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A note on convex renorming and fragmentability

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Abstract. Using the game approach to fragmentability, we give new and simpler proofs of the following known results: (a) If the Banach space admits an equivalent Kadec norm, then its weak topology is fragmented by a metric which is stronger than the norm topology. (b) If the Banach space admits an equivalent rotund norm, then its weak topology is fragmented by a metric. (c) If the Banach space is weakly locally uniformly rotund, then its weak topology is fragmented by a metric which is stronger than the norm topology.

Keywords. Fragmentability of Banach spaces; topological games; renorming of Banach spaces.

1. Introduction

Let (X, τ) be a topological space and ρ be a metric on X. Given $\epsilon > 0$, a nonempty subset A of X is said to be *fragmented by ρ down to ϵ* if each nonempty subset of A contains a nonempty τ–relatively open subset of ρ-diameter less than ϵ. A is called *fragmented by ρ* if A is fragmented by ρ down to ϵ for each $\epsilon > 0$. The set A is said to be *σ-fragmented by ρ* if for every $\epsilon > 0$, A can be expressed as $A = \cup_{n=1}^{\infty} A_{n,\epsilon}$ with each $A_{n,\epsilon}$ fragmented by ρ down to ϵ.

The notion of fragmentability was originally introduced in [3] to investigate the existence of nice selections for upper semicontinuous compact-valued mappings. The notion of σ-fragmentability appeared in [1] in order to study Banach spaces, the weak topology of which is σ-fragmented by the norm (such Banach spaces are said to be σ-fragmentable). Since then, these two concepts have been playing an important role in the study of the geometry of Banach spaces.

Kenderov and Moors [4] used the following topological game to characterize fragmentability of a topological space X: Two players Σ and Ω alternatively select subsets of X. Σ starts the game by choosing some nonempty subset A_1 of X. Then Ω chooses some nonempty relatively open subset B_1 of A_1. In general, if the selection $B_n \neq \emptyset$ of the player Ω is already specified, the player Σ makes the next move by selecting an arbitrary nonempty set A_{n+1} contained in B_n. Continuing the game the two players generate a sequence of sets

$$A_1 \supset B_1 \supset \cdots \supset A_n \supset B_n \supset \cdots$$

which is called a play and is denoted by $p = (A_i, B_i)_{i=1}^{\infty}$. If

$$p_1 = (A_1), \ldots, p_n = (A_1, B_1, \ldots, A_n)$$

are the first 'n' move of some play (of the game), then p_n is called the nth *partial play* of the game. The player Ω is said to have won the play p if $\cap_{i=1}^{\infty} A_i = \cap_{i=1}^{\infty} B_i$ contains at most one point. Otherwise the player $\sum$ is said to be the winner in this play. Under the term *strategy s for* Ω*-player*, we mean a rule by means of which the player Ω makes his/her choices. More precisely, the strategy s is a sequence of mappings $s = \{s_n\}_{n\geq 1}$, which are defined inductively as follows: s_1 assigns to each possible first move A_1 of Σ-player a nonempty relatively open subset $B_1 = s_1(A_1)$. Therefore, the domain of s_1 is the set of all nonempty subsets of X and s_1 assigns to each such an element a nonempty relatively open subset of it. The domain of s_2 consists of triples of the type (A_1, B_1, A_2), where A_1 is from the domain of s_1, $B_1 = s_1(A_1)$ and A_2 is an arbitrary nonempty subset of B_1. s_2 assigns to such a triple a nonempty relatively open subset $B_2 = s_2(A_1, B_1, A_2)$ of A_2. In general, the domain of s_{n+1} consists of partial plays of the type

$$(A_1, \ldots, A_i, B_i, A_{i+1}, \ldots, A_{n+1}),$$

where, for every $i \leq n$, $(A_1, \ldots, A_i)$ is from the domain of s_i, $B_i = s_i(A_1, \ldots, A_i)$ and A_{n+1} is an arbitrary nonempty subset of B_n. To every element from its domain s_{n+1} assigns a nonempty relatively open subset B_{n+1} of A_{n+1}.

