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A view on Bhattacharyya bounds for inverse Gaussian distributions

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Abstract Shanbhag (J Appl Probab 9:580–587, 1972; Theory Probab Appl 24:430–433, 1979) showed that the diagonality of the Bhattacharyya matrix characterizes the set of Normal, Poisson, Binomial, negative Binomial, Gamma or Meixner hypergeometric distributions. In this note, using Shanbhag (J Appl Probab 9:580–587, 1972; Theory Probab Appl 24:430–433, 1979) and Pommeret (J Multivar Anal 63:105–118, 1997) techniques, we evaluated the general form of the 5×5 Bhattacharyya matrix in the natural exponential family satisfying $f(x|\theta) = \frac{\exp\{xg(\theta)\}}{\beta(g(\theta))}\psi(x)$ with cubic variance function (NEF-CVF) of θ. We see that the matrix is not diagonal like distribution with quadratic variance function and has off-diagonal elements. In addition, we calculate the 5×5 Bhattacharyya matrix for inverse Gaussian distribution and evaluated different Bhattacharyya bounds for the variance of estimator of the failure rate, coefficient of variation, mode and moment generating function due to inverse Gaussian distribution.

Keywords Bhattacharyya matrix · Bhattacharyya bound · Inverse Gaussian distribution · Failure rate · Coefficient variation · Mode · Moment generating function · Natural exponential family

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1 Introduction

A lower bound for the variance of an estimator is one of the fundamental things in the estimation theory because it gives us an idea about the accuracy of an estimator. It is well known in statistical inference that the Cramér–Rao inequality establishes a lower bound for the variance of an unbiased estimator. But the inequality in question does not tell us how sharp it is, or how nearly it could be attained. It states that, under regularity conditions, the variance of any estimator can not be smaller than a certain quantity.

An important inequality to follow the Cramér–Rao inequality is that of a Bhattacharyya (1946, 1947, 1948). Bhattacharyya inequality achieves a greater lower bound for the variance of an unbiased estimator of a parametric function, and it becomes sharper and sharper as the order of the Bhattacharyya matrix increases. Other results related to the Bhattacharyya bounds are given by Blight and Rao (1974). They considered the Bhattacharrya bounds corresponding to the variance of the minimum variance unbiased estimator (MVUE) of $\tau(\theta)$ as a function of the parameter θ when the sampling distribution is a member of an exponential family with density $f(x|\theta)$, which has the property

$$\frac{\partial \log f(x|\theta)}{\partial \theta} = V^{-1}(\theta)(x-\theta),$$

where $V(\theta) = c_0 + c_1\theta + c_2\theta^2$, for some constants c_0, c_1 and c_2.

Using certain results of the Seth (1949) and Shanbhag (1972, 1979), they have shown that, under some regularity conditions, the Bhattacharyya bounds converge to the variance itself. They also provided a table computing the Bhattacharyya function (the (i, i)th element of the Bhattacharyya matrix) explicity for all exponential family distribution except the Meixner hyper geometric distribution. Alzaid (1987) presented the Bhattacharyya function for this distribution.

Using their result, Blight and Rao (1974) also gave the Bhattacharyya bounds for the variance of the MVUE with examples from negative Binomial and exponential distributions. Apparently, the same result was rediscovered by Khan (1984).

Alharbi et al. (1997) presented a characteristic property for the normal distribution based on the structure of the off-diagonal elements of Bhattacharyya matrix is construct.

Shanbhag and Kapoor (1993) and Mohtashami Borzadaran (2006) attempted to identify the class of distributions for which a 2×2 Bhattacharyya matrix is diagonal.

Alharbi (1994) and Bartosewicz (1980) have given an extension of the Bhattacharyya bound to multiparameter cases. They also gave an application of this result when independent samples are taken from the exponential distribution, and evaluated numerically the values of the first four generalized the Bhattacharyya bounds for the best unbiased estimator of $P_r(Y < X)$. Furthermore, Pommeret (1997) obtained Bhattacharyya bound in multivariate case.

Fosam (1993) has provided an alternative method of obtaining the result of Letac and Mora (1990) based on the structure of 3×3 Bhattacharyya matrices.

Recently, Tanaka (2003) determined the family of distributions attaining the Bhattacharyya bound when an exponential family is involved in the family of distributions attaining the Bhattacharyya bound, which is an extension of Theorem 3.1 in Tanaka (2003) in some sense.

