

## General relativistic hydrodynamic flows around a static compact object in final stages of accretion flow

J Ghanbari and M Shaghaghian

Department of Physics, School of Sciences, Ferdowsi University of Mashhad, Mashhad, Iran  
E-mail: ghanbari@ferdowsi.um.ac.ir , m.shaghaghian@gmail.com

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### Abstract

Dynamics of stationary axisymmetric configuration of the viscous accreting fluids surrounding a non-rotating compact object in final stages of accretion flow is presented here. For the special case of thin disk approximation, the relativistic fluid equations ignoring self-gravity of the disk are derived in Schwarzschild geometry. For two different state equations, two sets of self-consistent analytical solutions of fully relativistic fluid equations are obtained separately. The effect of bulk viscosity coefficient on the physical functions are investigated for each state equation, as well as the bounds that exert on the free parameters due to the condition of accretion flow in the last stages. The solutions found show that the radial and azimuthal velocities, density and pressure of the fluid increase inwards for both state equations. Also, viscosity has no effect on the velocities and density distributions in both state equations. Two state equations show different types of behavior with respect to the bulk viscosity coefficient. For  $\bar{p} = K$  state equation, if there is no bulk viscosity, the pressure remains constant throughout the disk, whereas with increasing bulk viscosity the pressure falls off in the inner regions but soon stabilizes at an almost constant value. However, for  $\bar{p} = \rho c^2$  state equation, the pressure is never constant, even in the absence of bulk viscosity. The larger the value of  $\eta_b$ , the higher the value of pressure in the inner regions.

**Keywords:** accretion, accretion disks, general relativity, hydrodynamics

### 1. Introduction

Plasma processes in the vicinity of the compact objects are believed to be the main mechanisms for the generation of energy in such objects. Falling matter which releases gravitational potential energy, heats the gas and generates radiation. The process of angular momentum removal of the infalling material operates on slower timescales as compared to free-fall time, so the infalling gas with sufficiently high angular momentum can form a disk-like structure around a central gravitating body. All accretion flow models require that angular momentum is removed from the flow in some way so that material can flow inwards. If the velocity field varies significantly in directions orthogonal to the flow (shearing motions), shear viscous effects will come into play (Frank et al. 1992). Shear viscosity which is an angular momentum transport mechanism, operates on a finite time-scale  $\tau_v$ . It becomes equal or larger than the accretion time-scale  $\tau_{acc}$  as the accreted material approaches the horizon, before the final plunge (Peitz & Appl 1997). Consequently, ignoring the shear viscosity is a wise assumption in last stages of

accretion flow while most of the gas orbital angular momentum already has been removed and the fluids' radial inflow is several times faster than its rotation.

Study of the axisymmetric stationary magnetofluid configuration around compact objects in the context of high-energy astrophysics is of long-standing considerable theoretical interest in both Newtonian and relativistic limits (e.g., Kaburaki 1986; Tripathy et al. 1990; Takahashi et al. 1990; Banerjee et al. 1995, 1997; Ghosh 2000; Tomimatsu & Takahashi 2001). If the gravitational field is strong enough, as in the vicinity of a compact object, the Newtonian description of gravity is only a rough approximation and it is natural to expect the effects of space-time curvature to affect the flows considered in a general relativistic framework. In studying the relativistic disks, there are two kinds of assumptions about the self-gravity of the disks. One is to assume that self-gravity is a source of space-time curvature and refers to solution of the Einstein's field equations (Lynden-Bell & Pineault 1978; Cai & Shu 2002). The other is that the self-gravity is ignorable in comparison with the gravitation of the central compact object and it is assumed that the basic

geometry just produced by the central object is not disturbed by the self-gravity of the disk (Prasanna 1989; Takahashi et al. 1990; Kudoh & Kaburaki 1996; Gammie & McKinney 2003; Anton et al. 2006).

Basic equations governing the dynamics of an axisymmetric stationary magnetofluid disk around a compact object in the curved space-time are given by Prasanna et al. (1989). Considering viscosity, in those equations  $p$  changes to  $\bar{p} = p - \left( \eta_b - \frac{2}{3} \eta_s \right) \Theta$ , since  $\Theta = u^k{}_{;k}$  is invariant (Prasanna & Bhaskaran 1989; Bhaskaran & Prasanna 1990; Peitz & Appl 1997; Banerjee et al. 1997; Riffert 2000).

