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Shift Invariant Spaces and Shift Preserving Operators on Locally Compact Abelian Groups

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Abstract

We investigate shift invariant subspaces of $L^2(G)$, where G is a locally compact abelian group. We show that every shift invariant space can be decomposed as an orthogonal sum of spaces each of which is generated by a single function whose shifts form a Parseval frame. For a second countable locally compact abelian group G we prove a useful Hilbert space isomorphism, introduce range functions and give a characterization of shift invariant subspaces of $L^2(G)$ in terms of range functions. Finally, we investigate shift preserving operators on locally compact abelian groups. We show that there is a one-to-one correspondence between shift preserving operators and range operators on $L^2(G)$ where G is a locally compact abelian group.

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1 Introduction

In the last decade, shift invariant (SI)subspaces of $L^2(\mathbb{R}^n)$ have been studied from different aspects, by many authors such as: Aldroubi, Benedetto, Bownik, De Boor, DeVore, Li, Ron, Rzeszotnik, Shen, Weiss and Wilson, cf. [1, 2, 4, 5, 7, 8, 10, 24, 28]. This theory plays an important role in many areas, specially in the theory of wavelets, and multiresolution analysis. It has been used to show a new characterization of orthonormal wavelets conjectured by Weiss [26], a result originally proved in [6] by applying the techniques of [23, 24].

In this paper we investigate the structure of shift invariant subspaces of $L^2(G)$, where G is a locally compact abelian group. Our results generalize some of the results appearing in the literature on shift invariant spaces. Such a unified approach seems to be useful, since it describes the basic features of shift invariant spaces, and includes most of the special cases. The general structure of these spaces in $L^2(\mathbb{R}^n)$ was revealed in the work of de Boor, DeVore and Ron with the use of fiberization techniques based on range functions [5]. The study of analogous spaces for $L^2(\mathbb{T}, H)$ with values in a separable Hilbert space H, in terms of range functions, is quite classical and goes back to Helson [15]. Recently Bownik gave a characterization of shift invariant subspaces of $L^2(\mathbb{R}^n)$ following an idea from Helson's book [15]. So far the theory of SI spaces has been investigated on $\mathbb{R}^n$ but to work with other concrete examples of locally compact abelian (LCA) groups, it is essential for the theory to be extended to the general setting. Some general properties of SI spaces on LCA groups, have been studied by the authors [19]. The present paper is devoted to the study of structural properties of SI spaces on second countable LCA groups using a range function approach.

A bounded linear operator $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is called shift preserving (which will be abbreviated to "SP") if $UT_k = T_kU$ for all $k \in \mathbb{Z}^n$, where T_k is the shift operator. As a special case of a shift operator is the time delay operator $T_k : l^2 \to l^2$ defined by $T_k u(n) = u(n-k)$, $u \in l^2$, k, $n \in \mathbb{Z}$ where the action is to delay the signal u by k units. A digital filter U is a SP operator on l^2. In other words a filter is a time invariant operator in which delaying the input by k units of time is just to delay the output by k units. These operators play an important role in signal processing, such as to analyse, code, reconstruct signals and so on. They are often used to extract required frequency components from signals. For example, high frequency components of a signal usually contain the noise and fluctuations, which often have to be removed from the signal using different kinds of filters. For more details and examples of filters cf. [17, 9].

SP operators on $\mathbb{R}^n$ have been studied by Bownik in [7]. He gave a characterization of these operators in terms of range operators. Our goal in this paper is to investigate SP operators on locally compact abelian (which will be abbreviated to "LCA") groups. The major result in this paper is a novel characterization of SP operators on $L^2(G)$, where G is a LCA group. This allows us to handel SP operators (specially filters) on $L^2(G)$ in a unified manner. As an application of this approach, one is able to extend several results from the theory of filters on $\mathbb{R}^n$ to a general LCA group.

2 Notations and Preliminary Results

Throughout this paper we assume that G is a locally compact abelian group. It is well known that such a group possesses a Haar measure μ that is unique up to a multiplication by constants. We refer to the usual text books about locally compact groups [13, 16]. We shall denote the measure of a measurable set A by $|A|$. Let $\hat{G}$ denote the dual group of G equipped with the compact convergence topology. The elements of $\hat{G}$ which we usually denote by ξ, are characters on G, but one can also regard elements of G as characters on $\hat{G}$. More precisely $\hat{\hat{G}} = G$ [13, Pontrjagin Duality Theorem].