A play $p = (A_i, B_i)_{i\geq 1}$ is called an s-play if $B_i = s_i(p_i)$ for each $i \geq 1$. s is called a *winning strategy* for the player Ω if he/she wins every s-play. If the space X is fragmentable by a metric $d(\cdot, \cdot)$, then Ω has an obvious winning strategy s. Indeed, to each partial play p_n this strategy puts into correspondence some nonempty subset $B_n \subset A_n$ which is relatively open in A_n and has d-diameter less than $1/n$. Clearly, the set $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ has at most one point because it has zero d-diameter. Kenderov and Moors have shown that the existence of a winning strategy for the player Ω characterizes fragmentability, that is,

Theorem 1.1 [4]. *The topological space X is fragmentable if and only if the player* Ω *has a winning strategy.*

Of special interest is the case when the topology generated by the fragmenting metric contains the original topology of the space (in this case it is said that *X is fragmented by a metric which is stronger than its topology*).

Theorem 1.2 [4]. *The topological space X is fragmentable by a metric stronger than its topology if and only if the player* Ω *has a strategy a such that, for every s-play* $p = (A_i, B_i)_{i\geq 1}$ *the intersection* $\cap_{i=1}^{\infty} A_i = \cap_{i=1}^{\infty} B_i$ *is either empty or contains just one point* x_0 *and for every neighborhood U of* x_0 *there exists some k such that* $A_i \subset U$ *for all* $i > k$.

This characterization of fragmentability has some applications (see e.g. [4–6]). In [5], it is shown that fragmentability and σ-fragmentability of the weak topology in a Banach space are related to each other in the following way:

Theorem 1.3 ([5], Theorems 1.3, 1.4 and 2.1). *For a Banach space X the following are equivalent*:

(i) $(X, \textit{weak})$ *is* σ*-fragmented by the norm (i.e. X is* σ*-fragmented)*;
(ii) $(X, \textit{weak})$ *is fragmented by a metric which is stronger than the weak topology*;
(iii) $(X, \textit{weak})$ *is fragmented by a metric which is stronger than the norm topology*;
(iv) *There exists a strategy s for the player* Ω *in* $(X, \textit{weak})$ *such that, for every s-play* $p = (A_i, B_i)_{i\geq 1}$ *either* $\cap_{i\geq 1} B_i = \emptyset$ *or* $\lim_{i\to\infty}$ *norm-diam* $(B_i) = 0$.

(v) *There exists a strategy s for the player Ω in $(X$, weak$)$ such that, for every s-play $p = (A_i, B_i)_{i\geq 1}$ either $\bigcap_{i\geq 1} B_i = \emptyset$ or every sequence $\{x_i\}_{i\geq 1}$ with $x_i \in B_i$, $i \geq 1$ has a weak cluster point.*

Moreover, we have the following: The norm $\|\cdot\|$ of a Banach space X is said to be *Kadec* if the norm topology and the weak topology coincide on the unit sphere $\{x \in X\colon \|x\| = 1\}$. In [2], it was shown that every Banach space with Kadec norm is σ-fragmented. It follows that there exists a strategy for the player Ω satisfying condition (iv) from the theorem of Kenderov and Moors. In the next section, we will directly construct such a strategy (without using the theorem of Kenderov and Moors).

The norm $\|\cdot\|$ of a Banach space X is said to be *rotund* (*or strictly convex*) if the unit sphere $\{x \in X : \|x\| = 1\}$ does not contain nontrivial line segments. Ribarska has shown in [7] that the weak topology of a rotund Banach space is fragmented by a metric. By the abovementioned characterization of fragmentability it follows that the player Ω has a winning strategy. In the next section we will directly define such a strategy (without using the result of Ribarska and the mentioned theorem of Kenderov and Moors). Moreover, if the norm of X is weakly locally uniformly rotund, then the strategy we construct satisfies condition (v) from the above theorem of Kenderov and Moors. Recall that the Banach space X is called *locally uniformly rotund* (*resp. weakly locally uniformly rotund*) if $\lim_{n\to\infty} \|x_n - x\| = 0$ (resp. *weak–* $\lim(x_n - x) = 0$, whenever $\lim_{n\to\infty} \|(x_n + x)/2\| = \lim_{n\to\infty} \|x_n\| = \|x\|$.

