrix} H_{,\xi} & 0 & 0 \\ H_{,\eta} & 0 & 0 \\ H_{,\zeta} & 0 & 0 \end{bmatrix}$$

$$[J]_y = \begin{bmatrix} 0 & H_{,\xi} & 0 \\ 0 & H_{,\eta} & 0 \\ 0 & H_{,\zeta} & 0 \end{bmatrix}, \quad [J]_z = \begin{bmatrix} 0 & 0 & H_{,\xi} \\ 0 & 0 & H_{,\eta} \\ 0 & 0 & H_{,\zeta} \end{bmatrix} \tag{17}$$

and

$$H_{,\xi} = \Sigma N_{i,\xi} f_k(r_i, s_i, t_i)$$

$$H_{,\eta} = \Sigma N_{i,\eta} f_k(r_i, s_i, t_i)$$

$$H_{,\zeta} = \Sigma N_{i,\zeta} f_k(r_i, s_i, t_i). \tag{18}$$

In the above formulation, $[J]$, $[J]_x$, $[J]_y$, and $[J]_z$ show the initial Jacobian matrix derivatives with respect to master node coordinates X_k, Y_k, and Z_k, respectively. Now, it can be easily proved that the determinant of the Jacobian matrix has a linear relation with the design changes as follows:

$$|\hat{J}| = |J| + |J|_x \delta X_K + |J|_y \delta Y_K + |J|_z \delta Z_K, \quad (19)$$

where

$$|J|_x = \begin{vmatrix} H_{,\xi} & y_{,\xi} & z_{,\xi} \\ H_{,\eta} & y_{,\eta} & Z_{,\eta} \\ H_{,\zeta} & y_{,\zeta} & z_{,\zeta} \end{vmatrix}, \quad |J|_y = \begin{vmatrix} x_{,\xi} & H_{,\xi} & z_{,\xi} \\ x_{,\eta} & H_{,\eta} & z_{,\eta} \\ x_{,\zeta} & H_{,\zeta} & z_{,\zeta} \end{vmatrix},$$

$$|J|_z = \begin{vmatrix} x_{,\xi} & y_{,\xi} & H_{,\xi} \\ x_{,\eta} & y_{,\eta} & H_{,\eta} \\ x_{,\zeta} & y_{,\zeta} & H_{,\zeta} \end{vmatrix} \quad (20)$$

are derivatives of $|J|$ with respect to nodal coordinates X_k, Y_k, and Z_k, respectively. Some attention to eqn (20) will give a valid relation between these derivatives. In order to show this relationship, a simultaneous system of equations is defined as

$$\begin{bmatrix} x_{,\xi} & y_{,\xi} & z_{,\xi} \\ x_{,\eta} & y_{,\eta} & z_{,\eta} \\ x_{,\zeta} & y_{,\zeta} & z_{,\zeta} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} H_{,\xi} \\ H_{,\eta} \\ H_{,\zeta} \end{bmatrix}. \quad (21)$$

It is obvious that the coefficient matrix of eqn (21) is the Jacobian matrix $[J]$ and the right-hand side of the equation is previously defined by eqn (18). Since the inverse of the Jacobian matrix is available in the analysis process, the unknowns h_1, h_2, and h_3 can be easily obtained. Therefore, the derivatives of $|J|$ can be computed as

$$|J|_x = h_1 \cdot |J|, \quad |J|_y = h_2 \cdot |J|, \quad |J|_z = h_3 \cdot |J|. \quad (22)$$

Equation (22) shows that the derivatives $|J|_x$, $|J|_y$, and $|J|_z$ are dependent on each other and they can be computed simultaneously.

3.2. *Derivatives of strain matrix*

The other component that is required for the calculation of derivatives of the element stiffness matrix is the derivative of strain matrix $[B]$. In three-dimensional elasticity, the strain matrix $[B]$ is given as follows:

$$[B] = [B_1 \cdots B_i \cdots B_n] \quad (23)$$

$$[B_i] = \begin{bmatrix} N_{i,x} & 0 & 0 \\ 0 & N_{i,y} & 0 \\ 0 & 0 & N_{i,z} \\ N_{i,y} & N_{i,x} & 0 \\ 0 & N_{i,z} & N_{i,y} \\ N_{i,z} & 0 & N_{i,x} \end{bmatrix}, \quad (24)$$

where $N_{i,x}$, $N_{i,y}$, and $N_{i,z}$ are the derivatives of finite element shape functions with respect to its global coordinates. These derivatives can be obtained in terms of local coordinates as follows:

$$\begin{bmatrix} N_{i,x} \\ N_{i,y} \\ N_{i,z} \end{bmatrix} = [J]^{-1} \begin{bmatrix} N_{i,\xi} \\ N_{i,\eta} \\ N_{i,\zeta} \end{bmatrix}. \quad (25)$$