Prasanna (1989) demonstrated that an initial spherical accretion (i.e., accretion with initial zero angular momentum and radially falling at infinity) onto a rotating black hole may acquire angular velocity entirely due to the inertial frame dragging induced by the space-time surrounding the compact object. As it approaches the horizon, a thin disk structure forms on the equatorial plane of the central compact object before the final plunge. In this scenario, the pressure balance at the marginally stable orbit is provided by the equality of the radiation pressure and the hydrostatic gas pressure. Furthermore, the bulk viscosity does contribute in the dynamical equations through the nonzero radial velocity. The existence of such solutions of exact equations gives us enough motivation to study the dynamics of the accreting fluids around a non-rotating compact object in last instants of accretion flow. We take up the study of the dynamics of the disk-like structure around a static compact object in final stages of accretion flow. Once the specific angular momentum of the gas is small enough in a way that in those instants, accretion flow more nearly resembles free-fall on to the central compact object. To clarify the problem one may assume that previously an accretion disk has been formed in the gravitational field surrounding a non-rotating compact object, and now we would like to investigate the fluids' dynamical treatments in last stages of accretion flow, for two different state equations.

The paper is organized as follows:

In Section 2, governing equations are derived according to the special assumptions, the self-consistent analytical solutions for the system's physical functions for two different state equations are found in Section. 3 The conclusions and possible generalizations are presented in Section 4.

## 2. Basic equations

The general system that considered is the axisymmetric stationary viscous fluid disk in the gravitational field surrounding a static compact object. We ignore the very slow increase in the mass of the central object due to accretion as well as the self-gravity of the disk. So, the space-time curved by the central compact object is introduced by Schwarzschild geometry and then the fluids are supported entirely by the object's gravitational

field. The line element is written as

$$ds^2 = \left( 1 - \frac{2m}{r} \right) c^2 dt^2 - \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where  $m = \frac{GM}{c^2}$ , with  $G$  as the universal gravitation constant,  $M$  the mass of the central object, and  $c$  the velocity of light. In this notation,  $2m$  is the Schwarzschild radius. The energy-momentum tensor for an imperfect non-magnetofluid is given by

$$T^{ij} = \left( \rho + \frac{\bar{p}}{c^2} \right) u^i u^j - \frac{\bar{p}}{c^2} g^{ij} + \frac{2}{c^2} \eta_s \sigma^{ij},$$

with  $\bar{p} = p - \left( \eta_b - \frac{2}{3} \eta_s \right) \Theta$  and  $\Theta = u^k{}_{;k}$ . Here  $\rho$ ,  $p$ ,  $\eta_b$  and  $\eta_s$  denote the mass density, gas pressure, bulk and shear viscosity coefficients, respectively. Note that the Roman indices run from 0 to 3 and the Greek ones run from 1 to 3.

$$\sigma^{ij} = \frac{1}{2} \left( u^{i;k} h_k^j + u^{j;k} h_k^i \right) - \frac{1}{3} \Theta h^{ij},$$

is the shear tensor wherein  $h^{ij} = g^{ij} - u^i u^j$  is the projection tensor. The process of momentum transport in radial direction gives the bulk viscosity and in directions orthogonal to the flow (shearing motions) it gives the shear viscosity. The momentum transport process in each direction is important only when there is large gradient in that component of velocity. Namely, if this gradient is small enough, the related viscosity can generally be neglected (Frank et al., 1992). Accordingly, neglecting the shear viscosity is a wise assumption in last stages of accretion flow while most of the gas orbital angular momentum already has been removed. The governing dynamical equations are obtained by considering the conservation law (Prasanna, Tripathy & Das 1989),

$$T^{ij}{}_{;j} = 0.$$

We are interested in expressing the equations in the orthonormal tetrad frame appropriate to Schwarzschild metric:

$$\lambda^i{}_{(\alpha)} = \text{diag} \left[ \left( 1 - \frac{2m}{r} \right)^{-1/2}, \left( 1 - \frac{2m}{r} \right)^{1/2}, \frac{1}{r}, \frac{1}{r \sin \theta} \right].$$

