Nonlinear dynamic analysis by Dynamic Relaxation method

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Abstract. Numerical integration is an efficient approach for nonlinear dynamic analysis. In this paper, general category of the implicit integration errors will be discussed. In order to decrease the errors, Dynamic Relaxation method with modified time step (MFT) will be used. This procedure leads to an alternative algorithm which is very general and can be utilized with any implicit integration scheme. For numerical verification of the proposed technique, some single and multi degrees of freedom nonlinear dynamic systems will be analyzed. Moreover, results are compared with both exact and other available solutions. Suitable accuracy, high efficiency, simplicity, vector operations and automatic procedures are the main merits of the new algorithm in solving nonlinear dynamic problems.

Keywords: Modified Dynamic Relaxation; implicit time integration; nonlinear dynamic analysis.

1. Introduction

Dynamic loads are among the important loads which affect the structures. In this case, considerable forces are applied to the system at a very short period of time as compared to the other loadings. As a result, acceleration and velocity will be generated in the structure. Dynamic analysis is used to consider these effects. Displacement, velocity and acceleration time history are the results of such analysis. On the other hand, these quantities are the answers of system of differential equations, formulated by dynamic equilibrium in each degree of freedom. There are different schemes for formulating these systems, with particular advantages in the special classes of problems. For example, Newton's second law of motion, principle of virtual work (Clough and Penzien 1993) or Hamilton's principle (Ozkul 2004) can be noted. As a result, the structural dynamics equation is

$$[M]^{n+1}\{\ddot{D}\}^{n+1} + [C]^{n+1}\{\dot{D}\}^{n+1} + \{f(D^{n+1})\} = \{P(t^{n+1})\} \tag{1}$$

Where, $[M]^{n+1}, [C]^{n+1}, \{f(D^{n+1})\}$ and $\{P(t^{n+1})\}$ are mass matrix, damping matrix, internal and external forces vectors, respectively. Also, $\{D\}^{n+1}$ is the nodal displacement vector and super dots denotes differential with respect to time. All of these quantities are calculated at time t^{n+1}. The solution of Eq. (1) needs the following initial values,

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$$\{D(0)\} = \{D_0\}, \qquad \{\dot{D}(0)\} = \{\dot{D}_0\} \tag{2}$$

Where $\{D_0\}$ and $\{\dot{D}_0\}$ are known as displacement and velocity at $t = 0$, respectively. If mass, damping or internal force is a nonlinear function of the nodal displacement, it requires nonlinear analysis. Generally, Eq. (1) can be solved by analytical or numerical techniques. The ability of the analytical methods is limited and they are mostly applied to the linear systems using modal analysis. On the other hand, numerical schemes are widely used in linear and nonlinear dynamic analysis of structures, known as step by step time integration algorithms. Basically, there are two general classes of numerical algorithms, i.e., Implicit and Explicit. Both of these procedures are based on dividing the total time period into several time steps.

In the explicit methods, the displacement and velocity predictions are formulated only based on previous time steps information. Acceleration will also be calculated by substituting these quantities into the dynamic equilibrium relationship at the current time step. For diagonal mass matrices, explicit integration will be accomplished by vector operations. Simplicity and low computational efforts are the main advantages of these procedures. However, the probability of the numerical instability is high. To overcome to this difficulty, the time step should be selected small enough. Higher order integration can also be useful in reducing the numerical instability. Generalized weighted residual approach, SSpj scheme, β_m algorithm, Hoff-Taylor technique and Zhai's method are well known explicit procedures.

In the generalized weighted residual approach, introduced by Zienkiewicz, p order polynomial is assumed for displacement over p previous time steps (Zienkiewicz *et al.* 1984). Constant time step size, inconsistent starting values and difficulties in nonlinear problems are the main disadvantages of the Zienkiewicz's method. The SSpj algorithm based on weighted residual formulation reduces these difficulties (Wood 1984). This single step method involves three parameters which directly related to the choice of weighted functions. The GNpj method (generalized Newmark) is another time integration approach like SSpj technique. On the other hand, the β_m method defined by Katona and Zienkiewicz uses Taylor series and finite difference approximations for displacement and velocity in the derivatives of the Newmark expression (Katona and Zienkiewicz 1985). In each step of the β_m, m derivatives of displacement are calculated where m indicates the order of the method. Moreover, Hoff and Taylor presented numerical integration scheme by considering two polynomial functions for displacement and velocity (Hoff and Taylor 1990). By substituting these functions in the reduction form of dynamic equilibrium relationship (first order equation of motion), two equations are achieved for calculating the unknown parameters. This method is conditionally stable and its accuracy is related to the integration parameters. Some other explicit techniques have been introduced in (Penry and Wood 1985, Hulbert 1994, Hulbert and Chung 1996, Zhai 1996).

Implicit methods are formulated by changing the dynamic equation to the equivalent static system. Therefore, displacement, velocity and acceleration satisfy the dynamic equilibrium equation simultaneously. This behavior helps to obtain the accurate results from numerical integration and permits to consider large time step. The implicit algorithms are accomplished by matrix operations. This would increase the cost and the computational time because large scale simultaneous algebraic equations must be solved in each time step. It could be worse in nonlinear analysis when answers have to be obtained from a nonlinear system of equations. In this case, numerical integration errors will be grown up if unsuitable equation solver is used. Generally, equation solvers have important effect on the implicit methods.

Many of implicit approaches are specific case of the generalized-α method (Chung and Hulbert

1993). This scheme which was proposed by Hulbert and Chung is the generalization of the HHT-α and WBZ-α algorithms, where acceleration is considered at $t^{n+1-\alpha_m}$ and displacement, velocity and internal force are taken at $t^{n+1-\alpha_f}$. These quantities are linearly interpolated between two time stations, t^n and t^{n+1}. The generalized-α method also uses the Newmark-β difference approximation equations contain two parameters β and γ. This technique has a second order accuracy and its numerical time integration is conditionally stable. If $\alpha_f = 0$ and $\alpha_m = 0$ are considered, the Newmark-β algorithm will be obtained. Another interesting implicit approach is Wilson-θ technique. In this algorithm, linear acceleration is considered over extended time step ($\theta\Delta t$), where Δt is time increment. If dynamic equilibrium equation is considered at time $t^{n+\theta}$ (where θ controls the accuracy and stability and it is between 0 and 1), the implicit collocation methods are obtained. Higher order constant acceleration method is another implicit numerical integration which formulated by combining Newmark constant acceleration scheme with multi time step approaches (Kim *et al.* 1997). For this purpose, each time step is divided into some equal time step and it can be proved that the accuracy is twice than the number of sub increments. Although this method is unconditionally stable, its formulation is only valid for linear dynamic behavior. In nonlinear cases, an iterative process is used. On the other hand, the generalized integration approach was presented by Tamma *et al.*, in which a higher order polynomial is considered for time weighted function (Tamma *et al.* 2001). The related coefficients are calculated by minimizing weighted residual errors (Galerkin method). This method is very general so that single and multi time step approaches such as Newmark, Wilson-θ, collocation methods and other similar techniques can be obtained by variation weighted time fields. Furthermore, Modak and Sotelinio presented single step integration with nine free parameters (Modak and Sotelino 2002). The Newmark method with complex time step, defined by Fung is another implicit integration scheme (Fung 1998). In one of the latest implicit methods, Green function has been used (Soares and Mansur 2005). Finally, a new group of implicit integrations has also been presented by Bathe (2005). This method which is called composite time integration has a good efficiency in nonlinear dynamics finite elements problems.

It should be noted that many of the explicit and implicit methods could be converted to each other by variation of the integration parameters. On the other hand, combination of these procedures, called predictor-corrector scheme, is also used. This algorithm eliminates some difficulties of the implicit and explicit formulations.

In this paper, numerical errors of the implicit integrations are discussed in nonlinear dynamic analysis. After classifying the error sources, an alternative solution will be suggested to decrease the errors. For this purpose, Dynamic Relaxation method with modified fictitious time step (MFT) is combined with the implicit integration procedures. As a result, an effective algorithm will be suggested for nonlinear dynamic analysis. Moreover, numerical examples from finite element and finite difference are utilized to verify the efficiency of the proposed approach.