The formulation of strain matrix for the deformed finite element can be obtained by calculating the above derivatives in terms of design changes δX_k, δY_k, and δZ_k. However, as mentioned previously, the derivatives $N_{i,\xi}$, $N_{i,\eta}$, and $N_{i,\zeta}$ are only functions of the local coordinates of numerical integration sampling points and are independent of design changes. Therefore, only the inverse of the Jacobian matrix, $[J]^{-1}$, must be calculated for the deformed finite element. The inverse of Jacobian matrix $[J]^{-1}$ associated with the deformed finite element has the following form

$$[\hat{J}]^{-1} = \frac{[\hat{J}]^a}{|\hat{J}|}. \quad (26)$$

Here $[\hat{J}]^a$ is the adjoint of the Jacobian matrix and its definition is available in numerical calculus. It can be proved that $[\hat{J}]^a$ has a linear form in terms of the design changes as follows:

$$[\hat{J}]^a = [J]^a + [J_x]^a \delta X_k + [J_y]^a \delta Y_k + [J_z]^a \delta Z_k, \quad (27)$$

where

$$[J_x]^a = \begin{bmatrix} 0 \\ T_z \\ -T_y \end{bmatrix}, \quad [J_y]^a = \begin{bmatrix} -T_z \\ 0 \\ T_x \end{bmatrix}, \quad [J_z]^a = \begin{bmatrix} T_y \\ -T_x \\ 0 \end{bmatrix}, \quad (28)$$

and

$$T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} x_{,\eta} & H_{,\eta} \\ x_{,\zeta} & H_{,\zeta} \end{vmatrix} & -\begin{vmatrix} x_{,\xi} & H_{,\xi} \\ x_{,\zeta} & H_{,\zeta} \end{vmatrix} & \begin{vmatrix} x_{,\xi} & H_{,\xi} \\ x_{,\eta} & H_{,\eta} \end{vmatrix} \\ \begin{vmatrix} y_{,\eta} & H_{,\eta} \\ y_{,\zeta} & H_{,\zeta} \end{vmatrix} & -\begin{vmatrix} y_{,\xi} & H_{,\xi} \\ y_{,\zeta} & H_{,\zeta} \end{vmatrix} & \begin{vmatrix} y_{,\xi} & H_{,\xi} \\ y_{,\eta} & H_{,\eta} \end{vmatrix} \\ \begin{vmatrix} z_{,\eta} & H_{,\eta} \\ z_{,\zeta} & H_{,\zeta} \end{vmatrix} & -\begin{vmatrix} y_{,\xi} & H_{,\xi} \\ y_{,\zeta} & H_{,\zeta} \end{vmatrix} & \begin{vmatrix} y_{,\xi} & H_{,\xi} \\ y_{,\eta} & H_{,\eta} \end{vmatrix} \end{bmatrix}. \quad (29)$$

Now the derivatives of shape functions $\hat{N}_{i,x}$, $\hat{N}_{i,y}$, and $\hat{N}_{i,z}$ associated with the deformed finite element can be easily found from the following equations

$$\begin{bmatrix} \hat{N}_{i,x} \\ \hat{N}_{i,y} \\ \hat{N}_{i,z} \end{bmatrix} = \frac{1}{|\hat{J}|} [[J]^a + [J_x]^a \delta X_k + [J_y]^a \delta Y_k + [J_z]^a \delta Zk] \begin{bmatrix} N_{i,\xi} \\ N_{i,\eta} \\ N_{i,\zeta} \end{bmatrix} = \frac{1}{|\hat{J}|} \begin{bmatrix} \hat{N}_{i,x} \\ \hat{N}_{i,y} \\ \hat{N}_{i,z} \end{bmatrix}^a. \quad (30)$$