2. Description of the strategies

Lemma 1. *Let X be a Banach space with Kadec norm. Then, for every $\epsilon > 0$ and $x \in X$, there exists some positive number $\alpha_{\epsilon,x}$ and a weakly open set $W_{\epsilon,x} \ni x$ such that $\|y - x\| < \epsilon$ whenever $y \in W_{\epsilon,x}$ and $|\|y\| - \|x\|| \leq \alpha_{\epsilon,x}$.*

Proof. If $x = 0$, it suffices to put $W_{\epsilon,x} = X$ and to take as $\alpha_{\epsilon,x}$ any positive number smaller than $\epsilon/2$. Suppose $x \neq 0$ and take a convex weakly open neighborhood G of x such that the norm diameter of $G \cap \{z\colon \|z\| = \|x\|\}$ is less than $\epsilon/2$. Define $\alpha_{\epsilon,x} > 0$ to be smaller than $\epsilon/2$, $\|x\|$ and such that $\alpha_{\epsilon,x} B \subset (G - x)/2$ (as usual B stands for the closed unit ball of X). Put $W_{\epsilon,x} := x + (G - x)/2 = (x + G)/2$. Let $y \in W_{\epsilon,x}$ and $|\|y\| - \|x\|| < \alpha_{\epsilon,x}$. Then we have

$$(\|x\|/\|y\|)y = ((\|x\|/\|y\|)y - y) + y = (\|x\| - \|y\|)y/\|y\| + y$$
$$\in |\|y\| - \|x\||B + W_{\epsilon,x} \subset \alpha_{\epsilon,x} B + W_{\epsilon,x} \subset (G - x)/2 + (G + x)/2 = G.$$

Hence $\|(\|x\|/\|y\|)y - x)\| < \epsilon/2$. Finally we have

$$\|y - x\| \leq \|y - (\|x\|/\|y\|)y\| + \|(\|x\|/\|y\|)y - x\| < \alpha_{\epsilon,x} + \epsilon/2 < \epsilon.$$

□

We also need the following result:

Lemma 2 ([5], Proposition 2.1). *If the closed unit ball B of a Banach space X admits a strategy s with the property* (iv) *of Theorem* 1.3, *then the whole space also admits such a strategy.*

Theorem 2.1. *Let X be a Banach space with Kadec norm. Then there exists a strategy s for the player Ω in $(B, weak)$ such that, for every s-play $p = (A_i, B_i)_{i\geq 1}$ either $\cap_{i\geq 1} B_i = \emptyset$ or* $\lim_{i\to\infty}$ *norm-diam* $(B_i) = 0$.

Proof. Let $\|\cdot\|$ denote the Kadec norm on X and A_1 be the first choice of Σ-player. By Lemma 2, we may assume that $A_1 \subset B$, where B denotes the closed unit ball of X. Put

$$\rho_1 = \sup\{\|x\| : x \in A_1\} \quad \text{and} \quad \epsilon_1 = 1.$$

Two cases may happen.

(1) There is an element $x_1 \in A_1$ such that $\alpha_{\epsilon_1, x_1} + \|x_1\| > \rho_1$. Then we take such a point x_1 and define $s_1(A_1) = B_1 := W_{\epsilon_1, x_1} \cap A_1 \backslash (\|x_1\| - \alpha_{\epsilon_1, x_1})B$ and $\epsilon_2 := \epsilon_1/2$. Then for each $y \in B_1$, $\|y\| \leq \rho_1 < \alpha_{\epsilon_1, x_1} + \|x_1\|$ and $\|y\| \geq \|x_1\| - \alpha_{\epsilon_1, x_1}$. Therefore, by Lemma 1, $\|y - x_1\| < \epsilon_1$. Hence $\| \ \| - \mathrm{diam}(B_1) < 2\epsilon_1$.

(2) For every $x \in A_1$, $\alpha_{\epsilon_1, x} + \|x\| \leq \rho_1$. Then,

$$s_1(A_1) = B_1 := A_1 \backslash (1/2)\rho_1 B$$

and set $\epsilon_2 = \epsilon_1$. Suppose the mappings $(s_i)_{i\leq n}$ participating in the definition of a strategy for player Ω have already been defined. Let $(A_i, B_i)_{1\leq i\leq n}$ be a partial play which is generated by the strategy mappings defined so far. This partial play is accompanied by the numbers $\{\epsilon_i\}_{1\leq i\leq n}$ and the points $x_1, \ldots, x_n$. If A_{n+1} is the next move of the player Σ, we put

$$\rho_{n+1} = \sup\{\|x\| : x \in A_{n+1}\}$$

and consider the following two possible cases:

(1) There exists an element $x_{n+1} \in A_{n+1}$, such that $\alpha_{\epsilon_{n+1}, x_{n+1}} + \|x_{n+1}\| > \rho_{n+1}$. In this case, we take such a point x_{n+1}, define

$$s_{n+1}(A_1, \ldots, A_{n+1}) = B_{n+1} := W_{\epsilon_{n+1}, x_{n+1}} \cap A_{n+1} \backslash (\|x_{n+1}\| - \alpha_{\epsilon_n, x_{n+1}})B$$

and set $\epsilon_{n+2} = \epsilon_{n+1}/2$. As above one shows that in this case $\| \ \| - \mathrm{diam}(B_{n+1}) < 2\epsilon_{n+1}$.