2. Numerical errors

The weakness of the numerical integration is its approximate solution. The result's accuracy depends on the several factors. In other words, the sources of the numerical errors are very different but generally they may be divided into the two main groups. The first group is related to the integration specifications and its accuracy and exists in all kinds of the numerical dynamic structural analysis with linear and nonlinear behaviors, known as integration errors. The second group of the errors depends on the dynamic analysis formulation and its assumptions and may happen in nonlinear analysis when implicit integration is used. This group can be nominated by unsuitable assumptions errors. In the following, the specifications of each group are discussed (Soroushian *et al.* 2005).

2.1 Integration error

In numerical integration, approximated time functions are used for displacement and velocity. The ability and accuracy of these functions play an important role in reducing the errors. Therefore, researchers try to introduce new integration procedures to increase the efficiency and accuracy of the numerical dynamic analysis approaches. Instance effect of the integration errors is appeared during every time step. Because of small time increment, this error does not have a considerable effect. However, the most important influence of the integration errors is its residual effects. In step by step techniques, numerical integration is performed based on the previous time step information. Because of approximation of the integration, initial values of the new increment will not be accurate and have some errors. By increasing analysis time, this effect will be extended so that in the final increments, large portion of errors will be related to the residual effects. To overcome these difficulties, smaller time step or higher order integration is suggested. However, using small time steps could increase the cost and the computational time drastically.

On the other hand, applying higher order derivatives in continuity conditions leads to the higher accuracy integration. Basically, there are two approaches to create continuity conditions for higher order derivatives. In the first method, displacement, velocity and acceleration at each time step are formulated based on their higher order derivatives at the beginning of the increment. Therefore, a time-polynomial function is assumed for displacement in each step. A similar idea is used in the higher order single step schemes, such as the β_m method and Hoff-Taylor formulation.

The second method to satisfy higher order derivative continuity can be based on memorized analysis. These schemes are classified as multi-time step algorithms. The well known generalized weighted residual approach is based on this idea. Some higher order numerical time integrations were also proposed by the authors that employ the accelerations of several previous time steps. It should be noted that the integration errors exist in all explicit and implicit techniques when linear or nonlinear problems are solved.

2.2 Unsuitable assumptions

In each step of the implicit formulation, dynamic equilibrium relationship is changed to the following static equivalent system

$$[S]_{EQ}^{n+1}\{D\}^{n+1} = \{P\}_{EQ}^{n+1} \tag{3}$$

In this equation, $[S]_{EQ}^{n+1}$, $\{D\}^{n+1}$ and $\{P\}_{EQ}^{n+1}$ are the equivalent secant stiffness matrix, the nodal displacement vector and equivalent load vector, at time t^{n+1}, respectively. If nonlinear dynamic effects are considered, equivalent system will be nonlinear. In other words, $[S]_{EQ}^{n+1}$ and $\{P\}_{EQ}^{n+1}$ will be functions of displacement vector at time t^{n+1}. Therefore, a nonlinear system must be solved in each time increment. Because of the large number of time steps and low efficiency of the common nonlinear solvers, the cost of the implicit nonlinear dynamic analysis will increase drastically. For this reason, some assumptions are applied to the numerical solution. Neglecting the nonlinear effects in each time step is the simplest approach. In other words, variation of the dynamic specifications during the time increment is ignored and $[S]_{EQ}^{n+1}$ and $\{P\}_{EQ}^{n+1}$ are constructed at the beginning of the step and kept constant during the time increment (Paz 1978). This approach leads to the linear system of equations; however, such assumptions increase the amount of errors. Furthermore, these errors may create some residual effects that appear in the following time steps.

Using nonlinear equation solvers is a way to control these errors. Hence, Newton-Raphson liked methods are used in common nonlinear dynamic analysis (Rao 2002, Rao *et al.* 2003, Rao 2005, Chen 2000, Kim *et al.* 2005, Qu *et al.* 2001, Lei and Qui 2000). Although these algorithms may reduce the errors, the analysis cost may be very high, because in each iteration of these schemes, a linear system of equations should be solved. Moreover, in the Newton-Raphson liked methods, inverse of the tangent stiffness is usually used. Therefore, zero or undefined stiffness matrix which is ordinary in nonlinear behaviors causes numerical instability. The arc length procedures can somedeal eliminate this difficulty (Lee *et al.* 2003). As a result, the Newton-Raphson schemes will not be suitable enough for all kinds of problems, especially in intense nonlinearities. An alternative approach to replace the Newton-Raphson technique is Dynamic Relaxation method.

3. Modified Dynamic Relaxation method

Dynamic Relaxation (DR) method is an iterative scheme which can be utilized for solving a system of simultaneous equations. This technique is based on the second order Richardson rule, developed by Frankel (1950). Physically, the DR method can also be described as steady state response of the artificial dynamic system. Therefore, mathematical and physical theories are utilized both together in DR formulation. According to the Dynamic Relaxation method, equivalent static system, Eq. (3), should be shifted to an assumed dynamic space by adding artificial inertia and damping forces as follows

$$[M]_{DR}^{k}\{A\}^{k}+[C]_{DR}^{k}\{V\}^{k}+[S]_{EQ}^{n+1,k}\{D\}^{n+1,k}=\{P\}_{EQ}^{n+1,k} \tag{4}$$

Where $\{V\}^k$ and $\{A\}^k$ are the artificial velocity and acceleration vectors and $[M]_{DR}^{k}$ and $[C]_{DR}^{k}$ are the artificial mass and damping matrices in k^{th} iteration of DR, respectively. The steady state response of this artificial dynamic system is the solution of Eq. (3), because in steady state response, inertia and damping forces will be zero. There are different approaches to derive iterative relationships of DR. In the common formulations, such as Papadrakakis scheme and Undewood procedure, mass and damping matrices are assumed to be diagonal. In addition, the explicit central finite difference integration is used. Consequently, DR iterative relations are obtained as follows (Papadrakakis 1881, Undewood 1983)

$$v_i^{k+1/2}=\frac{2m_{ii,DR}^{k}-C_{ii,DR}^{k}\tau_{DR}^{k}}{2m_{ii,DR}^{k}+C_{ii,DR}^{k}\tau_{DR}^{k}}v_i^{k-1/2}+\frac{2\tau_{DR}^{k}}{(2m_{ii,DR}^{k}+C_{ii,DR}^{k}\tau_{DR}^{k})}r_i^k,\quad i=1,2,\ldots,DOF \tag{5}$$

$$D_i^{n+1,k+1}=D_i^{n+1,k}+\tau_{DR}^{k+1}v_i^{k+1/2},\quad i=1,2,\ldots,DOF \tag{6}$$

Where, DOF is number of degrees-of-freedom and τ_{DR}^{k} specifies artificial time step. Factors

$m^k_{ii,DR}$ and $C^k_{ii,DR}$ indicate i^{th} element of diagonal mass and damping matrices, respectively. The residual force in k^{th} iteration of DR method can also be written as

$$\{R\}^k = \{P\}_{EQ}^{n+1,k} - [S]_{EQ}^{n+1,k}\{D\}^{n+1,k} \tag{7}$$

In the explicit DR technique introduced by Underwood, artificial damping matrix is assumed to be function of the artificial mass matrix as follows (Undewood 1983)

$$[C]^k_{DR} = c^k_{DR}[M]^k_{DR}, \tag{8}$$

Here, c^k_{DR} is damping factor in k^{th} iteration. Other quantities for explicit DR formulation have been proposed by other researchers (Papadrakakis 1881). Generally the Dynamic Relaxation is a conditionally stable method. Hence, artificial mass, damping factor and time step are defined in such away that stability is guaranteed and convergence rate reaches to maximum value. By using the Gerschgörin's circle theory, artificial mass matrix may be obtained as follows (Undewood 1983)

$$m^k_{ii,DR} > \frac{(\tau^k_{DR})^2}{4}\sum_{j=1}^{DOF}\left|s^k_{ij,EQ}\right| \qquad i = 1,2,\ldots,DOF \tag{9}$$

Structural dynamic theories indicate that the convergence rate to the steady state response will be maximum, if the damping is critical. Therefore, the artificial damping factor is estimated from Rayleigh's principle (Zhang and Yu 1989)

$$c^k_{DR} = 2\sqrt{\frac{(\{D\}^{n+1,k})^T[S]_{EQ}^{n+1,k}\{D\}^{n+1,k}}{(\{D\}^{n+1,k})^T[M]^k\{D\}^{n+1,k}}} \tag{10}$$