Then, the spatial 3-velocity  $V^\alpha$  defined through the relation  $u^\alpha = \frac{u^0}{c} V^\alpha$ , is achieved in local Lorentz frame as

$$V(r) = \left( 1 - \frac{2m}{r} \right)^{-1} V^r,$$

$$V(\theta) = r \left( 1 - \frac{2m}{r} \right)^{-1/2} V^\theta,$$

$$V(\phi) = r \sin \theta \left(1 - \frac{2m}{r}\right)^{-1/2} V^\phi.$$

It is assumed that the disk is thin and confined to the equatorial plane of the central object ( $\Theta = \frac{\pi}{2}$ ) (Tripathy et al., 1990; Kudoh & Kaburaki 1996). Furthermore, the meridional flow is ignored ( $V^\theta = 0$ ). With these assumptions and also the conditions that the disk is assumed to be stationary ( $\partial_t \equiv 0$ ) and axisymmetric ( $\partial_\phi \equiv 0$ ), the equations are simplified and given by the continuity equation

$$\left(\rho + \frac{\bar{p}}{c^2}\right) \left[ \frac{dV(r)}{dr} + \frac{2}{r} V(r) \right] + V(r) \frac{d}{dr} \left( \rho - \frac{\bar{p}}{c^2} \right) = 0, \quad (1)$$

the radial component

$$\left(\rho + \frac{\bar{p}}{c^2}\right) (u^0)^2 \left[ V(r) \frac{dV(r)}{dr} + \frac{mc^2}{r^2} \left(1 - \frac{2m}{r}\right)^{-1} \times \left[ 1 - \frac{(V(r))^2}{c^2} \right] - \frac{(V^\phi)^2}{r} \right] + \left(1 - \frac{2m}{r}\right)^{-1} \frac{d\bar{p}}{dr} = 0, \quad (2)$$

and the azimuthal component of momentum equation

$$\frac{dV^\phi}{dr} + \frac{1}{r} \left(1 - \frac{2m}{r}\right)^{-1} \left(1 - \frac{3m}{r}\right) V^\phi = 0. \quad (3)$$

The solution of eq. (3) is

$$V^\phi = \frac{L}{r} \left(1 - \frac{2m}{r}\right)^{1/2}, \quad l = \frac{L}{cm}, \quad (4)$$

where  $L$  being an integration constant and  $l$  which behaves as a free parameter, is called the angular momentum parameter.

### 3. Considering the state equation

Substituting  $V^\phi$  (eq. (4)) in eq. (2), there remains just two equations (eqs. (1)-(2)) and three variables. Obviously, to close the system it is necessary to include a state equation. We consider two different state equations:

#### 3.1. $\bar{p} = K$

Where  $K$  is constant throughout the disk. It means that if there is no bulk viscosity, the pressure remains constant over the whole accretion flow (Prasanna 1989). Thus, the continuity and radial momentum equations (eqs. (1)-(2)) take the form

$$\bar{p} \left[ \frac{dV(r)}{dr} + \frac{2}{r} V(r) \right] + V(r) \frac{d\bar{p}}{dr} = 0, \quad (5)$$

$$\frac{d}{dr} \left( V(r) \right)^2 - \frac{2m}{r^2} \left(1 - \frac{2m}{r}\right)^{-1} \left( V(r) \right)^2 = -\frac{2m c^2}{r^2} \left(1 - \frac{2m}{r}\right)^{-1} + \frac{2L^2}{r^3} \left(1 - \frac{2m}{r}\right), \quad (6)$$

integrating them yields

$$V(r) = \sqrt{\frac{2m c^2}{r} - \frac{L^2}{r^2} \left(1 - \frac{2m}{r}\right)}, \quad (7)$$

$$\bar{\rho} = \frac{K_1}{r^2 V(r)} = \frac{K_1}{r^2 \sqrt{\frac{2m c^2}{r} - \frac{L^2}{r^2} \left(1 - \frac{2m}{r}\right)}}, \quad (8)$$

wherein  $\bar{\rho} = \rho + \frac{\bar{p}}{c^2}$ . The radial inflow velocity is assumed to be positive which its positivity indicates on the direction toward the central object. We have

$$\rho = \bar{\rho} - \frac{\bar{p}}{c^2} = \frac{K_1}{r^2 \sqrt{\frac{2m c^2}{r} - \frac{L^2}{r^2} \left(1 - \frac{2m}{r}\right)}} - \frac{K}{c^2}, \quad (9)$$

here  $K_1$  and  $K$  are constant. The generalized definition for the rate of accretion  $\dot{M}$  in relativity (Bhaskaran & Prasanna 1990) is

$$\dot{M} = \left(\rho + \frac{\bar{p}}{c^2}\right) \left(1 - \frac{2m}{r}\right) \left(1 - \frac{V^2}{c^2}\right)^{-1} r^2 V(r), \quad (10)$$

as inferred from eq. (9),  $K_1$  will be the rate of accretion, i.e.  $K_1 = \dot{M}$  and

$$K = p_R - \eta_b \Theta|_{r=R}. \quad (11)$$

Note that, it is assumed that the hydrostatic gas pressure in the inner edge is equal to the radiation pressure ( $p_R = \frac{1}{3} a T^4$  in the local thermodynamical equilibrium LTE) (Prasanna 1989). To complete the definitions, it's time to expand  $\Theta$

$$\Theta = u^k{}_{;k} = u^r{}_{,r} + \frac{2}{r} u^r = u^0{}_{,r} \frac{V^r}{c} + \frac{u^0}{c} V^r{}_{,r} + \frac{2}{r} \frac{u^0}{c} V^r,$$