(2) For every point $x \in A_{n+1}$, $\alpha_{\epsilon_{n+1}, x} + \|x\| \leq \rho_{n+1}$. In this case, we define

$$s_{n+1}(A_1, \ldots, A_{n+1}) = B_{n+1} := A_{n+1} \Big\backslash \left(1 - \frac{1}{(n+2)}\right) \rho_{n+1} B$$

and set $\epsilon_{n+2} = \epsilon_{n+1}$. In this way the strategy $s = (s_i)_{i\geq 1}$ for the Ω-player is already defined.

Suppose $(A_i, B_i)_{i\geq 1}$ is an s-play with $x \in \cap_{n\geq 1} A_n$ and $\lim_{n\to\infty} \|\cdot\| - \mathrm{diam}(B_n) \neq 0$. Then there exists some $\delta > 0$, such that $\|\cdot\| - \mathrm{diam}(B_n) > \delta$ for each $n \in N$. This means that for all but finitely many n, the case (2) happens and thus $\{\epsilon_n\}$ is eventually constant: $\epsilon_n = \epsilon > 0$ for all $n > k$. Since $x \in \cap_{n\geq 1} A_n$,

$$\left(1 - \frac{1}{n}\right) \rho_n < \|x\| < \rho_n, \quad \text{for all} \quad n.$$

Let $\rho_n \searrow \rho$. Then the above inequality shows that $\|x\| = \rho$. On the other hand, $\alpha_{\epsilon, x} + \|x\| = \alpha_{\epsilon_n, x} + \|x\| \leq \rho_n$ for $n > k$ which implies the contradiction $\alpha_{\epsilon, x} + \|x\| = \|x\|$. □

Remark 2.2. Lemma 1 directly implies that Banach spaces with Kadec norm are σ-fragmentable. Actually, Theorem 2.3 of [2] indirectly implies that every Kadec renormable Banach space X has a countable cover by sets of small local norm diameter, i.e., for each $\varepsilon > 0$, it is possible to write $X = \cup_{n \in N} X_{n,\epsilon}$ such that for each $n \in N$ and $x \in X_{n,\epsilon}$, there exists an open neighborhood V_x, of x such that the norm diameter of $V_x \cap X_{n,\epsilon}$ is less then ϵ. Using Lemma 1, we can give another proof of this result.

PROPOSITION 2.3

Let X be a Banach space with Kadec norm. Then for every $\epsilon > 0$ *there exists a countable cover of* X, $X = \cup_{i \geq 0} X_i$, *such that, for every* $x \in X_i$, *there exists a weakly open neighborhood* W *of* x *such that* $W \cap X_i$ *is contained in* $x + \epsilon B$, *in particular the points of* X_i *have weak neighborhoods with norm-diameter smaller than* 2ϵ.

Proof. Given $\epsilon > 0$ consider, for $k = 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$, the sets $X_{kn} = \{x \in X : \alpha_{\epsilon,x} > 2/k$, and $n/k \leq \|x\| \leq (n+1)/k\}$. Clearly, X is covered by X_{kn}. Put $W := W_{\epsilon,x}$. By Lemma 1 the set $W \cap X_{kn}$ is contained in $x + \epsilon B$. □

Theorem 2.4. *Let X be a Banach space.*

(a) *If the norm of* X *is rotund, then* $(X, weak)$ *is fragmentable by a metric.*
(b) *If the norm of* X *is weakly locally uniformly rotund, then* $(X, weak)$ *is fragmented by a metric which is stronger than the norm topology.*