Some researchers have introduced damping factor for each node, separately (Zhang *et al.* 1994). In the most common DR algorithms, artificial time step is assumed to be equal to one. However, there are schemes for optimum and automatic selection of the time step. One of these techniques, which have been introduced by one of the authors, is based on the minimization of the residual force. For this purpose, residual force function is constructed as follows (Kadkhodayan *et al.* 2007)

$$RFF = \sum_{i=1}^{DOF}(r_i^{k+1})^2 \tag{11}$$

Where, RFF is the residual force function in $k+1^{th}$ iteration of DR method. By utilizing central finite difference approach, the residual force can be written as below

$$r_i^{k+1} = r_i^k - \tau^{k+1}\dot{f}_i^{k+1/2} \qquad i = 1,2,\ldots,DOF \tag{12}$$

Here, $\dot{f}_i^{k+1/2}$ is the internal force increment of i^{th} degree-of-freedom in mid point of the artificial time step. This quantity can be estimated as follows

$$\dot{f}_i^{k+1/2} \approx \sum_{j=1}^{DOF}s^k_{ij,EQ}\dot{V}_j^{k+1/2} \qquad i = 1,2,\ldots,DOF \tag{13}$$

If the residual force function is minimized, artificial time step will be obtained

$$\frac{\partial RFF}{\partial \tau^{k+1}} = 0 \Rightarrow \tau_{MFT}^{k+1} = \frac{\sum_{i=1}^{DOF} r_i^k \dot{f}_i^{k+1/2}}{\sum_{i=1}^{DOF} (\dot{f}_i^{k+1/2})^2} \tag{14}$$

By using second derivative test, it can be proved mathematically that the above time step minimizes the residual force function in each iteration of DR method. By using this time step, the convergence rate will increase and the analysis time will reduce (Kadkhodayan *et al.* 2007). It should be noted that the convergence of DR iterations is evaluated by the residual force and the artificial kinematics energy criteria of the system. In the following part, modified DR algorithm is presented for nonlinear dynamic analysis. These steps are iterated for each time increment of the implicit numerical integration.

(a) Start new time step of implicit numerical integration ($n = n + 1$).
(b) $k = 0$.
(c) Assume values for initial artificial velocity (null vector), initial displacement (converged displacement at the previous time step), artificial time step ($\tau^0 = 1$) and convergence criterion for residual force ($e_R = 1e-6$) and kinematics energy ($e_K = 1e-12$). Here, e_R and e_K are acceptable errors of residual force and kinematics energy of the system, respectively.
(d) Construct equivalent stiffness matrix and equivalent force vector by using implicit integration relationships.
(e) Calculate residual force vector using Eq. (7).
(f) If $\sum_{i=1}^{DOF} (r_i^k)^2 \le e_R$, go to (o), otherwise, continue.
(g) Construct artificial diagonal mass matrix using Eq. (9).
(h) Calculate artificial damping factor from Eq. (10).
(i) Update artificial velocity vector using Eq. (5).
(j) If $\sum_{i=1}^{DOF} (V_i^{k+1/2})^2 \le e_K$, go to (o), otherwise, continue.
(k) Determine internal force increment vector from Eq. (13).
(l) Calculate modified time step (τ_{MFT}^{k+1}) using Eq. (14).
(m)Update displacement vector using Eq. (6).
(n) $k = k + 1$ and return to (d).
(o) Calculate velocity and acceleration vector by using implicit integration relationships.
(p) Print results of the current time step.
(q) If dynamic analysis time is not complete, go to (a), otherwise, stop.

The above algorithm shows the main advantages of explicit DR method such as simplicity, vector operations and unique procedure for both linear and nonlinear systems. Since convergence to the solution is accomplished by the residual force (not by tangent stiffness), DR algorithm has also high efficiency and suitable accuracy in critical analyses and intense nonlinearities. It should be noted that in common nonlinear equation solvers, such as Newton-Raphson techniques, tangent stiffness is used. When the tangent stiffness approaches near zero or it is undefined, numerical instability will occur. Moreover, a linear system of equations must be solved in each iteration of the Newton-

Raphson procedures. Consequently, operations will be in matrix form and the analysis cost will be high. This effect is very considerable in step by step numerical time integrations because of large number of increments. For these reasons, in the common nonlinear dynamic analysis, the nonlinear effects along each time step are neglected and equivalent stiffness matrix and equivalent load vector are constructed at the beginning of the increment and kept constant during each time step, i.e., $[S]^n_{EQ}$ and $[P]^n_{EQ}$ are used for $n + 1^{\text{th}}$ time step. It is clear that such assumptions cause numerical errors in nonlinear dynamic analyses. For reducing these errors, the modified Dynamic Relaxation scheme is utilized which was presented in the above algorithm.

4. Numerical examples and discussion

To verify numerical efficiency of the proposed algorithm, some nonlinear dynamic systems from finite element and finite difference are analyzed. Evaluating the effect of the Dynamic Relaxation method in error reduction of nonlinear dynamic analysis is the main goal. For this purpose, implicit Newmark-β and Wilson-θ procedures are utilized. The Newmark-β integration is used for both constant acceleration (Trapezoidal rule) and linear acceleration. If Dynamic Relaxation method is combined with these procedures, three algorithms, Wilson-DR, Trapezoidal-DR and Newmark-DR, will be created. Here, Newmark-DR shows that linear acceleration method plus Dynamic Relaxation technique is used. By neglecting the nonlinear effects in each time step (ordinary nonlinear dynamic analyses), these numerical integrations will be named: Wilson-NR, Trapezoidal-NR and Newmark-NR algorithms. In order to do analysis, a computer program, using Fortran Power Station software, has been written by the authors.

In the following examples, numerical dynamic analyses are performed for long times (several times greater than period of the system). The reason for this subject is that in such great times the effect of residual errors is considerable. Therefore, one can simply and clearly compare the proposed algorithms with the previous methods.

4.1 Van Der Pol equation

For the first example, behavior of a triode oscillator discussed by Van Der Pol with the following governing equation and initial values is considered,

$$\begin{aligned} &\ddot{D} - 0.1(1 - D^2)\dot{D} + D = 0 \\ &D(0) = 2 \qquad \dot{D}(0) = 0 \end{aligned} \tag{15}$$

The exact solution of this nonlinear system is obtained by perturbation methods (Anvoner 1970) The quasi exact period of the system is 6.287 seconds. Therefore, the time step of numerical dynamic analysis is considered as 0.1 second. Fig. 1 displays the displacement-time histories for times between 54 and 60 seconds. It is clear that the proposed algorithm which uses Dynamic Relaxation method (Wilson-DR, Trapezoidal-DR and Newmark-DR), has a good agreement with the exact solution. Other methods along with initial stiffness were not able to trace the exact response of this nonlinear vibration. Hence, by combining DR with implicit integrations numerical errors decease considerably.