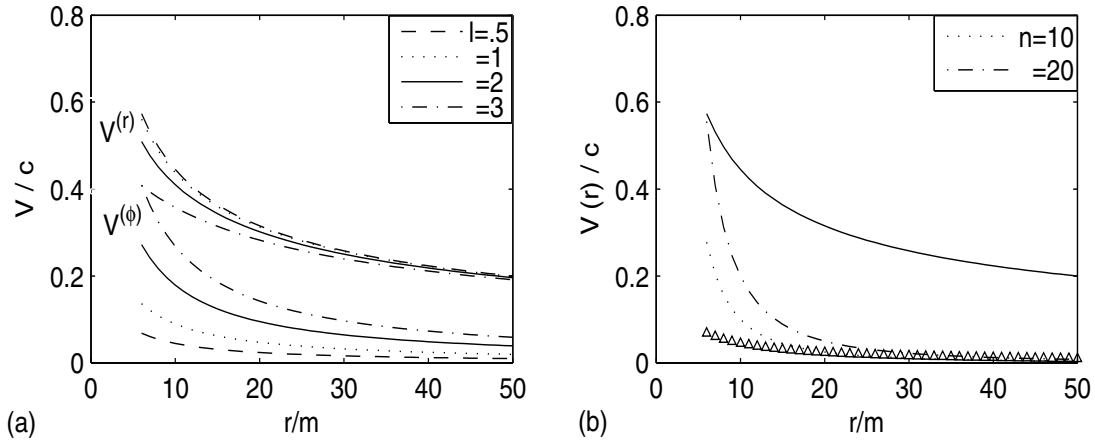
wherein

$$u^0 = \left(1 - \frac{2m}{r}\right)^{-1/2} \left(1 - \frac{V^2}{c^2}\right)^{-1/2},$$

and

$$u^0{}_{,r} = u^0 \left[ \left(1 - \frac{V^2}{c^2}\right)^{-1} \frac{V}{c^2} \frac{\partial V}{\partial r} - \frac{m}{r^2} \left(1 - \frac{2m}{r}\right)^{-1} \right],$$

and the fluid total velocity is defined as



**Figure 1.** (a) Velocities profiles of  $\bar{p} = K$  state equation for different values of  $l$  represented in legend. (b) Azimuthal velocity (' $\Delta$  sign') and Radial velocity (solid line indicates  $\bar{p} = K$  and dotted lines indicate  $\bar{p} = \rho c^2$  state equation for two values of  $n$  represented in legend) profiles in the case of  $l = 0.5$  and  $R = 6m$ .

$$V^2 = \left(V(r)\right)^2 + \left(V(\phi)\right)^2 = \frac{2mc^2}{r}.$$

Eventually,  $\Theta$  can be written as

$$\Theta = \frac{u^0}{c} \left\{ V(r) \left[ \left(1 - \frac{V^2}{c^2}\right)^{-1} \left(1 - \frac{2m}{r}\right) \frac{V}{c^2} \frac{dV}{dr} + \frac{2}{r} \left(1 - \frac{2m}{r}\right) + \frac{m}{r^2} + V(r) \left(1 - \frac{2m}{r}\right) \right] \right\}, \quad (12)$$

so the pressure is achieved

$$p = \bar{p} + \eta_b \Theta = K + \eta_b \Theta. \quad (13)$$

To determine  $K$ , it is necessary to assign the appropriate temperature to the marginally stable orbit. The observations have shown that at essentially subcritical fluxes  $\dot{M} = 10^{-12} - 10^{-10} M_{sun} / yr$ , the luminosity of the disk is of the order of  $L = 10^{34} - 10^{36} erg/s$ , and maximal surface temperatures are of the order of  $T_s = 3 \times 10^5 - 10^6 K$  in the inner regions of the disk where most of the energy is released (Shakura & Sunyaev 1973). Therefore, assignment of these numerical values for our calculations should not be an odd choice:

$M = 4 M_{sun}$ ,  $\dot{M} = 10^{-12} M_{sun} / yr$ , and  $T = 10^5 K$  in the inner edge located at  $R = 6m$ .

