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Some Aspects of Discrete Hazard Rate Function in Telescopic Families

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Abstract: In this paper some reliability concepts in the telescopic family of distributions are compared. The telescopic family is named after the telescopic series in mathematics and represents an interesting class of discrete life time distributions. The telescopic family is introduced and also some conditions being equivalent to the IFR property are presented.

Keywords: Extended exponential distribution, reversed hazard rate, second hazard rate, mean residual life.

1 Introduction

There are many situations where a continuous time is inappropriate for describing the lifetime of devices and other systems. For example, the life time of many devices in industry such as switches and mechanical tools, depend essentially on the number of times that they are turned on and off or the number of shocks they receive. In such cases, the time to failure is often more appropriately represented by the number of times they are used before they fail, which is a discrete random variable. Salvia and Bollinger (1982) have discussed the hazard functions of discrete distributions. In a large number of papers such as Barlow et al. (1963), Barlow and Proschan (1981), Bracquemond and Gaudoin (2003) and Lai and Xie (2006) the properties of discrete distributions are derived and characterization results are given.

Particular interest is given to the geometric distribution as an often applied discrete life distribution that corresponding to its continuous counterpart, the exponential distribution, has a constant failure rate. The telescopic[1] family of discrete probability distributions contains the geometric distribution as well as the discrete Weibull one and is therefore of special significance for the discrete analysis of reliability.

In this paper, we show that many continuous distribution have a discrete analog that generally inherits many properties from their continuous relatives. The discrete distributions are of interest as any continuous distribution can be looked upon as an approximation of the discrete reality.

[1]In mathematics, a telescopic series refers to a series whose sum can be found can be determined as almost every term cancels either the preceding or the succeeding one.

2 The Family of Telescopic Distributions

In this section, the telescopic family of discrete distributions is introduced following Rezaei Roknabadi (2000, 2006).

Definition:
A discrete non-negative random variable X has a telescopic distribution and is denoted by $X \sim T(q, k_\theta)$, if its probability mass function is of the form:

$$f_{X|\{(q,k_\theta)\}}(x) = q^{k_\theta(x)} - q^{k_\theta(x+1)} \qquad \text{for } x = 0, 1, 2, \ldots \tag{1}$$

where $0 < q < 1$ and $k_\theta(x)$ is a strictly increasing function of x with $k_\theta(0) = 0$ and $k_\theta(x) \to \infty$ as $x \to \infty$.

The sum $\sum_{x=0}^{\infty} f_{X|\{(q,k_\theta)\}}(x)$ of the telescopic series equals one, implying that $f_{X|\{(q,k_\theta)\}}(x)$ is in fact a probability mass function. The geometric and the discrete Weibull families of distributions belong to this class. It is interesting to note that each member of the telescopic family of distributions that we consider has a continuous analog with similar properties.

Let Y be a non-negative continuous random variable with distribution function:

$$G_{Y|\{(\alpha,k_\theta)\}}(y) = 1 - e^{-\alpha k_\theta(y)} \qquad \text{for } y > 0 \tag{2}$$

where $\alpha > 0$ and θ is the value of a parameter vector (which may contains α) and $k_\theta(y)$ is as in (1). The density function of Y is:

$$g_{Y|\{\alpha,k_\theta\}}(y) = \alpha k'_\theta(y) e^{-\alpha k_\theta(y)} \qquad \text{for } y > 0 \tag{3}$$

where

$$k'_\theta(y) = \frac{d}{dy} k_\theta(y) \tag{4}$$

The class of distributions (2) denoted by $EE(\alpha, k)$ and called *extended exponential family* contains many well known continuous life time distributions such as exponential, Rayleigh, Weibull, Linear-exponential, Gomperts, Rue, Brittle-Fracture and Wear-out.

The discrete version of these continuous distributions are members of the telescopic family of distributions defined by a probability mass function of the form (1) as shown in the following Theorem 1.

Theorem 1:
Let Y be a continuous random variable distributed as (2), and let $X = [Y]$ where $[a]$ means integer part of a, then $X \sim T(q, k_\theta)$.

Proof:
We have

$$\begin{aligned} f_X(x) &= P_X(\{x\}) = \int_x^{x+1} dG_{Y|\{\alpha,k_\theta\}}(y) = G_X(x+1) - G_X(x) \\ &= q^{k_\theta(x)} - q^{k_\theta(x+1)} = f_{X|\{(q,k_\theta)\}}(x) \quad \text{for } x = 0,1,2,\ldots \end{aligned} \tag{5}$$

with $q = e^{-\alpha}$. □

3 Reliability Concepts for the Telescopic Family

In this section, the similarities and differences of the reliability properties of the telescopic distributions and their continuous analogs are listed. However, at first the general properties of the telescopic family of distributions are given:

- Reliability function:

$$R_{X|\{(q,k_\theta)\}}(x) = P_{X|\{(q,k_\theta)\}}(\{y|y \geq x\}) = q^{k_\theta(x)} \qquad \text{for } x = 0,1,2,\ldots \tag{6}$$

- Hazard rate function:

$$h_{X|\{(q,k_\theta)\}}(x) = P_{X_x|\{(q,k_\theta)\}}(\{x\}) = 1 - q^{k_\theta(x+1)-k_\theta(x)} \quad \text{for } x = 0,1,2,\ldots \tag{7}$$

  where X_x denotes the life time X on condition that there is no failure before life time x.