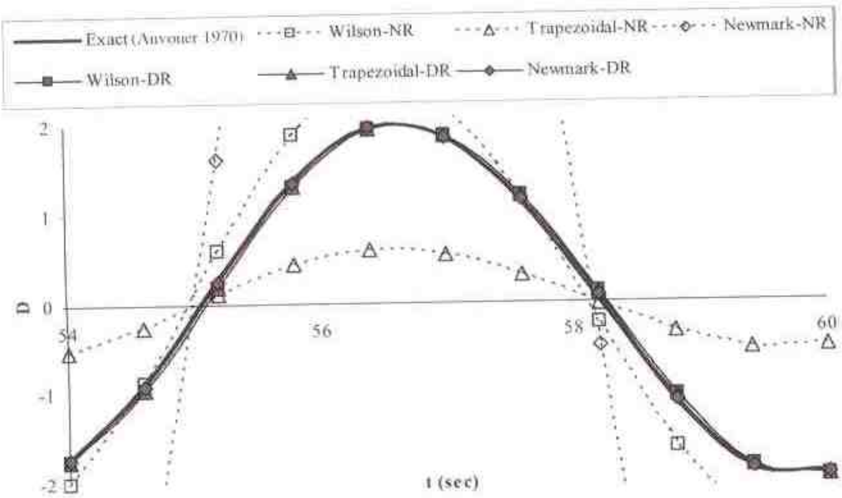

Fig. 1 Displacement-time response for Van Der Pol equation

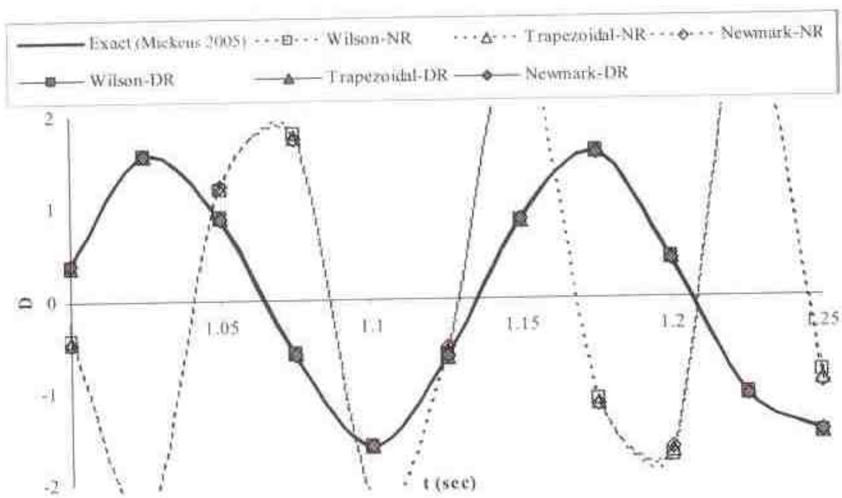

Fig. 2 Displacement-time response for nonlinear free vibration

4.2 Nonlinear free vibration

The undamped free vibration system with cube hardening is solved here. The system governing equation of motion and its initial values are given below (Zhai 1996)

$$\ddot{D} + 100D + 1000D^3 = 0$$

$$D(0) = 0.0 \qquad \dot{D}(0) = 60 \tag{16}$$

Considering the quasi exact period (0.1419 second), the time step of numerical analysis is taken as 0.00125 second. The exact solution is obtained by non-standard finite difference methods (Mickens 2005). For the time period between 1 and 1.25 seconds, the displacement-time responses of the system are plotted in Fig. 2. As seen in Fig. 2, utilizing DR gives more accurate results as compare with the other procedures so that quasi exact solution has been completely presented by Wilson-DR, Trapezoidal-DR and Newmark-DR.

4.3 Elastic-plastic free vibration

The nonlinear free vibration of a dynamic system with the following governing equation of motion and initial values is going to be solved (Hoff and Taylor 1990)

$$\begin{gathered} \ddot{D} + f(D) = 0 \\ D(0) = 0 \qquad \dot{D}(0) = 25 \end{gathered} \tag{17}$$

Here, $f(D)$ is an elastic-plastic function of the internal force, defined as follows

$$f(D) = \begin{cases} 100D & |D| \le 2.0 \\ 200 & D > 2.0 \\ -200 & D < -2.0 \end{cases} \tag{18}$$

The quasi-exact solution is obtained by using a non-standard procedure (Mickens 2005). Because the estimation of the exact period is 0.6709 second, the time step is considered as 0.0125 second for the numerical dynamic analysis. The displacement-time responses are plotted in Fig. 3 for times between 7 and 7.7 seconds. Because of long time analysis, numerical errors cause that common techniques such as Wilson-NR, Trapezoidal-NR and Newmark-NR have a phase lag with the exact solution. However, using DR method eliminates this difficulty. As a result, the proposed algorithms (Wilson-DR, Trapezoidal-DR and Newmark-DR) are completely compatible with the exact solution.

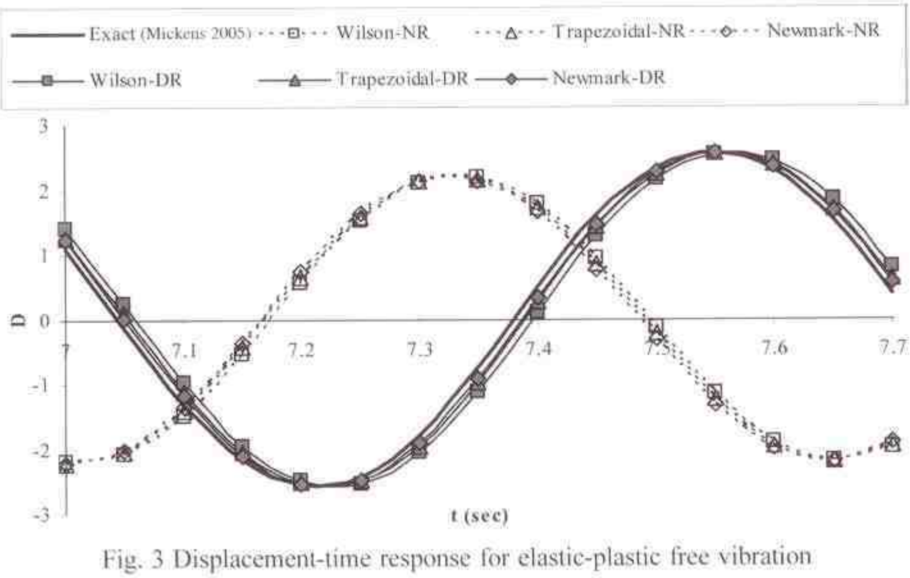

Fig. 3 Displacement-time response for elastic-plastic free vibration

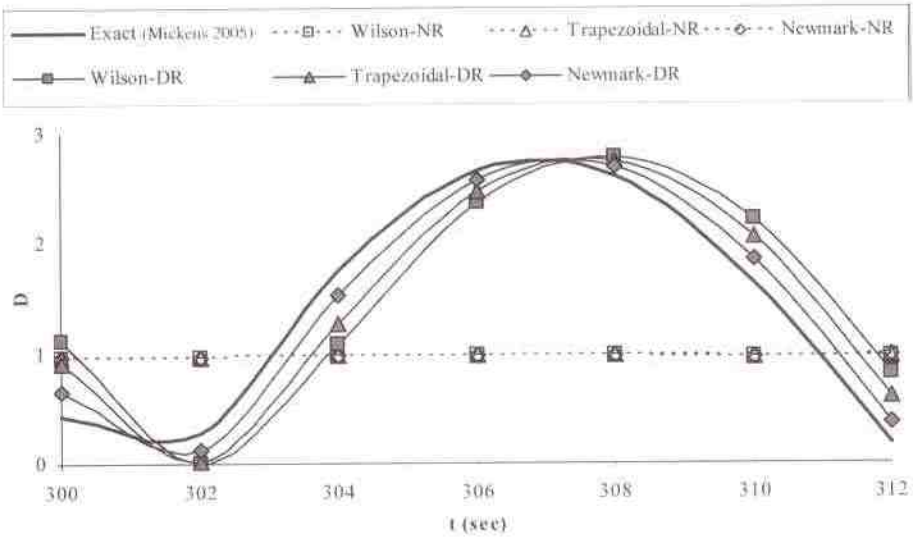

Fig. 4 Displacement-time response for undamped forced vibration

4.4 Undamped forced vibration

The undamped forced vibration of a softening spring with zero initial conditions is considered (Hoff and Taylor 1990)

$$\ddot{D}+\tanh(D) = 0.75$$
$$D(0) = 0 \qquad \dot{D}(0) = 0 \tag{19}$$

This system is nonlinear and its quasi-exact period is 11.5866 seconds. For numerical analysis, the time step is assumed to be 0. 2 second. The exact solution is also obtained by applying non-standard finite difference methods (central finite difference with a very small time step size) (Mickens 2005). Fig. 4 displays the displacement-time histories for time period between 300 and 312 seconds. Although the effect of residual errors is considerable large because of long time analysis, the proposed algorithms have also had suitable efficiency on the reduction of residual errors. Here, the residual errors prevent the system from vibration and hold it so that the displacement-time responses of ordinary methods such as Wilson-NR, Trapezoidal-NR and Newmark-NR become a horizontal line (Fig. 4). However, the modified Dynamic Relaxation method reduces these errors and general style of the exact vibration is obtained.