Let the Fourier transform $\hat{}: L^1(G) \longrightarrow C_0(\hat{G}),\ f \longrightarrow \hat{f}$, be defined by $\hat{f}(\xi) = \int_G f(x)\overline{\xi}(x)dx$. The Fourier transform can be extended to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$ known as the Plancherel transform [13, The Plancherel Theorem].

Suppose G is a locally compact abelian group and H is a closed subgroup of G. Let G/H be the quotient group whose Haar measure is μ (which is unique up to a constant factor). If this factor is suitably chosen we have

$$\int_G f(x)dx = \int_{G/H}\int_H f(xy)dyd\mu(xH) \qquad f \in L^1(G).$$

This identity is known as Weil's formula [13].

A subgroup L of G is called a uniform lattice if it is discrete and co-compact (i.e G/L is compact). The subgroup $L^\perp = \{\xi \in \hat{G};\ \xi(L) = \{1\}\}$ is called the annihilator of L in $\hat{G}$.

Let L be a uniform lattice in G. Then the identities $L^\perp = \widehat{G/L}$ and $\hat{G}/L^\perp = \hat{L}$, together with the fact that a locally compact abelian group is compact if and only if its dual group is discrete [13], imply that the subgroup $L^{\perp}$ is a uniform lattice in $\hat{G}$ (see also [21, 27]).

We now define a shift invariant space in $L^2(G)$.

Definition 2.1 *Let G be a locally compact abelian group and L be a uniform lattice in G. A closed subspace $V \subseteq L^2(G)$ is called a shift invariant space (with respect to L) if $f \in V$ implies $T_k f \in V$, for any $k \in L$, where T_k is the translation operator defined by $T_k f(x) = f(k^{-1}x)$ for all $x \in G$. For $\varphi \in L^2(G)$, $\overline{span}\{T_k\varphi;\ k \in L\}$ is called the principle shift invariant space generated by φ and will be denoted by V_φ.*

Let $\varphi \in L^2(G)$. We denote by $L^2(\hat{L}, w_\varphi)$ the space of all functions $r : \hat{L} \longrightarrow \mathbb{C}$, which satisfy $\int_{\hat{L}} |r(\xi)|^2 w_\varphi(\xi) d\xi < \infty$, where

$$w_\varphi(\xi) = \sum_{\eta \in L^{\perp}} |\hat{\varphi}(\xi\eta)|^2. \tag{1}$$

Note that $w_\varphi \in L^1(\hat{L})$. Indeed, by Weil's formula and The Plancherel Theorem $\int_{\hat{L}} \sum_{\eta \in L^{\perp}} |\hat{\varphi}(\xi\eta)|^2 d\xi = \int_{\hat{G}} |\hat{\varphi}(\xi)|^2 d\xi = \|\varphi\|^2$. In this case $\|r\|^2_{L^2(\hat{L},w)} = \int_{\hat{L}} |r(\xi)|^2 w_\varphi(\xi) d\xi$ is a norm in $L^2(\hat{L}, w)$.

The following proposition gives a characterization of elements in a principle shift invariant subspace of $L^2(G)$ in terms of their Fourier transforms.

Proposition 2.2 *Let $\varphi \in L^2(G)$. Then $f \in V_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$, for some $r \in L^2(\hat{L}, w_\varphi)$.*

Proof: Consider $A_\varphi = span\{T_k\varphi;\ k \in L\}$, then $V_\varphi = \overline{A_\varphi}$. For $f \in A_\varphi$ let $f(x) = \sum_{i=1}^n a_i \varphi(k_i^{-1}x),\ a_i \in \mathbb{C},\ k_i \in L,\ 1 \leq i \leq n$, for some $n \in \mathbb{N}$. Then we have

$$\hat{f}(\xi) = \sum_{i=1}^{n} a_i \overline{\xi}(k_i)\hat{\varphi}(\xi) = r(\xi)\hat{\varphi}(\xi), \tag{2}$$

where $r(\xi) = \sum_{i=1}^n a_i \overline{\xi}(k_i)$. Conversely every trigonometric polynomial will give us a function $f \in A_\varphi$, via formula (2). So $f \in A_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$ where r is a trigonometric polynomial. Denote the set of all trigonometric polynomials by P. The operator $U : A_\varphi \longrightarrow P$ given by