The last equation shows that derivatives $N_{i,x}$, $N_{i,y}$, and $N_{i,z}$ have a nonlinear form in terms of design changes ∂X_k, ∂Y_k, and ∂Z_k due to the appearance of $|\hat{J}|$ in the denominator of eqn (30). The nonlinearity appears on the strain matrix $[\hat{B}]$ as well since its components are formed from the shape functions derivatives. Although this nonlinearity of $[\hat{B}]$ may cause some diffculty in the calculations of its derivatives, it is clear that both the numerator and the denominator of eqn (30) are linear. Hence, based upon the defined variables, the strain matrix for the deformed finite element can be considered in the following form

$$[\hat{B}]=\frac{[\hat{B}]^a}{|\hat{J}|}=\frac{[B]^a+[B_x]^a\delta X_k+[B_y]^a\delta Y_k+[B_z]^a\delta Z_k}{|J|+|J|_x\delta X_k+|J|_y\delta Y_k+|J|_z\delta Z_k},\tag{31}$$

where

$$[\hat{B}]^a=[\hat{B}_1\cdots\hat{B}_i\cdots\hat{B}_n]^a\tag{32}$$

and

$$[\hat{B}_i]^a=\begin{bmatrix}\hat{N}_{i,x}&0&0\\0&\hat{N}_{i,y}&0\\0&0&\hat{N}_{i,z}\\\hat{N}_{i,y}&\hat{N}_{i,x}&0\\0&\hat{N}_{i,z}&\hat{N}_{i,y}\\\hat{N}_{i,z}&0&\hat{N}_{i,x}\end{bmatrix}^a.\tag{33}$$

Also it can be proved that matrices $[B_x]^a$, $[B_y]^a$, and $[B_z]^a$, which are the derivatives of $[B]^a$ with respect to the kth master node coordinates, are as follows:

$$[B_{ix}]^a=\begin{bmatrix}0&0&0\\0&T_{nz}&0\\0&0&-T_{ny}\\T_{nz}&0&0\\0&-T_{ny}&T_{nz}\\-T_{ny}&0&0\end{bmatrix}$$

$$[B_{iy}]^a=\begin{bmatrix}-T_{nz}&0&0\\0&0&0\\0&0&T_{nx}\\0&-T_{nz}&0\\0&T_{nx}&0\\T_{nx}&0&-T_{nz}\end{bmatrix}$$

$$[B_{iz}]^a=\begin{bmatrix}T_{ny}&0&0\\0&-T_{nx}&0\\0&0&0\\-T_{nx}&T_{ny}&0\\0&0&-T_{nx}\\0&0&T_{ny}\end{bmatrix},\tag{34}$$

where

$$[T_n]=\begin{bmatrix}T_{nx}\\T_{ny}\\T_{nx}\end{bmatrix}=[T]\begin{bmatrix}N_{i,\xi}\\N_{i,\eta}\\N_{i,\zeta}\end{bmatrix}.\tag{35}$$

3.3. *Derivatives of stiffness matrix*

Up to here the strain matrix $[\hat{B}]$ and the determinant of the Jacobian matrix $|\hat{J}|$ associated with the deformed finite element are formulated in terms of design changes δX_k, δY_k, and δZ_k. Hence, the stiffness matrix of the deformed finite element can be formulated. Here only a formulation with respect to δX_k is presented and, because of similarity, formulations with respect to δY_k and δZ_k are not considered

$$[\hat{k}]_e=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}[\hat{B}]^T[D][\hat{B}]|\hat{J}|\,d\xi\,d\eta\,d\zeta=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}\frac{([\hat{B}]^a)^T[D][\hat{B}]^a}{|\hat{J}|}d\xi\,d\eta\,d\zeta$$
$$=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}\frac{[G]\delta X_k^2+([E]+[E]^T)\delta X_k+[F]}{|J|+|J|_x\delta X_k}d\xi\,d\eta\,d\zeta,\tag{36}$$

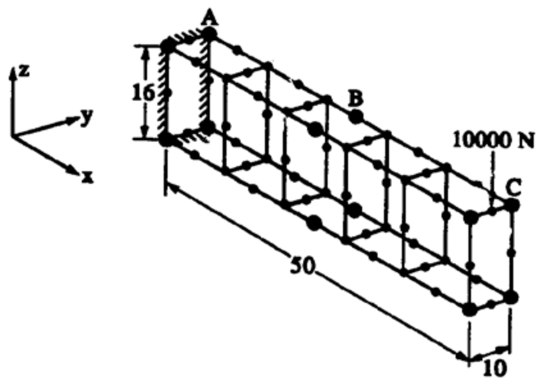

Fig. 2. Three-dimensional cantilever beam (all dimensions in millimeters).

