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An analytical algorithm for unsteady nonlinear convective-radiative equation

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Abstract

In this work, an effective analytical algorithm based on the parametric iteration method (PIM) is proposed for finding highly accurate approximate analytical solution of a nonlinear problem arising in heat transfer, that is, unsteady nonlinear convective-radiative equation containing two small parameters ε_1 and ε_2 .

1 Introduction

Heat transfer equations are such phenomena which mostly occur nonlinearly, hence solving them has been one of the most time-consuming and difficult affairs among the researchers working on heat transfer. Here, an analytical algorithm based on the parametric iteration method (or fractional iteration method) [Ghorbani08] will be proposed in solving the unsteady nonlinear convective-radiative equation. The numerical results reveal that the developed algorithm is a very simple, effective and much more accurate. In this work we consider the heat transfer in a lumped system of combined convective-radiative heat transfers. The specific heat coefficient is linear with temperature as follows [Ganji06]:

$$C = C_\alpha\Big(1 + \beta(T - T_\alpha)\Big). \tag{1}$$

Let the system have volume ν, surface area A, density ρ, specific heat c, emissivity E and initial temperature T_i. At $t = 0$, the system is exposed to an environment with convective heat transfer with coefficient $\hbar$ and the temperature T_α. The system also loses heat through radiation and the effective sink temperature is T_s. The cooling equation of the system with the following initial condition is as follows:

$$\rho\nu c\frac{dT}{dT} + \hbar A(T - T_\alpha) + E\sigma A(T^4 - T_s^4) = 0, \quad T(t = 0) = T_i. \tag{2}$$

To solve the equation we would do the following change of parameters:

$$u=\frac{T}{T_i},\quad u_\alpha=\frac{T_\alpha}{T_i},\quad x=\frac{t\hbar A}{\rho\nu c_\alpha},\quad \varepsilon_1=\beta T_i,\quad \varepsilon_2=\frac{E\sigma T_i^3}{\hbar},\quad u_s=\frac{T_s}{T_i}. \tag{3}$$

After the parameters are changed, the system heat transfer equation will result the following:

$$[1+\varepsilon_1(u-u_\alpha)]\frac{du}{dx}+(u-u_\alpha)+\varepsilon_2(u^4-u_s^4)=0, \tag{4}$$

$$x=0\longrightarrow u=0. \tag{5}$$

For simplicity, we assume the case $u_\alpha=u_s=0$. So we have:

$$[1+\varepsilon_1 u(x)]\frac{du}{dx}+u(x)+\varepsilon_2 u^4(x)=0, \tag{6}$$

$$x=0\longrightarrow u=0. \tag{7}$$

2 Analysis of the algorithm

In this section, we first describe PIM for solving Eq. (6). Based on the presented PIM, then, an effective algorithm is proposed for handling Eq. (6).

The basic character of PIM is to construct a family of iterative processes for Eq. (6) as follows [Ghorbani08]:

$$u_{n+1}(x)=u_n(x)+h\int_{x_0}^{x}\left(\frac{du_n(t)}{dt}+\frac{u_n(t)+\varepsilon_2 u_n^4(t)}{[1+\varepsilon_1 u_n(t)]}\right)dt, \tag{8}$$

where $u_0(x)=u(x_0=0)=c(=1)$ is the initial guess and $h\neq 0$ denotes the so-called auxiliary parameter. The convergence of the iterative relation of (8) is ensured by Banach's fixed point theorem [Reed80], provided that the right hand side of (8) is a contractive mapping. As will be shown later in this work, the auxiliary parameter h can be properly chosen so that solution of (8) exists. Accordingly, the successive approximations $u_n(x),\quad n\geq 0$ of the solution $u(x)$ will be readily obtained by selecting the zeroth component. Consequently, the exact solution may be obtained by using $u(x)=\lim_{n\longrightarrow\infty}u_n(x)$.

In general, the application of PIM to nonlinear problems leads to the calculation of unneeded terms. To completely cancel these terms in each step, the following modification of PIM is suggested, which is called the truncated PIM (TP):

$$u_{n+1}(x)=u_n(x)+(1+h)[u_n(x)-u_{n-1}(x)]+h\int_{x_0}^{x}\Big(F_n(t)-F_{n-1}(t)\Big)dt,\quad x\in[x_0,T], \tag{9}$$

where $u_0(x)=c$, $F_{-1}(t)=0$ and $F_n(t),\quad n=0,1,2,...$ are obtained from the following:

$$\frac{u_n(t)+\varepsilon_2 u_n^4(t)}{[1+\varepsilon_1 u_n(t)]}=F_n(t)+\mathrm{O}[(t-x_0)^{n+1}]. \tag{10}$$