In this paper, we will concentrate on the structure and behavior of Bhattacharyya bound for NEF especially in the case of the inverse Gaussian distribution as a member of NEF-CVF. The variance of an estimator can be approximated by Bhattacharyya bounds, when the order of Bhattacharyya matrix be more than one, hence as simulations show for inverse Gaussian, this approximation is better than the approximation by Cramér–Rao bound.

2 Preliminaries

The Bhattacharyya inequality involves the covariance matrix of the random vector,

$$\frac{1}{f(X|\theta)}\left(f^{(1)}(X|\theta), f^{(2)}(X|\theta), \ldots, f^{(n)}(X|\theta)\right),$$

where $f^{(j)}(\cdot|\theta)$ is the jth derivative of the probability density function $f(\cdot|\theta)$ w.r.t. the parameter θ. The covariance matrix of the above random vector is referred to as the $n \times n$ Bhattacharyya matrix and n is the order of it. It is clear that the first diagonal ((1,1)th) element of the Bhattacharyya matrix is the Fisher information.

The Bhattacharyya inequality states that

$$\mathrm{Var}_\theta(T(X)) \geq \xi_\theta^t J^{-1} \xi_\theta, \tag{1}$$

where

- t refer to the transpose and $\xi_\theta = (\tau^{(1)}(\theta), \tau^{(2)}(\theta), \ldots, \tau^{(n)}(\theta))$,
- $\tau(\theta) = E_\theta(T(X))$ and $\tau^{(j)}(\theta) = \frac{\partial^j E_\theta(T(X))}{\partial\theta^j}$ for $j = 1, 2, \ldots, n$,
- J^{-1} is the inverse of the Bhattacharyya matrix, where $J = (J_{rs})$=(Cov$_\theta\{\frac{f^{(r)}(X|\theta)}{f(X|\theta)}, \frac{f^{(s)}(X|\theta)}{f(X|\theta)}\}$), such that $E_\theta(\frac{f^{(r)}(X|\theta)}{f(X|\theta)}) = 0$ for $r, s = 1, 2, \ldots, n$.

If we substitute $n = 1$ in (1), then it indeed reduces to the Cramér–Rao inequality.

The closeness of the bound for the variance of an unbiased estimator, depends on the order of the Bhattacharyya matrix in such a way that increasing the order implies that the size of the Bhattacharyya matrix becomes bigger and the inverse of it requires a greater effort for calculation. This behavior of the Bhattacharyya matrix is due to properties of the multiple correlation coefficient, which could be illustrated in special case $T(X) = X$ as follows:

Let $X = (X_1, X_2, \ldots, X_n)^t$ be a random vector with covariance matrix $\Sigma > 0$ (positive definite). Consider a partition of X and Σ given by

$$X = (X_1, Y)' \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12}' & \Sigma_{22} \end{bmatrix},$$

where $Y = (X_2, X_3, \ldots, X_n)^t$ and Σ is an $n \times n$ matrix, so that $Var(X_1) = \sigma_{11}$ and Σ_{22} is the covariance matrix of the random vector Y and σ_{12} is an $(n-1) \times 1$ vector of the covariances between X_1 and the component of Y. The maximum correlation between X_1 and any linear combination $b'Y$, is called the multiple correlation coefficient between X_1 and Y, and it is denoted by $\rho_{1\cdot 2\ldots n}$; this is attained for $b = \sigma_{12}\Sigma_{22}^{-1}$ (Anderson 1958). So, we have that

$$\rho_{1\cdot 2\ldots n} = \left(\frac{\sigma_{12}\Sigma_{22}^{-1}\sigma_{12}'}{\sigma_{11}}\right)^{1/2},$$

and in view of $\rho^2_{1\cdot 2\ldots n} \leq 1$, can get,

$$\sigma_{11} \geq \sigma_{12}\Sigma_{22}^{-1}\sigma_{12}',$$

which leads to the Bhattacharyya inequality. The lower bound for the variance when n is replaced by $n+1$ is greater than or equal to that obtained by the vector with n components. Hence, as the order of the matrix increases, the Bhattacharyya bound becomes sharper. Increasing the order of the matrix makes it more and more difficult to invert it. But if the matrix is diagonal or when the off-diagonal elements are equal, then one could easily invert it to obtain the corresponding bounds.

Let X be a non-degenerate r.v. distributed according to a distribution with density:

$$f(x|\theta) = \frac{\exp\{xg(\theta)\}}{\beta(g(\theta))}\psi(x), \tag{2}$$

where $\theta \in \Theta$, Θ is an open interval.