$U(f) = r$ is an isometry which is onto. In fact by using The Plancherel Theorem and Weil's formula we have

$$\begin{aligned} \|f\|_2^2 = \|\hat{f}\|_2^2 &= \int_{\hat{G}} |\hat{f}(\xi)|^2 d\xi \\ &= \int_{\hat{L}} \sum_{\eta \in L^\perp} |r(\xi)|^2 |\hat{\varphi}(\xi\eta)|^2 d\xi \\ &= \int_{\hat{L}} w_\varphi(\xi) |r(\xi)|^2 d\xi \\ &= \|r\|^2_{L^2(\hat{L}, w_\varphi)}. \end{aligned} \tag{3}$$

Therefore there is a unique isometry $\widetilde{U} : \overline{A_\varphi} \longrightarrow \overline{P}$, which extends U from V_φ onto $\overline{P} = L^2(\hat{L}, w_\varphi)$ (Note that for a compact abelian group G the set of all trigonometric polynomials is dense in $L^2(G)$ [25]).

Note that in the case $G = \mathbb{R}$, $\mathbb{Z}$ is a uniform lattice. In this case we have the following corollary which is also proved in [26, Theorem 1.2.4].

Corollary 2.3 *Let $V\varphi$ be a principle shift invariant subspace of $L^2(\mathbb{R})$. Then $f \in V_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$, for some $r \in L^2(\mathbb{T}, w_\varphi)$, where $w_\varphi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2$.*

3 The Structure of Shift invariant Spaces

It is natural to ask if for any given principle shift invariant space V we can find a function φ in $L^2(G)$ whose shifts are orthonormal. In general the answer is not affirmative. (However, we will find another kind of generator for every principle shift invariant space; see Corollary 3.8 below). In the following theorem we state a necessary and sufficient condition for shifts of a function φ in $L^2(G)$, to be an orthonormal system. Throughout this section the notations are as in Section 2.

Proposition 3.1 *[19] Suppose that $\varphi \in L^2(G)$, then $\{T_k\varphi;\ k \in L\}$ is an orthonormal system in $L^2(G)$ if and only if*

$$w_\varphi = 1 \ a.e. \ on \ \hat{G} \tag{4}$$

If V_φ is a principle shift invariant space and w_φ is given by (1), then the set *supp* w_φ is called the spectrum of V_φ and is denoted by Ω_φ (Note that by *supp* w_φ we mean the set of all ξ such that $w_\varphi(\xi) \neq 0$. Also our convention is that all measurable sets are determined up to a null set). In the case of Proposition 3.1, Ω_φ is equal to $\hat{G}$. The following example shows the existence of principle shift invariant spaces which do not satisfy this property.

Example 3.2 *Let $G = (\mathbb{R}^2, +)$, $L = \mathbb{Z}^2$, (so $L^\perp = \mathbb{Z}^2$), $E = [0, 1/2] \times [1/2, 3/2]$, and φ be given by $\hat{\varphi} = \chi_E$. Then $w_\varphi(\xi) = \sum_{k\in\mathbb{Z}^2} \chi_E(\xi + k)$. So $\Omega_\varphi = \bigcup_{k\in\mathbb{Z}^2}(E + k) \neq \mathbb{R}^2$.*

Now we shall determine how the information about orthogonality of V_{φ_1} and V_{φ_2} can be transferred into some other information about the generators φ_1 and φ_2 in $L^2(G)$.

Proposition 3.3 *[19] The spaces V_{φ_1} and V_{φ_2} are orthogonal if and only if*

$$\sum_{\eta\in L^\perp} \hat{\varphi}_1(\xi\eta)\overline{\hat{\varphi}_2}(\xi\eta) = 0 \quad a.e.\ \xi \in \hat{G}.$$

As a consequence of Propositions 3.1 and 3.3, we have the following corollary; (see also [28]).

Corollary 3.4 *(i)Suppose $\psi \in L^2(\mathbb{R})$. Then $\{\psi(. - k);\ k \in \mathbb{Z}\}$ is an orthonormal system if and only if $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + k)|^2 = 1$, for a.e. $\xi \in \mathbb{R}$.*

(ii)For any two functions φ, $\psi \in L^2(\mathbb{R})$ the sets $\{\varphi(. - k);\ k \in \mathbb{Z}\}$ and $\{\psi(. - k);\ k \in \mathbb{Z}\}$ are biorthogonal, if and only if $\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi+k)\overline{\hat{\psi}}(\xi+k) = 0$, for a.e. $\xi \in \mathbb{R}$.