4.5 3-D nonlinear oscillator

A 2 kg mass is attached to three springs having stiffness of K_1, K_2 and K_3 (i.e., 15 N/m, 10 N/m and 20 N/m) and original length (L) $10\sqrt{3}$ m. This structure is pinned at three points A, B and C (Fig. 5). The system is released from rest. By using Lagrange principle (Clough and Penzien 1993), fundamental equations of motions are obtained as follows

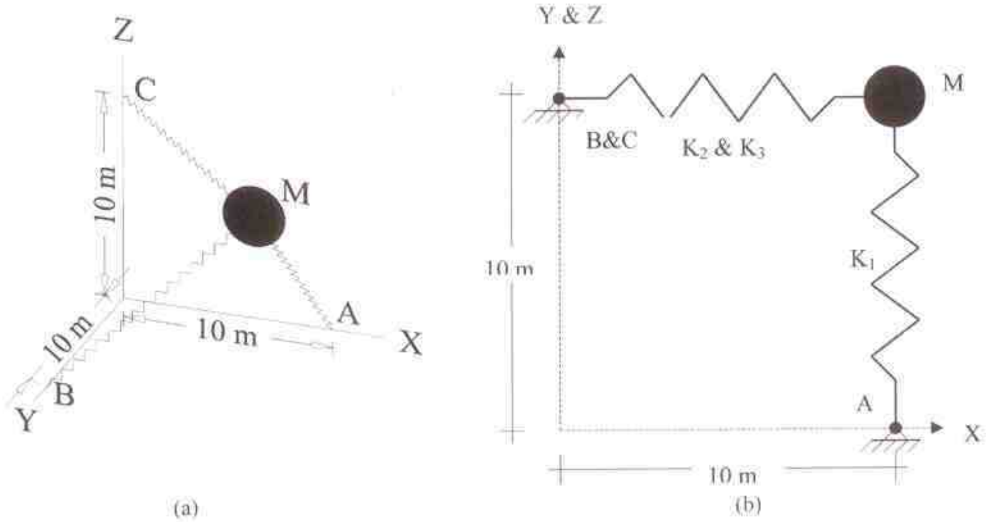

Fig. 5 Nonlinear 3-D oscillator; (a) 3-D view, (b) side view

$$
\begin{gathered}
M\ddot{x}+K_1\frac{\sqrt{x^2+(10+y)^2+(10+z)^2}-L}{\sqrt{x^2+(10+y)^2+(10+z)^2}}x+ \\
\left[K_2\frac{\sqrt{(10+x)^2+y^2+(10+z)^2}-L}{\sqrt{(10+x)^2+y^2+(10+z)^2}}+K_3\frac{\sqrt{(10+x)^2+(10+y)^2+z^2}-L}{\sqrt{(10+x)^2+(10+y)^2+z^2}}\right](10+x)=0 \\
M\ddot{y}+K_2\frac{\sqrt{(10+x)^2+y^2+(10+z)^2}-L}{\sqrt{(10+x)^2+y^2+(10+z)^2}}y+ \\
\left[K_1\frac{\sqrt{x^2+(10+y)^2+(10+z)^2}-L}{\sqrt{x^2+(10+y)^2+(10+z)^2}}+K_3\frac{\sqrt{(10+x)^2+(10+y)^2+z^2}-L}{\sqrt{(10+x)^2+(10+y)^2+z^2}}\right](10+y)=0 \\
M\ddot{z}+K_3\frac{\sqrt{(10+x)^2+(10+y)^2+z^2}-L}{\sqrt{(10+x)^2+(10+y)^2+z^2}}z+ \\
\left[K_1\frac{\sqrt{x^2+(10+y)^2+(10+z)^2}-L}{\sqrt{x^2+(10+y)^2+(10+z)^2}}+K_2\frac{\sqrt{(10+x)^2+y^2+(10+z)^2}-L}{\sqrt{(10+x)^2+y^2+(10+z)^2}}\right](10+z)\pm Mg=0
\end{gathered}
\tag{20}
$$

Where g is the gravity acceleration (g = 9.81 m/s^2). The quasi exact solution is obtained by selecting small time step (0.0005 second) in higher order implicit integrations (Bathe and Baig 2005). Figs. 6, 7 and 8 display displacement-time responses for x, y and z directions using time step 0.025 second, respectively. It is clear that the proposed algorithms (Wilson-DR, Trapezoidal-DR and Newmark-DR) converge to the quasi exact solution approximately, when time step is 0.025 second. But ordinary Wilson-approach (Wilson-NR), common Trapezoidal rule (Trapezoidal-NR) and ordinary Linear Newmark (Newmark-NR) method do not give accurate answer. In other words,