Considering a 3×3 Bhattacharyya matrix, the $(r, s)^{th}$ elements of this matrix is

$$J_{rs} = \text{Cov}\left(\frac{f^{(r)}}{f}, \frac{f^{(s)}}{f}\right) = E\left(\frac{f^{(r)}}{f}.\frac{f^{(s)}}{f}\right),$$

since $E(\frac{f^{(r)}}{f}) = 0$, $k = 1, 2, 3$.

Shanbhag (1972) proved that the 3×3 Bhattacharyya matrix is diagonal if and only if:

$$E_\theta(X) = c_{11} + c_{21}\theta$$
$$E_\theta(X^2) = c_{12} + c_{22}\theta + c_{32}\theta^2,$$

that implies

$$\text{Var}_\theta(X) = c_{13} + c_{23}\theta + c_{33}\theta^2,$$

and

$$g'(\theta) = \frac{c_{21}}{c_{13} + c_{23}\theta + c_{33}\theta^2},$$

where $c_{ij}, i, j = 1, 2, 3$ are constants independent of θ. By putting different values for the constants, Shanbhag (1972, 1979) characterized the Normal, Poisson, Gamma, Binomial and negative Binomial distributions in additional to the Meixner hypergeometric distribution. Further, it is observed that in all cases the $n \times n$ Bhattacharyya matrix is defined and is diagonal for all n.

3 Bhattacharyya matrix in natural exponential family with cubic variance function (NEF-CVF)

For finding the Bhattacharyya matrix in NEF-CVF, we first present two important Lemmas.

Lemma 1 (Fend 1959) *In the exponential family satisfying (2) for each* $r = 1, 2, \ldots$ *the function* $\frac{f^{(r)}(X|\theta)}{f(X|\theta)}$ *can be represented as*

$$\frac{f^{(r)}(X|\theta)}{f(X|\theta)} = \sum_{l=0}^{r} d_{rl}(\theta)x^l,$$

through the functions $d_{r0}(\theta), d_{r1}(\theta), \ldots, d_{rr}(\theta)$ *which depend on* θ *only.*

Lemma 2 (Mohtashami Borzadaran 2001) *If for the exponential family with form (2), g is thrice differentiable and* $Var_\theta(X)$ *is the kth degree polynomial in* $E_\theta(X)$*, then, under the assumption that* $E_\theta(X)$ *is linear in* θ*,* $E_\theta(X^r)$ *for* $r \geq 1$ *is a polynomial in* θ *of degree* $1 + (r-1)(k-1)$.

So, considering that $E(X)$ is a linear function of θ and $E(X^2)$ is a cubic function of θ (hence the variance is cubic function of θ) and $E(\frac{f^{(r)}(X|\theta)}{f(X|\theta)}) = 0$ for each r; in view of Pommeret (1997), for exponential family with density, $f(x|\theta)$ we can write:

$$\begin{aligned}
J_{st} &= E\left(\frac{f^{(s)}(X|\theta)}{f(X|\theta)}\frac{f^{(r)}(X|\theta)}{f(X|\theta)}\right) \\
&= \frac{\partial^s}{\partial m^s} E\left(\frac{f(X|m)}{f(X|\theta)}\frac{f^{(t)}(X|\theta)}{f(X|\theta)}\right)\Bigg|_{m=\theta} \\
&= \frac{\partial^s}{\partial m^s} \int \left(f(X|m)\frac{f^{(t)}(X|\theta)}{f(X|\theta)}\right)\Bigg|_{m=\theta} \\
&= \frac{\partial^s}{\partial m^s} E_m\left(\frac{f^{(t)}(X|\theta)}{f(X|\theta)}\right)\Bigg|_{m=\theta}.
\end{aligned}$$

Now, using Lemma 1, $\frac{f^{(t)}(X|\theta)}{f(X|\theta)}$ is a polynomial in X of degree t and from Lemma 2, $E_m(\frac{f^{(t)}(X|\theta)}{f(X|\theta)})$ is a polynomial in m with degree $2t-1$. Then, for all $s > 2t-1$, we obtain, $J_{st} = 0$. It follows that $J_{21} = J_{31} = J_{41} = J_{51} = J_{42} = J_{52} = 0$.