Definition 3.5 *Let $\mathcal{H}$ be a Hilbert space. A subset $X \subseteq \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist two numbers $0 < A \leq B < \infty$ so that*

$$A\|h\|^2 \leq \sum_{\eta\in X} |<h, \eta>|^2 \leq B\|h\|^2 \quad for \quad h \in \mathcal{H}. \tag{5}$$

If $A = B = 1$, X is called a Parseval frame.

Now we prove that every principle shift invariant space has generators whose shifts form a Parseval frame. The key is the following theorem.

Theorem 3.6 *[19] Let $\varphi \in L^2(G)$. The shifts of φ (with respect to L) form a Parseval frame for the space V_φ, if and only if*

$$w_\varphi = \chi_{\Omega_\varphi} \quad a.e.\ on\ \hat{G}. \tag{6}$$

Remark 3.7 *Equality (6) is obviously a more general version of equality (4) that characterizes the orthonormality of the system $\{T_k\varphi\}_{k\in L}$.*

Corollary 3.8 *[19] If V_φ is a principle shift invariant space and ψ is given by* $\hat{\psi}(\xi) = \begin{cases} \hat{\varphi}(\xi) w_\varphi(\xi)^{-1/2} & \xi \in \Omega \\ 0 & otherwise \end{cases}$, *then $\{T_k\psi, k \in L\}$ is a Parseval frame for V_φ.*

Definition 3.9 *If V_φ is a principle shift invariant space and the system $\{T_k\varphi,\ k \in L\}$ is a Parseval frame for V_φ, the function φ is called a Parseval frame generator of V_φ.*

Corollary 3.8 shows that every principle shift invariant space has a Parseval frame generator.

Now we show the existence of a decomposition of a shift invariant subspace of $L^2(G)$ into an orthogonal sum of spaces each of which is generated by a single function whose shifts form a Parseval frame.

Theorem 3.10 *[19] Let G be a locally compact abelian group and let L be a uniform lattice in G. If V is a shift invariant space in $L^2(G)$, then there exists a family of functions $\{\varphi_\alpha\}_{\alpha\in I}$ in $L^2(G)$ (where I is an index set), such that*

$$V = \bigoplus_{\alpha \in I} V_{\varphi_\alpha}, \tag{7}$$

and φ_α is a Parseval frame generator of the space V_{φ_α}. Moreover, $f \in V$ if and only if

$$\hat{f}(\xi) = \sum_{\alpha\in I} r_\alpha(\xi)\hat{\varphi_\alpha}(\xi), \tag{8}$$

and $\|f\|_2^2 = \sum_{\alpha\in I} \|r_\alpha\|^2_{L^2(\hat{L}\bigcap\Omega_{\varphi_\alpha}, w_{\varphi_\alpha})}$, where $r_\alpha \in L^2(\hat{L}\bigcap\Omega_{\varphi_\alpha}, w_{\varphi_\alpha})$ and Ω_{φ_α} is the spectrum of V_{φ_α}, for every $\alpha \in I$.

Remark 3.11 *Using the above theorem we can find a Parseval frame for every shift invariant subspace of $L^2(G)$:*

If $\{T_k\varphi_\alpha\}_{k\in L}$ is a Parseval frame for V_{φ_α}, for every $\alpha \in I$, then $\{T_k\varphi_\alpha\}_{k\in L,\alpha\in I}$ is a Parseval frame for the orthogonal sum $\bigoplus_{\alpha\in I} V_{\varphi_\alpha}$. Indeed, for every $f = \sum_{\beta\in I} P_\beta f \in \bigoplus_{\alpha\in I} V_{\varphi_\alpha}$, where P_β is the orthogonal projection onto V_{φ_β}, we have:

$$\begin{aligned}
\sum_{\alpha\in I}\sum_{k\in L} |<T_k\varphi_\alpha, f>|^2 &= \sum_{\alpha\in I}\sum_{\beta\in I}\sum_{k\in L} |<T_k\varphi_\alpha, P_\beta f>|^2 \\
&= \sum_{\alpha\in I}\sum_{k\in L} |<T_k\varphi_\alpha, P_\alpha f>|^2 \\
&= \sum_{\alpha\in I} \|P_\alpha f\|^2 \\
&= \|\sum_{\alpha\in I} P_\alpha f\|^2 \\
&= \|f\|^2.
\end{aligned}$$