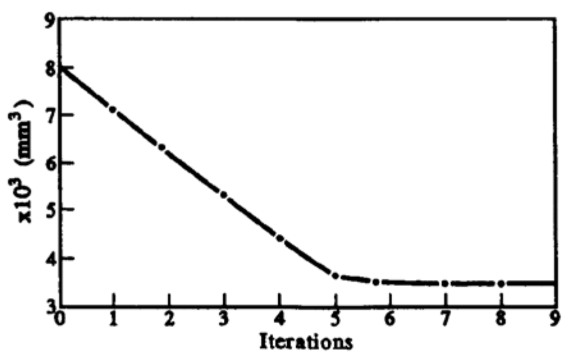

Fig. 3. Volume design history of cantilever beam.

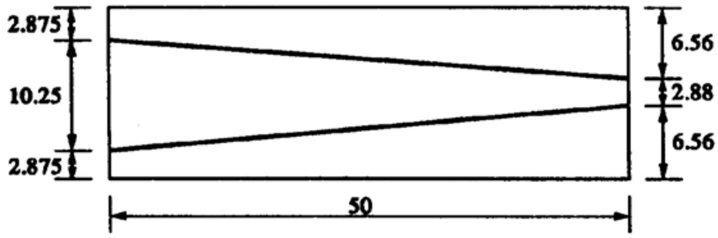

Fig. 4. Initial and final designs of the beam.

where

$$[\mathbf{E}] = ([B_x]^e)^T[D][B]^e$$

$$[\mathbf{F}] = ([B]^e)^T[D][B]^e$$

$$[\mathbf{G}] = ([B_x]^e)^T[D][B_x]^e. \quad (37)$$

Equation (36) shows the stiffness matrix of the deformed finite element. It is interesting to note that if design change δX_k takes a zero value, the formulation gives the stiffness matrix of the initial finite element. Now the general definition of *derivative* can be used to compute the first-order derivative of the stiffness matrix as follows:

$$\left[\frac{\partial k}{\partial X_k}\right]_e = \lim_{\delta X_k \to 0} \frac{[\hat{k}]_e - [k]_e}{\delta X_k} = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{([E]+[E]^T)}{|J|} - [F]\frac{|J|_x}{|J|^2}\right] d\xi\, d\eta\, d\zeta. \quad (38)$$

To this end, the first order analytical derivative of the stiffness matrix is calculated. It should be noted that eqn (30) is completely general and is applicable to each of the master nodes coordinates of the design element.

4. NUMERICAL EXAMPLES

4.1. *Three-dimensional beam*

Minimum weight design of a three-dimensional cantilever beam with rectangular cross-section is considered here. The initial shape of the beam and its loading are shown in Fig. 2. The width of the beam is constant and the position of its upper and lower surfaces, which are parallel to the x–y plane, are to be determined. Based upon the beam theory, the exact solution of the optimum shape is given as follows [15]:

$$z = \sqrt{\frac{6P(L-x)}{b\sigma}}, \quad (39)$$

where z is the height of the beam at a distance x from its clamped end, L is the length of the beam, b is the uniform width of the beam, and σ is the maximum allowable value of the bending stress.

The beam is modeled with five 20-node isoparametric finite elements. Using symmetry and constant width conditions, the shape of curve ABC determines the shape of the whole beam. Thus, the heights of the three nodes A, B, and C calculated from the midplane are selected as design variables. A concentrated load of 10,000 N is applied at the free end of the beam. Poisson's ratio, Young's modulus, and allowable bending stress are 0.3, 10.0×10^6, and 3000 MPa, respectively.

A sequential linear programming technique along with a combination of fixed and variable move limits is used to solve the design problem. The initial volume of the beam is 8000 mm^3 which is reduced to 3484 mm^3 at the optimum design. This volume reduction of the beam is achieved after nine iterations as shown in Fig. 3. The initial and final designs of the beam are presented in Fig. 4 and Table 1. It should be noted that the numerical solution of the problem is also presented by Yang and Botkin [16], as well as Kodiyalam and Vanderplaats [17]. In order to compare the results, all of these are presented in Table 1.

Table 1. Design variables for cantilever beam

Design variable	Initial	Final: This Paper	Final: Exact	Final: Ref. [16]	Final: Ref. [17]
A	8.0	5.125	5.000	5.006	5.154
B	8.0	3.582	3.536	3.619	3.540
C	8.0	1.440	—	1.440	1.630
Volume (mm^3)	8000	3484	—	3487	3488

4.2. *Three-dimensional bar*

As a second example, design of a three-dimensional steel bar with a square cross-section is considered.