But for other elements we have:

$$
\begin{aligned}
J_{23} = J_{32} &= E\left(\frac{f^{(3)}}{f}\cdot\sum_{l=0}^{2} d_{2l}(\theta)X^l\right) = \sum_{l=0}^{2} d_{2l}(\theta)E\left(\frac{f^{(3)}}{f}\cdot X^l\right)\\
&= d_{22}(\theta)\frac{\partial^3}{\partial\theta^3}E(X^2) \neq 0,\\
J_{34} = J_{43} &= E\left(\frac{f^{(4)}}{f}\cdot\sum_{l=0}^{3} d_{3l}(\theta)X^l\right) = \sum_{l=0}^{3} d_{3l}(\theta)E\left(\frac{f^{(4)}}{f}\cdot X^l\right)\\
&= d_{33}(\theta)\frac{\partial^4}{\partial\theta^4}E(X^3) \neq 0,\\
J_{35} = J_{53} &= E\left(\frac{f^{(5)}}{f}\cdot\sum_{l=0}^{3} d_{3l}(\theta)X^l\right) = \sum_{l=0}^{3} d_{3l}(\theta)E\left(\frac{f^{(5)}}{f}\cdot X^l\right)\\
&= d_{33}(\theta)\frac{\partial^5}{\partial\theta^5}E(X^3) \neq 0,\\
J_{45} = J_{54} &= \sum_{l=0}^{4} d_{4l}(\theta)E\left(\frac{f^{(5)}}{f}\cdot X^l\right)\\
&= d_{43}(\theta)\frac{\partial^5}{\partial\theta^5}E(X^3) + d_{44}(\theta)\frac{\partial^5}{\partial\theta^5}E(X^4) \neq 0,
\end{aligned}
\tag{3}
$$

and for diagonal elements:

$$
\begin{aligned}
J_{rr} &= E\left(\frac{f^{(r)}}{f}\cdot\sum_{l=0}^{r} d_{rl}(\theta)X^l\right) = \sum_{l=0}^{r} d_{rl}(\theta)E\left(\frac{f^{(r)}}{f}\cdot X^l\right)\\
&= \sum_{l=1}^{r} d_{rl}(\theta)\frac{\partial^r}{\partial\theta^r}E(X^l) \neq 0.
\end{aligned}
$$

So, the 5×5 Bhattacharyya matrix of the distribution with linear expectation and cubic variance function of θ has the following general form:

$$
\begin{bmatrix}
J_{11} & 0 & 0 & 0 & 0\\
 & J_{22} & J_{23} & 0 & 0\\
 & & J_{33} & J_{34} & J_{35}\\
 & & & J_{44} & J_{45}\\
 & & & & J_{55}
\end{bmatrix}.
\tag{4}
$$

As we see for obtaining the Bhattacharyya bounds for distributions with these properties, we should just find the non-central moments of X. In the following section, we obtain the 5×5 Bhattacharyya matrix for the inverse Gaussian distribution.

4 Bhattacharyya bounds in the inverse Gaussian distribution

The inverse Gaussian distribution is a very versatile positive-domain two-parametric probabilistic model having numerous applications in diverse fields. It originates as the distribution of the first passage time of the Brownian motion with positive drift. Further applications include lifetime models in connection with repairs, accelerated life testing, reliability problems and frailty models. The naming inverse Gaussian distribution is derived from the fact that its cumulant generating function is the inverse of that of the Gaussian distribution.

The standard or canonical two-parameter inverse Gaussian distribution has probability density function given by:

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{\frac{-\lambda}{2\theta^2 x}(x-\theta)^2\right\}, \quad x > 0,$$

where $\theta, \lambda > 0$ and we denote it by $IG(\theta, \lambda)$ (Seshadri 1988). In this distribution, we have $E(X) = \theta$, $Var(X) = \frac{\theta^3}{\lambda}$. The moment generating function (mgf) for $t < \frac{\lambda}{2\theta^2}$ is:

$$M_X(t) = \exp\left[\frac{\lambda}{\theta}\left(1 - \sqrt{1 - \frac{2\theta^2 t}{\lambda}}\right)\right].$$

Also, in this distribution we can show that the non-central moments are as follow:

$$E(X^r) = \theta^r \sum_{s=0}^{r-1} \frac{(r+s-1)!}{s!(r-s-1)!}\left(\frac{2\lambda}{\theta}\right)^{-s}.$$