4 A Hilbert Space Isomorphism

We show that $L^2(G)$ is isometrically isomorphic to the space $L^2(S_{L^\perp}, l^2(L^\perp))$ of square integrable functions from $S_{L^\perp}$ to $l^2(L^\perp)$. Notice that this space is just the direct integral $\int_A^\oplus H_\xi d\xi$, where $A = S_{L^\perp}$ and $H_\xi = l^2(L^\perp)$, for all $\xi \in S_{L^\perp}$ [13]. $L^2(S_{L^\perp}, l^2(L^\perp))$ is a Hilbert space with inner product $< f, g >= \int_{S_{L^\perp}} < f(\xi), g(\xi) >_{l^2(L^\perp)} d(\xi)$ [12, part II, Proposition 1.5].

Proposition 4.1 *[18] The mapping $\mathcal{T} : L^2(G) \longrightarrow L^2(S_{L^\perp}, l^2(L^\perp))$, defined by $\mathcal{T}f(\xi) = (\hat{f}(\xi\eta))_{\eta\in L^\perp}$ is an isometric isomorphism, between $L^2(G)$ and $L^2(S_{L^\perp}, l^2(L^\perp))$.*

Applying Proposition 4.1 to $G = \mathbb{R}^n$ and $L = \mathbb{Z}^n$, the following corollary which is [7, Proposition 1.2], is immediate.

Corollary 4.2 *[18] The mapping $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$ defined for $f \in L^2(\mathbb{R}^n)$ by $\mathcal{T}f : \mathbb{T}^n \to l^2(\mathbb{Z}^n)$, $\mathcal{T}f(x) = (\hat{f}(x + k))_{k\in\mathbb{Z}^n}$, is an isometric isomorphism between $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$.*

Consider $L^2(\hat{L}, l^2(L^\perp))$ as the direct integral $\int_A^\oplus l^2(L^\perp) d\lambda$, for $A = \hat{L}$ with its Haar measure λ. It is interesting to note that this space is also isometrically isomorphic to $L^2(G)$. To prove it we use a direct integral argument.

Proposition 4.3 *$L^2(\hat{L}, l^2(L^\perp))$ is isometrically isomorphic to $L^2(G)$.*

Proof: By [12, Part II, Proposition 1.11], we have

$$(\int_{\hat{L}}^\oplus \mathbb{C} d\lambda) \otimes l^2(L^\perp) \simeq \int_{\hat{L}}^\oplus (\mathbb{C} \otimes l^2(L^\perp)) d\lambda,$$

where $\otimes$ is the Hilbert space tensor product (see [22]). The right hand side is isometrically isomorphic to $\int_{\hat{L}}^\oplus l^2(L^\perp) d\lambda$. Therefore,

$$L^2(\hat{L}) \otimes l^2(L^\perp) \simeq L^2(\hat{L}, l^2(L^\perp)).$$

Let S_L denote a fundamental domain for L in G. We have $l^2(L^\perp) \simeq L^2(S_L)$, $L^2(\hat{L}) \simeq L^2(S_{L^\perp})$ [21, the proof of Theorem 3.1.7]. Thus,

$$L^2(S_{L^\perp}) \otimes L^2(S_L) \simeq L^2(\hat{L}, l^2(L^\perp)).$$

But $L^2(S_{L^\perp}) \otimes L^2(S_L) \simeq L^2(S_{L^\perp} \times S_L)$ [13, Theorem 7.16] (note that S_L and $S_{L^\perp}$ are of finite measure [21]), and $L^2(G) \simeq L^2(S_{L^\perp} \times S_L)$ [21, Theorem 1.3.7]. So $L^2(G) \simeq L^2(\hat{L}, l^2(L^\perp))$ (By $\simeq$ we mean "is isometrically isomorphic to").

As an immediate consequence of Propositions 4.1 and 4.3 we have:

Corollary 4.4 *Suppose G is a second countable LCA group, L is a uniform lattice in G and $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$. Then the three Hilbert spaces $L^2(G)$, $L^2(\hat{L}, l^2(L^\perp))$ and $L^2(S_{L^\perp}, l^2(L^\perp))$ are isometrically isomorphic.*

5 A Characterization of Shift-Invariant Spaces

Let G be a LCA group and L be a uniform lattice in G. A range function is a mapping

$$J : S_{L^\perp} \to \{\text{closed subspaces of } l^2(L^\perp)\}.$$

J is called measurable if the associated orthogonal projections $P(\xi) : l^2(L^\perp) \to J(\xi)$ are measurable i.e. $\xi \mapsto < P(\xi)a, b >$ is measurable for each $a, b \in l^2(L^\perp)$ (see [12]).