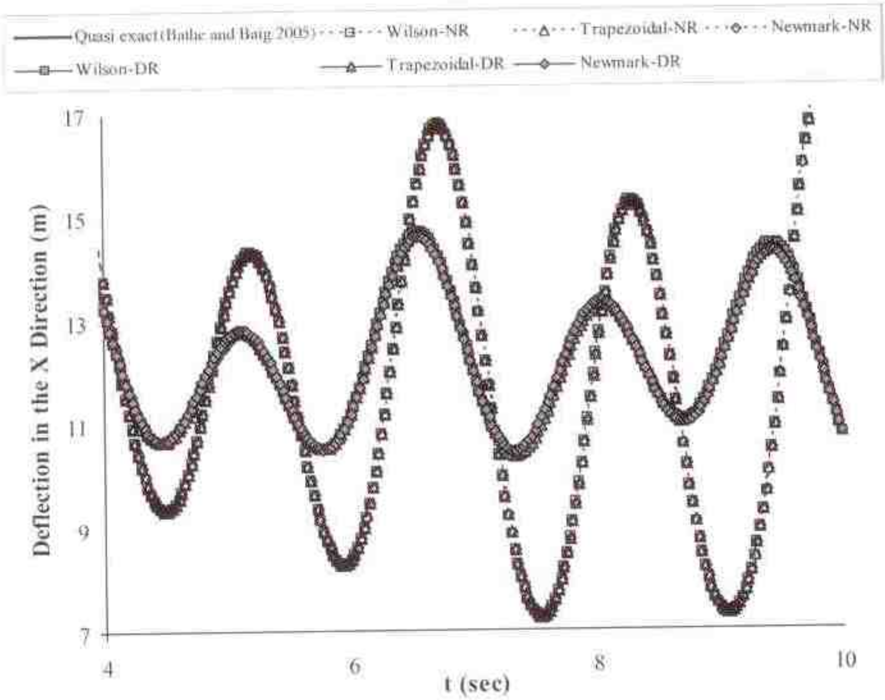

Fig. 6 Displacement-time responses of 3-D nonlinear oscillator for X direction

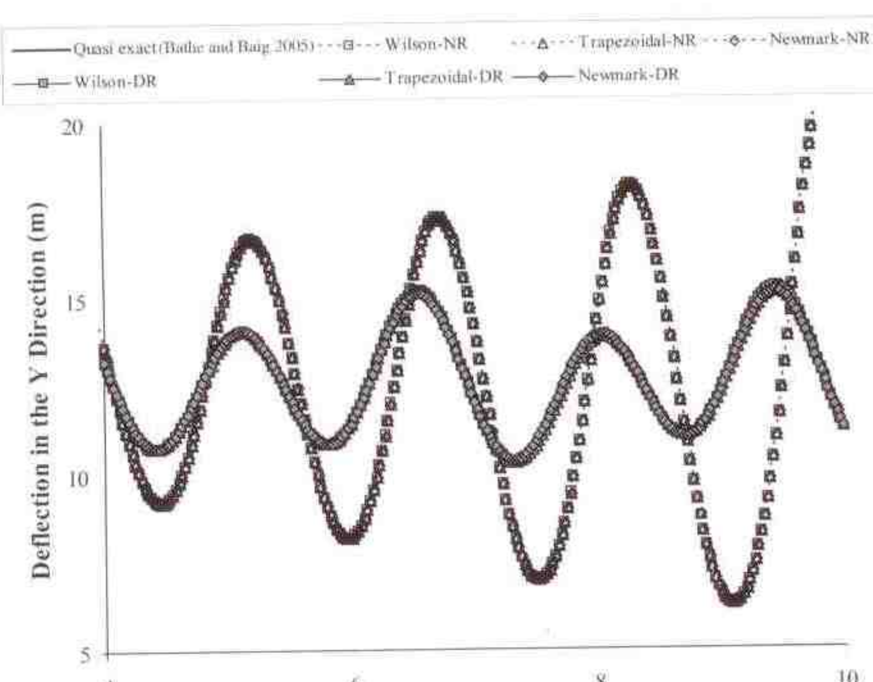

Fig. 7 Displacement-time resonses of 3-D nonlinear oscillator for Y direction

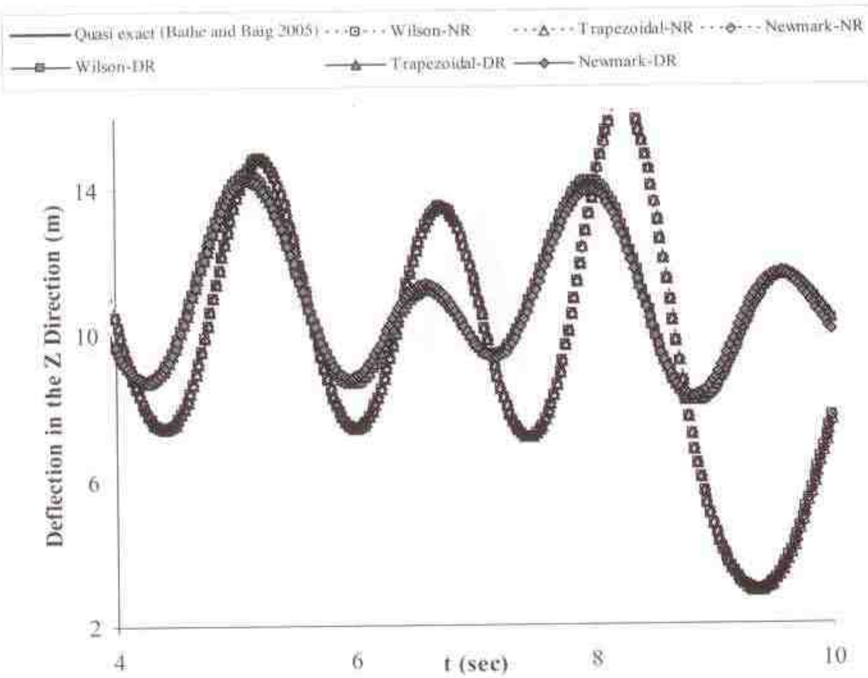

Fig. 8 Displacement-time responses of 3-D nonlinear oscillator for Z direction

numerical errors drift the results from the exact solution. To get better results, the time step should be reduced (less than 0.005 second). It is concluded that the proposed algorithms even with large time steps converge to the accurate solution. Hence, the cost and the analysis time decrease and the numerical stability is held.

4.6 Three story truss

Fig. 9 shows a three story steel space truss which is excited by horizontal impact as 30 m/sec at the top of structure (Qu *et al.* 2001). The mass density of members and the elastic modulus of the material are 7800 kg/m^3 and 2.06e11 Pa, respectively. This structure has 108 degrees of freedom and the section area of vertical members, horizontal members and braces are 3.7953×10^{-3} m^2, 1.3193×10^{-3} m^2 and 3.1903×10^{-4} m^2, respectively. Geometric nonlinear behavior is also considered for the dynamic analysis using total Lagrangian finite element formulation (Felippa 1999). The quasi exact solution of free vibration of the truss is found by implicit higher order integration along with small time step (0.00005 second) (Bathe and Baig 2005). For verification of DR capability in dynamic analysis, trapezoidal rule and linear Newmark approach are utilized. Figs. 10 and 11 display the displacement-time histories of horizontal (X) and vertical (Y) deflection of the top of the truss between 0.1 and 0.4 seconds, respectively. For more clarity, the horizontal and vertical time responses are plotted in Figs. 12 and 13 for times between 0.2 and 0.25 second, respectively. It is clear that proposed algorithms which use DR as equation solver converge to quasi exact solution even with large time step (0.001 second). On the other hand, ordinary methods which do not use DR, need smaller time step (0.0001 second). For instance, if time step is 0.001 second, Trapezoidal-NR and Newmark-NR are unstable. In this case, combination of DR method

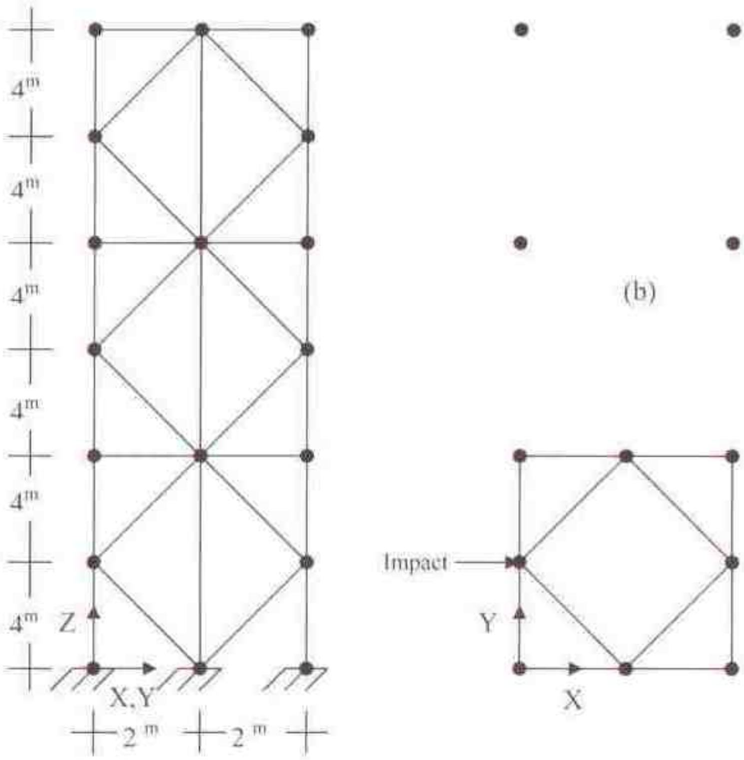

Fig. 9 Three story truss; (a) side view, (b) node coding in the elevations 4, 12 and 20 meter, (c) node coding in the elevations 8, 16 and 24 meter