Proof. According to Theorems 1.2 and 1.3 and Lemma 2, it is enough to show that in (B, *weak*) the player Ω has a winning strategy s such that, for every s-play $p = (A_i, B_i)_{i \geq 1}$, $\cap_{i \geq 1} B_i$ has at most one point and in case (b) either $\cap_{i \geq 1} B_i = \emptyset$ or every sequence $\{y_n\}$, $y_n \in B_n$, $n \geq 1$ is weakly convergent to the element of $\cap_{i \geq 1} B_i$. Let $\| \ \|$ be the equivalent norm on X and Σ start a game by choosing a nonempty subset A_1 of B. Define

$$\rho_1 = \sup\{\|x\| \ : \ x \in A_1\}.$$

Choose an element $x_1 \in A_1$ such that $\|x_1\| > \rho_1 - 1/2$ and find some $\mu_1 \in X^*$ such that $\|\mu_1\| = 1$ and $\mu_1(x_1) = \|x_1\|$. Define

$$s_1(A_1) = B_1 := \{x \in A_1 \colon \mu_1(x) > \rho_1 - 1/2\}$$

as the first choice of Ω-player. Then for each $x \in B_1$, we have

$$\rho_1 - 1/2 < \mu_1(x) \leq \|x\| \leq \rho_1.$$

Suppose that the finite sequence $\{x_k\}_{k \leq n}$ of points of X, $\{\mu_k\}_{k \leq n}$ of elements of X^*, and the partial play $p_n = (A_1, \ldots, B_n)$ have already been specified so that for each $x \in B_k$, $k \leq n$ the inequality

$$\rho_k - \frac{1}{k+1} < \mu_k(x) < \|x\| \leq \rho_k$$

holds. Let A_{n+1} be the answer of Σ-player to p_n. Put

$$\rho_{n+1} = \sup\{\|x\| : x \in A_{n+1}\}$$

and find some $x_{n+1} \in A_{n+1}$, $\|x_{n+1}\| > \rho_{n+1} - \frac{1}{n+2}$. Take some $\mu_{n+1} \in X^*$, $\|\mu_{n+1}\| = 1$ with $\mu_{n+1}(x_{n+1}) = \|x_{n+1}\|$ and define

$$s_{n+1}(A_1, \ldots, A_{n+1}) = B_{n+1} = \left\{x \in A_{n+1}\colon \mu_{n+1}(x) > \rho_{n+1} - \frac{1}{n+2}\right\},$$

as the next choice of the player Ω. Clearly, for each $x \in B_{n+1}$, the inequality

$$\rho_{n+1} - \frac{1}{n+2} < \mu_{n+1}(x) < \|x\| \leq \rho_{n+1}$$

holds. Thus, by induction on n, we have shown that the Ω-player can choose sets of the form

$$B_n = \left\{x \in A_n : \mu_n(x) > \rho_n - \frac{1}{n+1}\right\},$$

where $\|\mu_n\| = 1$ and $\rho_n = \sup\{\|x\| : x \in A_n\}$ for each $n \in N$.

Let $\cap_{n\geq 1} B_n \neq \emptyset$ and μ be a weak* cluster point of $\{\mu_n\}$. Then for each $x \in \cap_{n\geq 1} B_n$, the inequality

$$\rho_n - \frac{1}{n+1} < \mu_n(x) < \|x\| \leq \rho_n$$

for each $n \in N$ implies that $\mu(x) = \|x\| = \rho$, where ρ is the limit of the decreasing sequence $\{\rho_n\}$. It follows that for each $x, y \in \cap_{n\geq 1} B_n$, we have $\mu(x) = \|x\| = \|y\| = \mu(y)$. Rotundity of X implies that $x = y$, thus, in this case, $\cap_{n\geq 1} B_n$ has at most one point. In case (b), suppose that $x \in \cap_{n\geq 1} B_n$. If $y_n \in B_n$, the inequality

$$\rho_n - \frac{1}{n+1} < \frac{1}{2}\mu_n(x + y_n) \leq \frac{1}{2}\|x + y_n\| \leq \frac{1}{2}(\|x\| + \|y_n\|) \leq \rho_n$$

shows that $\lim_{n\to\infty} \|(x + y_n)/2\| = \lim_{n\to\infty} \|y_n\| = \|x\| = \rho$. Since $(X, \| \|)$ is weakly locally uniformly rotund, it follows that $\lim_{n\to\infty}(x - y_n) = 0$. By Theorem 1.2, the space is fragmented by a metric stronger than the weak topology. This completes the proof. □

Remark 2.5. It is well-known that locally uniformly rotund norms are Kadec. Therefore statement (b) from the above theorem follows from Theorem 2.1 as well.

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