The main result of this section is the following characterization theorem in $L^2(G)$.

Theorem 5.1 *Suppose G is a second countable LCA group, L is a uniform lattice in G, and $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$. A closed subspace $V \subseteq L^2(G)$ is SI (with respect to the uniform lattice L) if and only if $V = \{f \in L^2(G),\ \mathcal{T}f(\xi) \in J(\xi) \ for\ a.e\ \xi \in S_{L^\perp}\}$, where J is a measurable range function and $\mathcal{T}$ is the mapping as in Proposition 4.1. The correspondence between V and J is one to one under the convention that the range functions are identified if they are equal a.e. Moreover, if $V = S(\phi)$ for some countable set $\phi \subseteq L^2(G)$ then*

$$J(\xi) = \overline{span}\{\mathcal{T}\varphi(\xi);\ \varphi \in \phi\}. \tag{9}$$

We will prove this theorem in the sequel. For this, we need some preparations. We start with a definition.

Definition 5.2 *For a given range function J, we define the space*

$$M_J = \{\varphi \in L^2(S_{L^\perp}, l^2(L^\perp)),\ \varphi(\xi) \in J(\xi)\ for\ a.e.\ \xi \in S_{L^\perp}\}. \tag{10}$$

The following proposition entails that M_J defined by (10) is a Hilbert subspace of $L^2(S_{L^\perp}, l^2(L^\perp))$.

Proposition 5.3 *[18] Let J be a range function. Then M_J is a closed subspace of $L^2(S_{L^\perp}, l^2(L^\perp))$.*

The following lemma is needed in the proof of Theorem 6.1.

Lemma 5.4 *Let J be a measurable range function with associated orthogonal projection P. Let Q denote the orthogonal projection of $L^2(S_{L^\perp}, l^2(L^\perp))$ onto M_J. Then for any $\varphi \in L^2(S_{L^\perp}, l^2(L^\perp))$,*

$$(Q\varphi)(\xi) = P(\xi)(\varphi(\xi))$$

for a.e. $\xi \in S_{L^\perp}$.

Proof of Theorem 5.1. Suppose $V = S(\phi)$ is a SI space for some countable set $\phi \subseteq L^2(G)$, $M = \mathcal{T}V$ and $J(\xi)$ is given by (9). It's enough to show that $M = M_J$. Let $\varphi \in M$. Then there exists a sequence $\{\varphi_n\}$ converging to φ such that $\mathcal{T}^{-1}\varphi_n \in span\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$. Since $\mathcal{T}T_k\varphi(\xi) = ((\widehat{T_k\varphi})(\xi\eta))_{\eta\in L^\perp} = (\hat{\varphi}(\xi\eta)\overline{\xi}(k))_{\eta\in L^\perp} = \overline{\xi}(k)\mathcal{T}\varphi(\xi)$, thus $\varphi_n(\xi) \in J(\xi)$ and so $\varphi(\xi) \in J(\xi)$. This implies that $M \subseteq M_J$.
To show that $M_J \subseteq M$, we observe that $M^\perp = \{0\}$. Take any $\psi \in L^2(S_{L^\perp}, l^2(L^\perp))$ which is orthogonal to M. For any $\varphi \in \mathcal{T}\phi$ and $k \in L$, we have $M_k\varphi \in \mathcal{T}V$, where $M_k\varphi(\xi) = \overline{\xi}(k)\varphi(\xi)$, so $0 =< M_k\varphi, \psi >= \int_{S_{L^\perp}} \overline{\xi}(k) < \varphi(\xi), \psi(\xi) >_{l^2(L^\perp)} d\xi$. Hence $< \varphi(\xi), \psi(\xi) >= 0$ for a.e. $\xi \in S_{L^\perp}$ and any $\varphi \in \mathcal{T}\phi$. Thus $\psi(\xi) \in J(\xi)^\perp$for a.e. $\xi \in S_{L^\perp}$. This implies that there is no $0 \neq \psi \in M_J$ which is orthogonal to M. Therefore $M = M_J$. Moreover we need to show that J, given by (9) is measurable. Let $P(\xi)$ be the orthogonal projection of $l^2(L^\perp)$ onto $J(\xi)$ and $\psi \in L^2(S_{L^\perp}, l^2(L^\perp))$. By [12, part II, Proposition 1.9], It is enough to show that $\xi \mapsto P(\xi)\psi(\xi)$ is measurable. Let Q denote the orthogonal projection of $L^2(S_{L^\perp}, l^2(L^\perp))$ onto M. Since the map $\xi \mapsto Q\psi(\xi)$ is measurable, by Lemma 5.4, so is $\xi \mapsto P(\xi)\psi(\xi)$. Thus J is measurable.