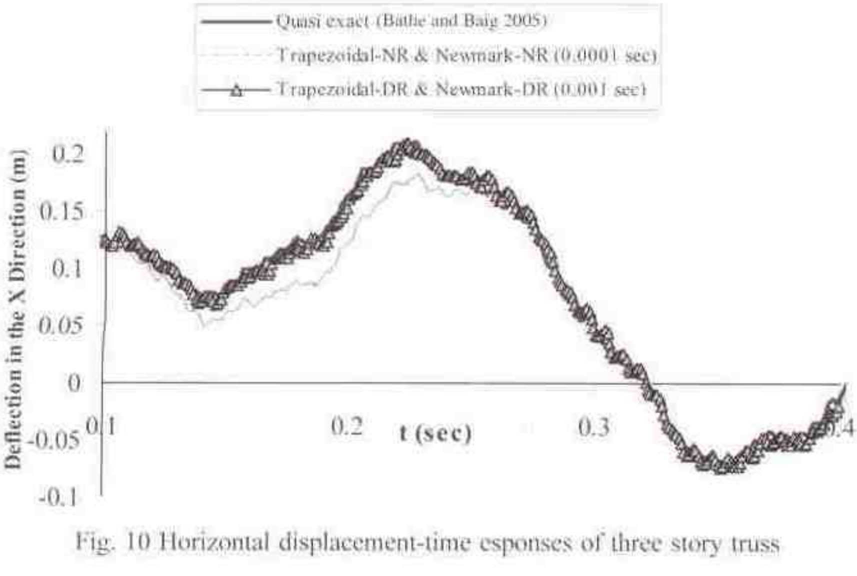

Fig. 10 Horizontal displacement-time esponses of three story truss

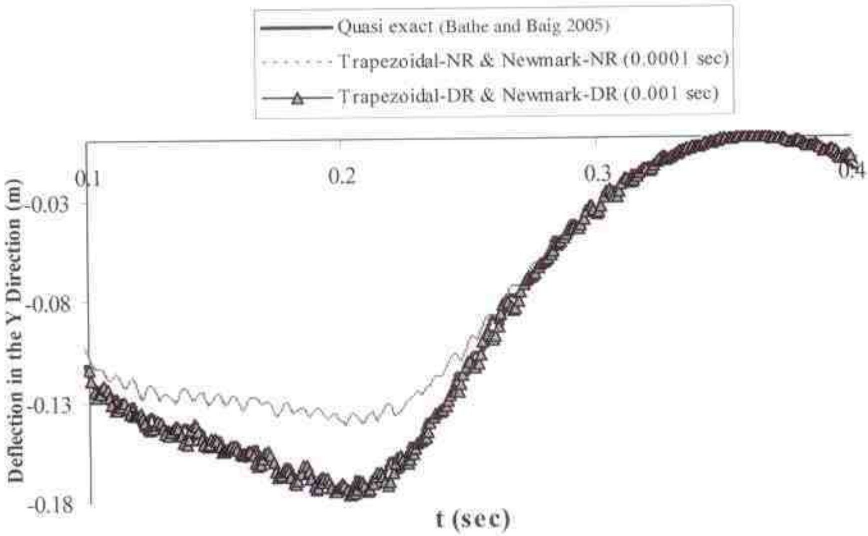

Fig. 11 Vertical displacement-time responses of three story truss

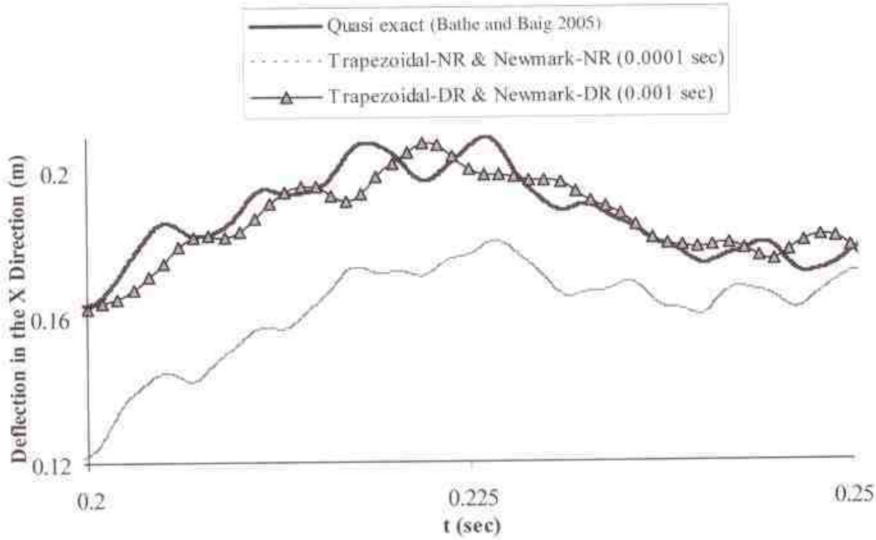

Fig. 12 Horizontal displacement-time responses of three story truss

along with these numerical time integrations (Trapezoidal-DR and Newmark-DR) reduces the errors so that results are near the exact solution. As seen in these figures, when time step is 0.0001 second, numerical errors still drift the results of common methods (Trapezoidal-NR and Newmark-NR) from the exact solution. It is concluded that when DR method is combined with implicit time integrations greater time steps can be used. Therefore, the cost and the computational time decrease. Moreover, the methods which use Newton-Raphson (NR) solver need more than ten times computer memory as compared with DR technique for guaranteeing numerical stability of time integrations.

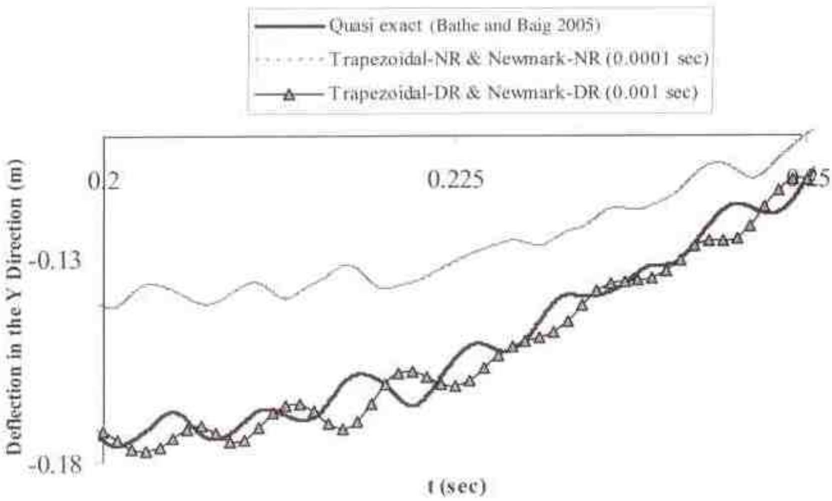

Fig. 13 Vertical displacement-time responses of three story truss

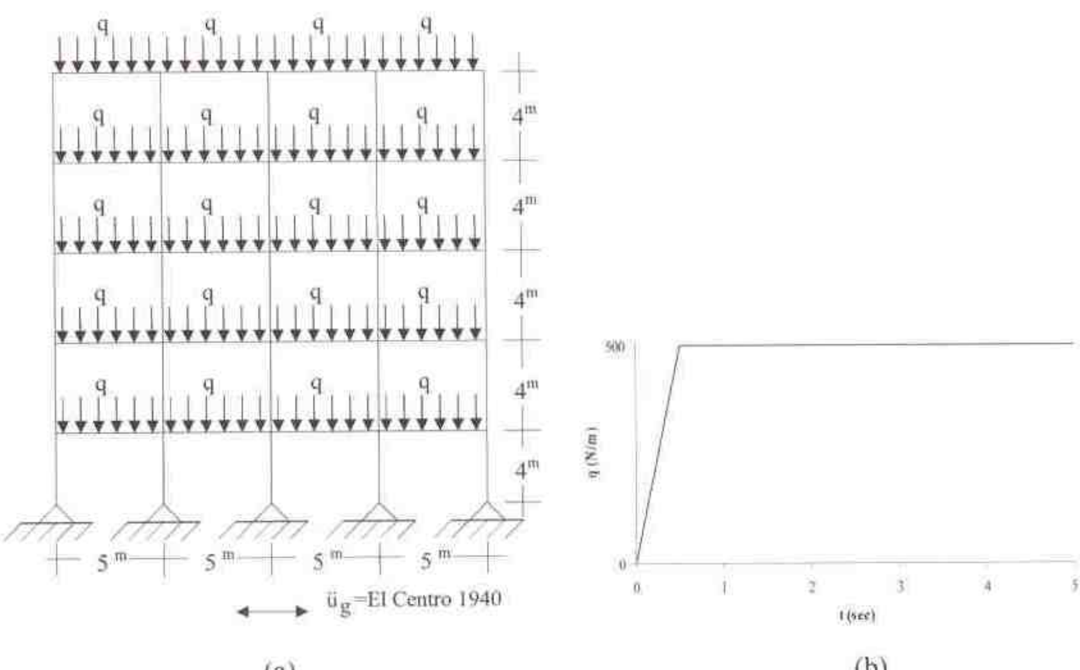

Fig. 14 Plane frame: (a) side view, (b) gravity time history for dynamic analysis

4.7 Plane frame

A steel moment resisting frame which is shown in Fig. 14(a) is analyzed with elastic geometric nonlinear behavior. In order to solve this problem, the co-rotational finite element model is used (Felippa 1999). This frame has 75 degrees of freedom and columns and beams are M10 × 9 and W6 × 9, respectively. Material mass density of 100 times the mass density of steel (i.e., $\rho = 790000$

kg/m^3) is assumed for the beams and columns to take the typical additional masses into account (i.e., slabs, floors and ceilings). Consistent mass matrix is constructed based on beams and columns mass matrices (Paz 1978). First, the uniform gravity loads are applied to the structure during 0.5 second as shown in Fig. 14(b). Then, this frame is subjected to a nonlinear response history analysis for earthquake base excitation, taken as the balanced 1940 El Centro earthquake record (Fig. 15). Wilson-θ and Linear Newmark method are used for dynamic analysis. The quasi exact solution is also obtained by implicit higher order integration, using small time step (0.00001 second) (Bathe and Baig 2005). For instance, Figs. 16 and 17 display displacement-time responses of vertical and horizontal deflection of the top of the frame (right hand side). If time step is 0.002 second, Wilson-NR and Newmark-NR are unstable. Hence, for these methods time step is reduced to 0.0001 second. However, Dynamic Relaxation technique gives the quasi exact solution when time step is taken as 0.002 second. In other words, time step for the proposed algorithms (Wilson-DR and