So, by using Lemmas 1 and 2 and supposing λ as a known value (fixed), we can calculate the 5×5 Bhattacharyya matrix elements which diagonal elements are:

$$\begin{aligned}
J_{11} &= \frac{\lambda}{\theta^3},\\
J_{22} &= \frac{6\lambda}{\theta^5} + \frac{2\lambda^2}{\theta^6},\\
J_{33} &= \frac{126\lambda}{\theta^7} + \frac{54\lambda^2}{\theta^8} + \frac{6\lambda^3}{\theta^9},\\
J_{44} &= \frac{6210\lambda}{\theta^9} + \frac{2664\lambda^2}{\theta^{10}} + \frac{432\lambda^3}{\theta^{11}} + \frac{24\lambda^4}{\theta^{12}},\\
J_{55} &= \frac{545400\lambda}{\theta^{11}} + \frac{232200\lambda^2}{\theta^{12}} + \frac{41400\lambda^3}{\theta^{13}} + \frac{3600\lambda^4}{\theta^{14}} + \frac{120\lambda^5}{\theta^{15}},
\end{aligned}$$

and the off-diagonal elements are:

$$J_{23} = \frac{6\lambda}{\theta^6},$$
$$J_{34} = \frac{360\lambda}{\theta^8} + \frac{72\lambda^2}{\theta^9},$$
$$J_{35} = \frac{360\lambda}{\theta^9},$$
$$J_{45} = \frac{31320\lambda}{\theta^{10}} + \frac{9360\lambda^2}{\theta^{11}} + \frac{720\lambda^3}{\theta^{12}}.$$

5 Bhattacharyya bounds for the variance of the estimator of the failure rate in $IG(\theta, \lambda)$ via simulation

One of the application of the inverse Gaussian distribution is in reliability problems; The failure rate of $IG(\theta, \lambda)$ is

$$Z(t) = Z(t; \theta, \lambda) = \frac{f_T(t; \theta, \lambda)}{1 - F_T(t; \theta, \lambda)} \quad \text{for } t, \theta, \lambda > 0$$

where

$$Z(t; \theta, \lambda) = \frac{\sqrt{\frac{\lambda}{2\pi t^3}} \exp\left\{-\frac{\lambda}{2\theta^2} \cdot \frac{(t-\theta)^2}{t}\right\}}{\Phi\left(\sqrt{\frac{\lambda}{t}} - \frac{\sqrt{\lambda t}}{\theta}\right) - \Phi\left(-\sqrt{\frac{\lambda}{t}} - \frac{\sqrt{\lambda t}}{\theta}\right) \cdot e^{\frac{2\lambda}{\theta}}}.$$

In Chhikara and Folks (1977) the failure rate of this distribution is shown to converge to $\frac{\lambda}{2\theta^2}$ as $t \to \infty$. So, here we want to approximate the variance of the estimator of the parameter function $\tau(\theta) = \frac{\lambda}{2\theta^2}$ using Bhattacharyya bounds, $B_1, B_2, \ldots, B_5$ are the first five Bhattacharyya bounds for different values of λ and θ, respectively that are presented in Table 1.

We see that, as the order of Bhattacharyya matrix increase, the Bhattacharyya bound get sharper and sharper. Here, the important point is that, sometimes finding

Table 1 Bhattacharyya bounds for the variance of the estimator of the failure rate in $IG(\theta, \lambda)$

θ	λ	B_1	B_2	B_3	B_4	B_5
2	1	0.125	0.7678	2.1845	4.49142	7.7735
4	2	0.0078	0.0479	0.1365	0.2807	0.4858
8	4	0.0004	0.0029	0.0085	0.0175	0.0303
1	2	0.5000	2.3000	5.3820	9.4517	14.2528
4	16	0.0009	0.0034	0.0068	0.0103	0.0138
0.6	4	1.1574	3.3125	5.5224	7.3926	8.8621

Table 2 Bhattacharyya bounds for the variance of the estimator of the coefficient variation of the $IG(\theta, \lambda)$

θ	λ	B_1	B_2	B_3	B_4	B_5
1	1	0.2500	0.2539	0.2545	0.2547	0.2548
1	2	0.125	0.12656	0.12676	0.12681	0.12683
2	1	1.0000	1.0178	1.0214	1.0227	1.0233
4	8	0.5000	0.50625	0.50706	0.50726	0.50732
0.6	0.5	0.1800	0.1829	0.1834	0.1836	0.1837

the variance of estimator is very difficult and therefore we need to approximate it, here Bhattacharyya bound is an appropriate one.