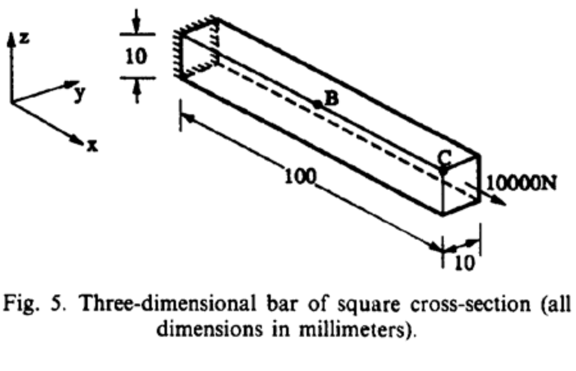

Fig. 5. Three-dimensional bar of square cross-section (all dimensions in millimeters).

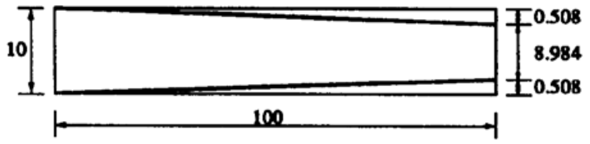

Fig. 6. Optimal shape of the bar under tension.

The bar is under an axial tension of 10,000 N as shown in Fig. 5. For imposed limitations, the dimensions of the clamped end of the bar are considered fixed. In this case, only the z-coordinates of master nodes B and C are assumed as design variables and all other coordinates are obtained from the symmetry conditions.

Maximum value of tensile stress at the bar cross-section is considered as a behavior constraint. Poisson's ratio, Young's modulus, and tensile allowable stress are 0.3, 2.0×10^5, and 140 MPa, respectively.

After 10 design iterations, the optimal shape of the bar is obtained as in Fig. 6 and Table 2. The initial volume of the bar (i.e. 10,000 mm^3) is reduced to 8903 mm^3 at the optimum design. The volume reduction history of the bar is shown in Fig. 7.

Table 2. Initial and final values of design variables

Design variable	Initial	Final
B	5.0	4.700
C	5.0	4.492

5. DISCUSSIONS AND CONCLUSIONS

In order to carry out sensitivity analysis in structural design, a finite difference or an analytical formulation can be used. The finite difference is simple and straightforward, but it requires solving finite element equilibrium equations several times. This disadvantage needs a large amount of computer work. Another alternative procedure to perform sensitivity analysis is based upon the implicit differentiation of equilibrium equations. This approach has been used in shape optimization of the structure successfully. In this technique, design element concept is found to be effective. According to the aforementioned method, the variation of the structural shape is defined by a number of nodes. In fact, only a limited number of predefined master nodes characterize the change of the finite element shapes. This procedure uses the coordinates of these master nodes as shape design variables.

The formulation presented in this paper calculates the derivatives of element stiffness matrix analytically. Structural stiffness derivatives are assembled from the element stiffnesses and are used to find pseudo-loads. By applying this load to the structure, desired displacement derivatives are calculated. In spite of the fact that the method is very general, this paper considers the formulation required for sensitivity analysis of three-dimensional isoparametric finite elements.

A simple and effective factoring technique is used throughout this formulation. According to the procedure, the derivatives of the Jacobian matrix have been found for use in numerical integration of isoparametric element stiffness. This matrix is written linearly in terms of design variables. The inverse of the Jacobian matrix along with the derivatives of its determinant are found through the analysis process. Another part which is required to compute element stiffness matrix in terms of design changes is the derivatives of strain matrix. This matrix has been evaluated from the presented formula. It is shown that the strain matrix for the deformed finite element has a nonlinear form. However, the matrix was decomposed into two parts. These parts, which occur in numerator and denomenator, have both linear variations in terms of design variables. After calculating all parts required to form the stiffness matrix of the deformed finite element, the derivatives of the stiffness matrix can be found easily.

Based upon the present formulation, a finite element analysis of a three-dimensional structure was done. The method shows the ability to optimize the structural shapes. It is straightforward to code the procedure or use it along with already available computer programs. Application of the performed sensitivity analysis demonstrates effciency of the formulation and its swiftness to reach the optimization solution.

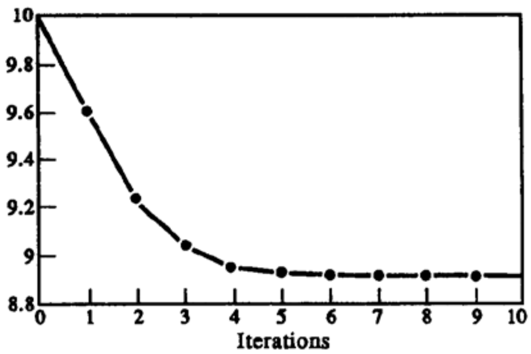

Fig. 7. Volume reduction of three-dimensional bar.

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