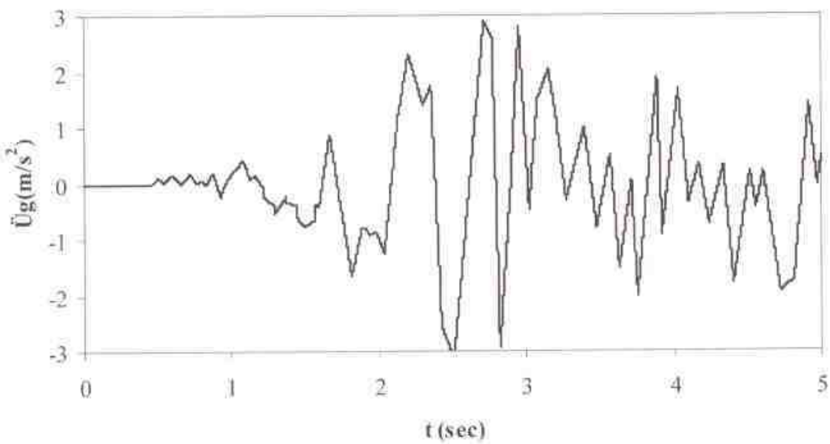

Fig. 15 Earthquake excitation history for dynamic analysis of the plane frame

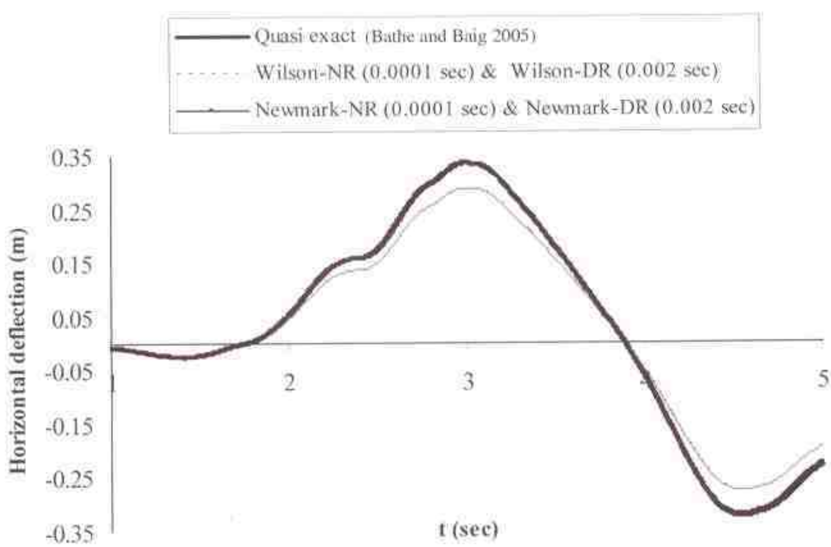

Fig. 16 Horizontal displacement-time responses for top of the frame

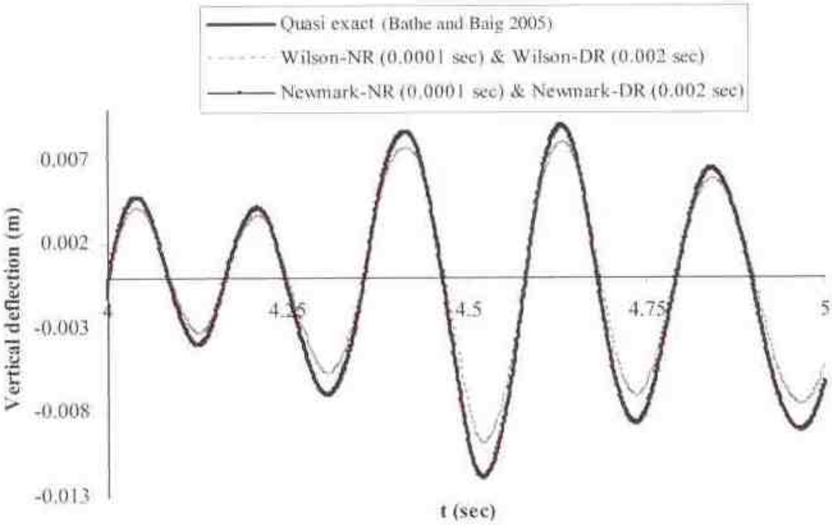

Fig. 17 Vertical displacement-time responses for top of the frame

Newmark-DR), are 20 times greater than the other methods. As a result, the computational time, the cost and the numerical instability decrease, especially in complicated and large structures like nonlinear frames. Hence, combination of DR scheme and implicit time integrations leads to an efficient procedure which reduces errors and controls instabilities in nonlinear dynamic analysis.

It is concluded from numerical examples that combining DR with implicit time integrations reduces the cost and required computer memory as compared with ordinary methods when seeking the same level of errors. In other words, more accurate results are obtained along with large time steps so that probability of numerical instabilities decreases.

Generally it can be seen from results that employing modified fictitious time step (MFT) in DR algorithm causes the average reduction in the required iterations for convergence up to 40% (Kadkhodayan *et al.* 2007). This reduction has a significant effect on the computational time of nonlinear analysis. Construction of internal force vector and tangent stiffness matrix is a complicated and time consuming procedure in nonlinear systems; hence, reduction of convergence iterations by using the MFT method decreases the analysis time significantly.

5. Conclusions

The Dynamic Relaxation method, which uses the modified time step (MFT), was combined by implicit numerical integrations such as Wilson-θ method, trapezoidal rule and Linear Newmark approach, for dynamic analysis. Single and multi degrees of freedom systems with nonlinear dynamic behavior were analyzed. By using Dynamic Relaxation method numerical errors decrease and more accurate results compared to Newton-Raphson analysis will be obtained. In fact, DR procedure controls and reduces the residual errors so that the efficiency and numerical stability of implicit numerical time integrations increases. Therefore, greater time step can be used and the cost and the computational time will decrease. These improvements are more considerable in long time analyses and intense nonlinear behaviors. Moreover, the modified Dynamic Relaxation method which is performed only by vector operations can be combined with any other implicit time integration and the proposed technique is not dependent on the time integration scheme.

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Notation

AE	: axial rigidity
$\{A\}^k$	: artificial acceleration vector in k^{th} iteration of Dynamic Relaxation algorithm
$[C]^{n+1}$	: structural damping matrix in n^{th} time step of dynamic analysis
$[C]_{DR}^k$	: artificial damping matrix in k^{th} iteration of Dynamic Relaxation algorithm
DOF	: number of degrees of freedom
$\{D\}, \{\dot{D}\}, \{\ddot{D}\}$	: vectors of displacement, velocity and acceleration, respectively
EI	: flexural rigidity
$\{f\}$	: vector of internal forces
$\{f\}$	: vector of internal force increment
$[M]^{n+1}$	: structural mass matrix in $n+1^{th}$ time step of dynamic analysis
$[M]_{DR}^k$	: artificial mass matrix in k^{th} iteration of Dynamic Relaxation algorithm
$\{P\}$	: vector of external dynamic loads
$\{P\}_{EQ}^{n+1}$	: equivalent force vector of dynamic system
RFF	: residual force function
$\{R\}$	: vector of residual force
$\{S\}_{EQ}^{n+1}$	: equivalent stiffness matrix of dynamic system
$\{V\}^k$	: artificial velocity vector in k^{th} iteration of dynamic relaxation algorithm

τ^{k}_{MFT} : modified time step in k^{th} iteration of Dynamic Relaxation algorithm

Superscripts

k : Iteration number in DR method

n : number of time increment in dynamic analysis

Subscripts

i : each degree of freedom of structure