6 Bhattacharyya bounds for the variance of the estimator of the coefficient variation of the $IG(\theta, \lambda)$ via simulation

Seshadri (1988) has found the unbiased estimation of the square of the coefficient of variation of the inverse Gaussian distribution and he has shown that the estimator is indeed a U-statistic. Also, he derived the exact distribution of the estimator.

As we know, the coefficient variation of the inverse Gaussian distribution is $\sqrt{\frac{\theta}{\lambda}}$.

$B_1, B_2, \ldots, B_5$ are the first five Bhattacharyya bounds for different values of λ and θ that are presented in Table 2.

We see that, as the order of Bhattacharyya matrix increasing, the Bhattacharyya bound get sharper and sharper and there is not big difference between B_1 and B_i, $i = 2, \ldots, 5$, in this case.

7 Bhattacharyya bounds for the variance of the estimator of the mode of the $IG(\theta, \lambda)$ via simulation

Noriaki and Kosei (1985) has found the unbiased estimation of the mode of the inverse Gaussian distribution.

As we know the mode of the inverse Gaussian distribution is $\theta\left[\left(1+\frac{9\theta^2}{4\lambda^2}\right)^{\frac{1}{2}}-\frac{3\theta}{2\lambda}\right]$.

$B_1, B_2, \ldots, B_5$ are the first five Bhattacharyya bounds for different values of λ and θ that are presented in Table 3.

Table 3 Bhattacharyya bounds for the variance of the estimator of the mode of the $IG(\theta, \lambda)$

θ	λ	B_1	B_2	B_3	B_4	B_5
1	1	0.0025	0.0147	0.0171	0.0175	0.0176
2	1	0.0005	0.0169	0.0384	0.0470	0.0500
4	2	0.00004	0.0025	0.0143	0.0308	0.0431
4	8	0.00010	0.00044	0.0020	0.0039	0.0048
0.5	2	0.0025	0.00311	0.00312	0.00313	0.00313
0.5	0.4	0.0117	0.0222	0.0231	0.0232	0.0233

Table 4 Bhattacharyya bounds for the variance of the estimator of the moment generating function of the $IG(\theta, \lambda)$

θ	λ	t	B_1	B_2	B_3	B_4	B_5
1	1	0.025	0.3082	0.32121	0.32124	0.32124	0.32124
2	2	0.025	0.003870	0.0038903	0.0038903	0.0038903	0.0038903
0.5	2	1	0.5536	0.57814	0.578491	0.578493	0.578493
0.5	1	0.5	0.08179	0.083322	0.083323	0.083323	0.83323
0.25	2	2	0.1397	0.14263	0.14266	0.14266	0.14266

We see that, as the order of Bhattacharyya matrix increase, the Bhattacharyya bound get sharper and sharper and it seen that the difference between B_1 and $B_i, i = 2, \ldots, 5$ is noticeable.

8 Bhattacharyya Bounds for the variance of the estimator of the moment generating function of the $IG(\theta, \lambda)$ via simulation

As it said in Sect. 4, Seshadri (1988) has shown that the moment generating function (mgf) for $t < \frac{\lambda}{2\theta^2}$, in inverse Gaussian distribution is:

$$M_X(t) = \exp\left[\frac{\lambda}{\theta}\left(1 - \sqrt{1 - \frac{2\theta^2 t}{\lambda}}\right)\right].$$

$B_1, B_2, \ldots, B_5$ are the first five Bhattacharyya bounds for different values of λ, θ and t, that are presented in Table 4.

Like all previous tables, we see that, as the order of Bhattacharyya matrix increase, the Bhattacharyya bound get sharper and sharper and it seen that the difference between B_1 and $B_i, i = 2, \ldots, 5$ is noticeable and the convergence is fast.

9 Conclusion

In this paper, we have shown that the Bhattacharyya bound is a good approximation for the variance of any estimator of the parameter function in inverse Gaussian distribution, as one element of the NEF-CVF that is not a member of NEF-QVF. As an example, we concentrate on the variance of the estimators of failure rate, coefficient variation, mode and moment generating function of the inverse Gaussian distribution.

So, most of time, Bhattacharyya bound that the order of Bhattacharyya matrix is more than one, is better approximation for the variance of an estimator. Increasing the order of Bhattacharyya bound lead us to a sharper bound, so the Bhattacharyya bound of order more than 1 is better than the Cramér–Rao bound.

Finding the same result for the other members of NEF-CVF like Abel, Tackas,... is the future of this research.

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