- Reversed hazard rate function:

$$rh_{X|\{(q,k_\theta)\}}(x) = P_{{}_xX|\{(q,k_\theta)\}}(\{x\}) = \frac{q^{k_\theta(x)-k_\theta(x+1)}}{1-q^{k_\theta(x+1)}} \quad \text{for } x = 0,1,2,\ldots \tag{8}$$

  where ${}_xX$ denotes the life time X on condition that it does not exceed the life time x.

The corresponding reliability functions for the random variable Y are obtained straightforward:

- Reliability function:

$$R_{Y|\{\alpha,k_\theta\}}(y) = 1 - G_{Y|\{\alpha,k_\theta\}}(y) = e^{-\alpha k_\theta(y)} \quad \text{for } y > 0 \tag{9}$$

- Hazard rate function:

$$h_{Y|\{\alpha,k_\theta\}}(y) = \frac{g_{Y|\{\alpha,k_\theta\}}(y)}{1 - G_{Y|\{\alpha,k_\theta\}}(y)} = \alpha k'_\theta(y) \quad \text{for } y > 0 \tag{10}$$

- Reversed hazard rate function:

$$rh_{Y|\{\alpha,k_\theta\}}(y) = \frac{g_{Y|\{\alpha,k_\theta\}}(y)}{G_{Y|\{\alpha,k_\theta\}}(y)} = \frac{\alpha k'_\theta(y) e^{-\alpha k_\theta(y)}}{1 - e^{k_\theta(y)}} \quad \text{for } y > 0 \tag{11}$$

Let $k^*_\theta(x) = k_\theta(x+1) - k_\theta(x)$, then X and Y are IFR, CFR or DFR (increasing, constant or decreasing failure rate) if $k^*_\theta(t)$ and $k'_\theta(t)$ are increasing, constant or decreasing functions of t respectively.

From the above functions the "mean time to failure" (MTTF) of the random variables X and Y are immediately obtained:

$$E[X|\{(q,k_\theta)\}] = \sum_{x=0}^{\infty} q^{k_\theta(x+1)} \tag{12}$$

$$E[Y|\{\alpha,k_\theta\}] = \int_0^\infty e^{-\alpha k_\theta(y)} dy \tag{13}$$

where $q = e^{-\alpha}$.

The mean residual life (MRL) functions of the random variables X and Y are as follows:

$$MRL_{X|\{(q,k_\theta)\}}(x) = E[X_x|\{(q,k_\theta)\} - x] = \sum_{i=x}^{\infty} q^{k_\theta(i+1)-k_\theta(x)} \tag{14}$$

$$MRL_{Y|\{\alpha,k_\theta\}}(y) = \int_y^\infty e^{-\alpha[k_\theta(t)-k_\theta(y)]} dt \tag{15}$$

The reversed mean residual life (RMR) function of Y is following Nanda et. al. (2003) defined as follows:

$$RMR_{Y|\{\alpha,k_\theta\}}(y) = E[y - {}_yY|\{\alpha,k_\theta\}] = \frac{\int_0^y (1 - e^{-\alpha k_\theta(t)} dt}{1 - e^{-\alpha k_\theta(y)}}. \tag{16}$$

Goliforushani and Asadi (2008) took up the concept of the reversed mean residual life and applied it to a discrete random variable X on the condition that its life time is shorter than x. We define the reversed mean residual life of a telescopic random variable X on the condition that its life time does not exceed x. Thus, we obtain:

$$RMR_{X|\{(q,k_\theta)\}}(x) = E[x - {}_xX|\{(q,k_\theta)\}] = \frac{\sum_{i=1}^{x}(1-q^{k_\theta(i)})}{1-q^{k_\theta(x+1)}}. \tag{17}$$

Finally, Roy and Gupta (1999) defined the second failure rate (SFR) function for discrete life time random variables as $r_X(x) = \ln \frac{R_X(x-1)}{R_X(x)}$. Based on this proposal, we define the second failure rate function for X as follows:

$$r_{X|\{(q,k_\theta)\}}(x) = \Big(k_\theta(x-1) - k_\theta(x)\Big) \ln q. \tag{18}$$

Evidently, the second failure rate function $r_{X|\{(q,k_\theta)\}}(x)$ of $X|\{(q,k_\theta)\}$ has similar monotonicity property as $h_{X|\{(q,k_\theta)\}}(x)$, since $\ln q$ and $k^*_\theta(x-1) = -[k_\theta(x-1) - k_\theta(x)]$ are negative.