Conversely, if J is a measurable range function and V is given by (9) then since $V = \mathcal{T}^{-1}M_J$, obviously it is a closed shift invariant space.

Suppose $M_{J_1} = M_{J_2}$ for some measurable range functions J_1 and J_2 with associated projections P_1 and P_2, respectively. Then $J_1(\xi) = J_2(\xi)$

for a.e. $\xi \in S_{L^\perp}$. Indeed, if we apply Lemma 5.4 to the constant function $\varphi(\xi) = e_\eta$, where $(e_\eta)_{\eta\in L^\perp}$ is the standard basis of $l^2(L^\perp)$, then we have $P_1(\xi)e_\eta = P_2(\xi)e_\eta$ for all $\eta \in L^\perp$ and a.e. $\xi \in S_{L^\perp}$. Therefore $P_1(\xi) = P_2(\xi)$ for a.e. $\xi \in S_{L^\perp}$. So the correspondence between V and J is one to one. $\square$

Now suppose that G is a second countable LCA group, L is a uniform lattice in G, $S_{L^\perp}$ is a fundamental domain for $L^\perp$, V is a shift invariant subspace of $L^2(G)$ with the associated range function J, and $P(\xi)$ is the projection onto $J(\xi)$, for $\xi \in S_{L^\perp}$. A range operator on J is a mapping R from the fundamental domain $S_{L^\perp}$ to the set of bounded linear operators on closed subspaces of $l^2(L^\perp)$, so that the domain of $R(\xi)$ is equal to $J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. R is called measurable if $\xi \mapsto < R(\xi)P(\xi)a, b >$ is a measurable scalar function for all $a, b \in l^2(L^\perp)$.

Example 5.5 *For applications the most important class of LCA groups is the class of compactly generated LCA Lie groups. By the Structure Theorem for compactly generated LCA Lie groups, these groups are of the form* $\mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^r \times F$, *where* $p, q, r \in \mathbb{N}_0$ *and* F *is a finite abelian group (see [16]). Let* $G = \mathbb{R}^a \times \mathbb{Z}^b \times \mathbb{T}^c \times \mathbb{Z}_d$ *for* $a, b, c, d \in \mathbb{N}$, *where* $\mathbb{Z}_d$ *is the finite abelian group* $\{0, 1, 2, ..., d-1\}$ *of residues modulo* d. *Fix* $\alpha \in \mathbb{N}$. *Then* $\hat{G} = \mathbb{R}^a \times \mathbb{Z}^c \times \mathbb{T}^b \times \mathbb{Z}_d$ *and* $L = \mathbb{Z}^a \times \alpha\mathbb{Z}^b \times \mathbb{Z}_d$ *is a uniform lattice in* G. *Thus* $L^\perp = \mathbb{Z}^a \times \mathbb{Z}^c \times \mathbb{Z}^b_\alpha$. *Obviously* $S_{L^\perp} := \mathbb{T}^a \times \alpha\mathbb{T}^b \times \mathbb{Z}_d$ *is a fundamental domain for* $L^\perp$ *in* $\hat{G}$. *Consider the orthonormal basis* $B := B_1 \otimes B_2 \otimes B_3 \otimes B_4$ *for* $L^2(G)$, *where* $B_1 = \{M_\gamma T_k \chi_{[0,1)};\ k, \gamma \in \mathbb{Z}^a\}$, *in which* $M_\gamma T_k \chi_{[0,1)}(x) = e^{2\pi i\gamma x}\chi_{[0,1)}(x-k)$ *for* $x \in \mathbb{R}^a$, $B_2 = \{\chi_{\{m\}}; m \in \mathbb{Z}^b\}$, $B_3 = \{e^{2\pi i l};\ l \in \mathbb{Z}^c\}$, $B_4 = \mathbb{Z}_d$. *Then* $V := \bigoplus_{\varphi\in B,\gamma\in L^\perp} V_{\varphi,\gamma}$, *in which* $V_{\varphi,\gamma} = \overline{span}\{M_\gamma T_k\varphi;\ k \in L\}$, $\varphi \in B, \gamma \in L^\perp$, *is a shift invariant subspace of* $L^2(G)$. *By Theorem 5.1,* $V = \{f \in L^2(G),\ (\hat{f}(\xi\eta))_{\eta\in L^\perp} \in J(\xi)\ for\ a.e\ \xi \in S_{L^\perp}\}$, *where* $J(\xi) = \{\mathcal{T}(M_\gamma\varphi)(\xi);\ \varphi \in B,\ \gamma \in L^\perp\} = \overline{span}\{\hat{\varphi}(\gamma^{-1}\xi\eta)_{\eta\in L^\perp};\ \varphi \in B,\ \gamma \in L^\perp\}$.