It is noteworthy to point out that the TP formula (9) can cancel all terms that are not needed. By using the TP algorithm (9), we obtain a series solution, which in practice is a truncated series solution. This series solution gives a good approximation to the exact solution in a small region of x. An easy and reliable way of ensuring validity of the approximations (9) for large x is to determine the solution in a sequence of subintervals of x, i.e., $I_i = [x_i, x_{i+1}]$ where $\Delta_i = x_{i+1} - x_i$, $i = 0, 1, ..., N-1$, with $x_N = T$. Therefore, in order to enlarge the convergence region of the solutions (9), we can construct the following piecewise approximations for Eq. (9), which is called the piecewise TP (PTP):

$$u_{i+1,k+1}(x) = u_{i+1,k}(x) + (1+h)[u_{i+1,k}(x) - u_{i+1,k-1}(x)] + h\int_{x_i}^{x}\Big(F_{i+1,k}(t) - F_{i+1,k-1}(t)\Big)dt, \quad (11)$$

$$u_{i+1,0}(x) = u_{i,n_i}(x_i) = c_i, \quad i = 0, 1, ..., N-1, \quad (12)$$

$$\frac{u_{i+1,k}(t) + \varepsilon_2 u^4_{i+1,k}(t)}{[1 + \varepsilon_1 u_{i+1,k}(t)]} = F_{i+1,k}(t) + \mathrm{O}[(t - x_i)^{k+1}], \quad k = 0, 1, ..., n_{i+1} - 1, \quad (13)$$

where $u_{0,n_0}(x_0) = u(x_0) = c_0$ and $F_{i+1,-1}(t) = 0,\ i = 0, 1, ..., N-1$. Then we can obtain the n_{i+1}-order PTP approximation $u_{i+1,n_{i+1}}(x)$ on I_i. Thus, in the light of (11)-(13), the approximate analytical solution of Eq. (6) on the entire interval $[x_0, T]$ can easily be obtained. It should be emphasized that the PTP technique provides analytical solutions in $[x_i, x_{i+1}]$, which are continuous at the end points of each interval, i.e., $u_{i,n_i}(x_i) = c_i = u_{i+1,n_{i+1}}(x_i),\ i = 1, ..., N-1$.

Following the present section, the n_{i+1}-order explicit approximate analytical solution via the PTP method for Eq. (6) can be written as:

$$u_{i+1,n_{i+1}}(x) = \sum_{m=0}^{n_{i+1}} \frac{\gamma_{i,m}(x_i, c_i, h)}{m!}(x - x_i)^m + \mathrm{O}[(x - x_i)^{n_{i+1}+1}],\ x \in [x_i, x_{i+1}], \quad (14)$$

where $\gamma_{i,m}(x_i, c_i, h)$ is a coefficient dependent of x_i, c_i and h. The expression (14) demonstrates that the numerical results of the n_{i+1}-order PTP method has the error per step is on the order of $(\Delta_i)^{n_{i+1}+1}$, while the total accumulated error is of order $(\Delta_i)^{n_{i+1}}$.

3 Geometric ways for choosing h and Δ_i

It is important to ensure that the numerical result obtained using the PTP algorithm (i.e., c_i), which is as a series in the auxiliary parameter h and the fixed step size $\Delta_i = \Delta x = \Delta$ is convergent in a large enough region whereby the convergence region and rate are dependent upon the h and Δ. Since we have a family of solution expressions in the auxiliary parameter h and the step size Δ, hence, regarding h and Δ as independent variables, a simple and practical way of selecting h with relation to Δ is to plot the curves of the resulting series (c_i) with respect to h and Δ. Thus, if the series is convergent, there exists a horizontal segment in its figure called *the $h\Delta$-curves* that corresponds to a region of h and Δ. For brevity, we call such a region *the valid region of h with relation to* Δ i.e., $R_{h\Delta}$. Accordingly, if we set h and Δ values in $R_{h\Delta}$, we are sure that the corresponding solution series converges. Therefore, these curves provide us with a convenient way to show the influence of h and Δ on the convergence of the PTP algorithm.

4 Numerical implementations

Now we apply the PTP algorithm proposed in Section 3 to Eq. (6), which ultimately shows the efficacy and accuracy of this method. We mention that Eq. (6) has been solved by some approximate analytical methods that the convergence region of the corresponding results is very small, see [Ganji06], while the proposed algorithm is free from this restriction.

According to (11)-(13), we can obtain a second-order PTP approximation in the subintervals I_i as follows:

$$u_{i+1,2}(x) = c_i + \frac{hc_i(1+\varepsilon_2 c_i^3)(2+h)}{1+\varepsilon_1 c_i}(x-x_i) + \frac{h^2 c_i(1+\varepsilon_2 c_i^3)(1+4\varepsilon_2 c_i^3+3\varepsilon_1\varepsilon_2 c_i^4)}{2(1+\varepsilon_1 c_i)^3}(x-x_i)^2, \quad (15)$$

where $x_0 = 0$, $c_0 = u(0) = 1$ and $c_{i+1} = u_{i+1,2}(x_{i+1})$, $\quad i = 0, 1, ..., N-1$. The absolute error (i.e., the difference between the PTP solution and the numerical solution) of the relation (15) for $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.25$, $N = 250,\ 500,\ 1000,\ 5000$ ($T = 50$) and $h = -1$ have been given in Fig. 1. Also, in Fig. 2, we have plotted the $h\Delta$-curves of (15) for $i = 2$, i.e., c_3. It is important to mention that only the second-order term of the PTP algorithm was used in evaluating the approximate analytical solution for Fig. 1, which is in excellent agreement with the numerical values in the large interval.

Figure 1: The absolue errors ($E_2(x) = |u_{Numer.}(x) - u_2(x)|$) of the 2th-order PTP solution when $h = -1$ for Eq. (6) with $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 0.25$.

In closing our analysis, we mention that a concrete modeling problem of heat transfer was tested by using the PTP algorithm proposed in this paper, and the obtained results have shown excellent performance.

References

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Figure 2: (Plotting the $h\Delta - curves$) The valid region of h with respect to Δ (Delta) for Eq. (6) with $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 0.25$ using the numerical result c_3 of the 2th-order PTP algorithm.

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