4 Increasing Failure Rate (IFR) Class in the Telescopic Family

Barlow et al. (1963) presented four equivalent conditions for a discrete random variable having an IFR distribution. The equivalent relations for the IFR property for the telescopic family are derived here based on the following theorem.

Theorem 2:
A telescopic distribution belongs to the IFR-class, if one of the following conditions hold:

(i) $k^*_\theta(x) = k_\theta(x+1) - k_\theta(x)$ is an increasing function of x.

(ii) For every x the sequence $\left\{q^{k_\theta(i+x)} - q^{k_\theta(x)}\right\}_{i=0,1,2,...}$ is decreasing.

(iii) For all $j_1, j_2, k_1, k_2 \in \{0,1,2,...\}$, such that $j_1 < j_2$ and $k_1 < k_2$,

$$k_\theta(j_1 - k_1) + k_\theta(j_2 - k_2) \leq k_\theta(j_2 - k_1) + k_\theta(j_1 - k_2)$$

which is equivalent to the condition of a Polya sequence of order 2 for the reliability function.

(iv) The sequence $\{k_\theta(x)\}_{x\geq 0}$ is convex. In other words for all $x_1, x_2, x_3 \in \{0,1,2,...\}$ such that $x_1 < x_2 < x_3$, we have the following inequalities:

$$\frac{k_\theta(x_2) - k_\theta(x_1)}{x_2 - x_1} \leq \frac{k_\theta(x_3) - k_\theta(x_1)}{x_3 - x_1} \leq \frac{k_\theta(x_3) - k_\theta(x_2)}{x_3 - x_2}$$

Remark 1: Since the hazard rate function depends only on $k_\theta(x)$, we can derive the conditions of IFR, DFR or CFR by solely considering the function $k_\theta(x)$ as can be seen from Theorem 1. The conditions for the DFR property can be obtained analogously.

In the following theorem, a further condition that is equivalent to the IFR property for telescopic distributions is derived:

Theorem 3:
Let

$$T_\theta(x) = \frac{1}{2}\Big(2k_\theta(x+1) - k_\theta(x) - k_\theta(x+2)\Big) \tag{19}$$

then the following statements about $h_X(x)$ for telescopic distributions holds:

(i) X is IFR (DFR) iff $T_\theta(x) > (<)0$, for all $x \geq 0$.

(ii) X is CFR iff $T_\theta(x) = 0$, for all $x \geq 0$.

Proof:
$T_\theta(x) > 0$ implies that,

$$\begin{aligned} S_\theta(x) &= q^{2k\theta(x+1)} - q^{k_\theta(x)+k_\theta(x+2)} \\ &= \left(q^{k_\theta(x)} - q^{k_\theta(x+1)}\right)^2 - \Big((q^{k_\theta(x)} - q^{k_\theta(x+1)}) - (q^{k_\theta(x+1)} - q^{k_\theta(x+2)})\Big)q^{k_\theta(x)} \\ &> 0. \end{aligned} \tag{20}$$

In accordance with the forms of $h_X(x)$, $R_X(x)$ and $f_X(x)$ being telescopic distributions, $S_\theta(x) > 0$ implies:

$$h_X(x) > 1 - \frac{f(x+1)}{f(x)} \tag{21}$$

On the other hand, we have:

$$\frac{f(x+1)}{f(x)} = \frac{(1 - h_X(x))\, h_X(x+1)}{h_X(x)} \tag{22}$$

Thus, in view of (21), we have for all $x \geq 0$:

$$h_X(x+1) > h_X(x) \tag{23}$$

Hence, the random variable X has an IFR distribution. When $T_\theta(x) < 0$, there is a similar condition for the DFR property, while $T_\theta(x) = 0$ is equivalent to $1 - h_X(x) = \frac{f(x+1)}{f(x)}$ for all $x \geq 0$ implying that $h_X(x)$ constant. □

Remark 2: In Theorem 3, $T_\theta(x) = 0$, leads to geometric distribution. Since, $k_\theta(x) = xk_\theta(1)$ for $x = 0, 1, 2, ...$ on using induction. Supposing $k_\theta(1) = a$, the probability mass function of telescopic distribution is :

$$f(x) = q^{k_\theta(x)} - q^{k_\theta(x+1)} = q^{ax}(1 - q^a),$$

which is the geometric distribution with parameter $1 - q^a$.

5 Summary

In this paper, some reliability properties for the distributions of the discrete telescopic family are derived and compared with those of the continuous analog. Furthermore, equivalent conditions for the IFR property in case of telescopic distributions are obtained. So far, discrete life time distributions play only a marginal role in reliability analysis. However, real world life time is either observed in discrete time points or it is measured by discrete quantities. Therefore, the focus of reliability analysis should turn to the more realistic discrete life time distributions with finite support that so far are hardly investigated in academic reliability theory, despite the fact that in real world everything is finite. Actually, many properties, for example, the loss of memory property as exhibited by the exponential and geometric distribution which are most frequently assumed in the life time analysis, have no analogous counterpart in the case of a finite support.

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