6 A Characterization of Shift Preserving operators

In this section the notation will be as in the previous section. The following theorem is a characterization of SP operators in terms of range operators which is proved in [20].

Theorem 6.1 *(The Characterization Theorem) [20] Suppose $V \subseteq L^2(G)$ is a shift invariant space and J is its associated range function. For every SP operator $U : V \to L^2(G)$, there exists a measurable range operator R on J such that*

$$(\mathcal{T} \circ U)f(\xi) = R(\xi)(\mathcal{T}f(\xi)) \quad \textit{for a.e. } \xi \in S_{L^\perp}, \textit{ for all } f \in V, \tag{11}$$

where $\mathcal{T}$ is the isometric isomorphism between $L^2(G)$ and $L^2(S_{L^\perp}, l^2(L^\perp))$. Conversely, given a measurable range operator R on J with

$$\textit{ess sup}_{\xi \in S_{L^\perp}} \|R(\xi)\| < \infty,$$

there is a bounded SP operator $U : V \to L^2(G)$, such that (11) holds. The correspondence between U and R is one-to-one under the usual convention that the range operators are identified if they are equal a.e.

An immediate consequence of Theorem 6.1 is [7, Theorem 4.5] which is obtained by putting $G = \mathbb{R}^n,\ L = \mathbb{Z}^n,\ L^\perp = \mathbb{Z}^n,\ S_{L^\perp} = \mathbb{T}^n$ in Theorem 6.1.

Example 6.2 *Let G be the second countable LCA group $\mathbb{R}^n \times \mathbb{Z}^n \times \mathbb{T}^n \times Z_n$, for $n \in \mathbb{N}$, where Z_n is the finite abelian group $\{1, 2, ..., n\}$ of residues modulo n. Then $L = \mathbb{Z}^n \times \mathbb{Z}^n \times \{1\} \times Z_n$ is a uniform lattice in G and $L^\perp = \widehat{G/L} = \mathbb{Z}^n \times \{1\} \times \mathbb{Z}^n \times \{1\}$. Let π be the left regular representation of G on $L^2(G)$ and $\psi \in L^2(G)$ be admissible (see [14]). Then the continuous wavelet transform, $V_\psi : L^2(G) \to L^2(G)$, defined by $V_\psi \varphi(x) =< \varphi, \pi(x)\psi >$ is obviously a SP operator, so by Theorem 6.1 there is a range operator R such that for every $f \in L^2(G)$,*
$R(\xi)(\mathcal{T}f(\xi)) = (\mathcal{T} \circ V_\psi)f(\xi) = ((\widehat{V_\psi f}(\xi\eta))_{\eta \in L^\perp} = (\hat{f}(\xi\eta)\overline{\hat{\psi}}(\xi\eta))_{\eta \in L^\perp}$.

Example 6.3 *Define $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Uf(x) = f(x) - f(x-1)$. Obviously U is a SP operator. By Theorem 6.1 there exists a range operator R so that $R(\xi)(\mathcal{T}f(\xi)) = (\mathcal{T} \circ U)f(\xi) = (\widehat{Uf}(\xi + k))_{k \in \mathbb{Z}} = (1 + \exp(i\xi))(\hat{f}(\xi + k))_{k \in \mathbb{Z}}$, for every $f \in L^2(\mathbb{R})$.*

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