Close to the black hole event horizon, the gas temperature and velocities become extremely high (Popham & Gammie 1998) and gradually fall off outwards (Figure 1a). As  $l$  increases, the fluids' rotation speeds up, whereas the radial inflow slows down, consequently the total fluid velocity remains constant with respect to  $l$ . Furthermore, as Figure 1a shows

clearly,  $l$  increasing makes the slope of the azimuthal velocity profile steep and that of the radial velocity profile gentle. This means that, in the process of ascending the velocities inward, an increase in  $l$  speeds up this process for the azimuthal velocity and slows it down for the radial velocity. Ignoring the shear viscosity as an angular momentum transport mechanism necessitates the small gradient for fluids' azimuthal velocity (Frank et al. 1992). This exerts an upper bound on  $l$ , since beyond  $l = 1$  the gradient becomes significant.

### 3.2. $\bar{p} = \rho c^2$

For this state equation, the equations (1) and (2) reduce to

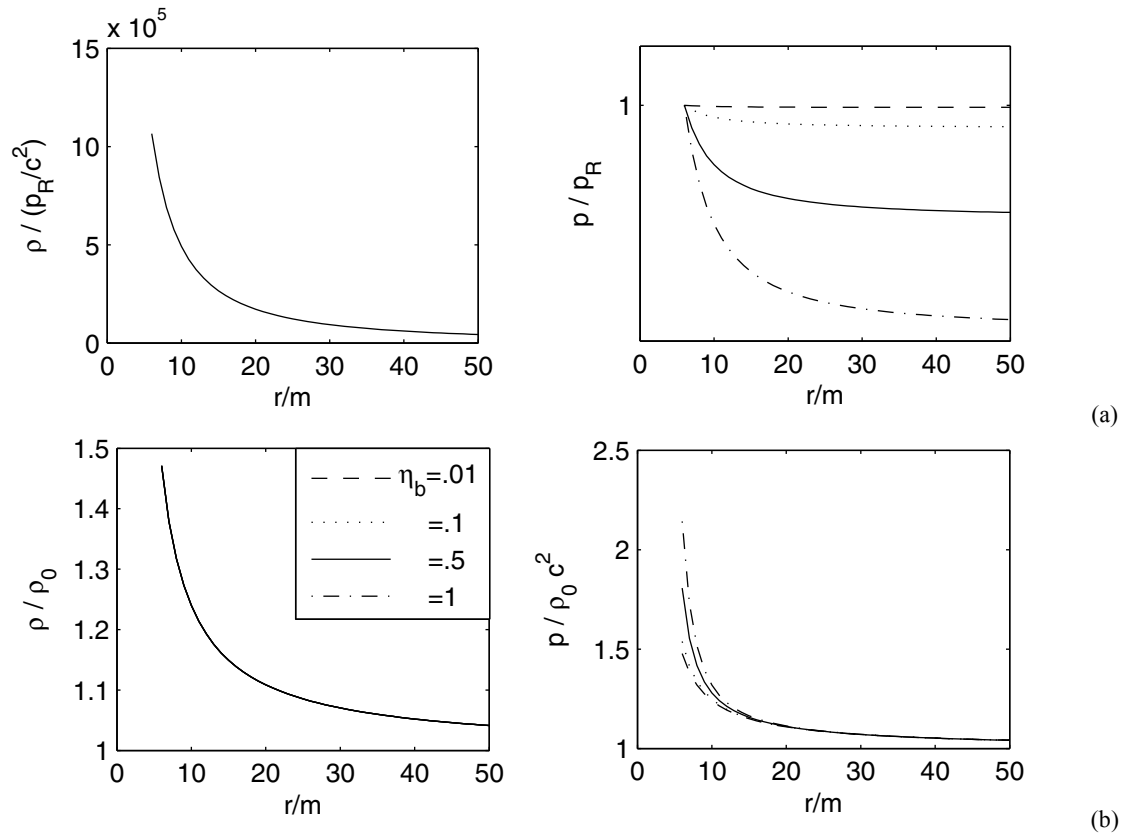
$$\frac{d}{dr} \ln V(r) = -\frac{2}{r}, \quad (14)$$

$$\frac{1}{c^2 - V^2} \left[ \frac{d}{dr} \left( V(r) \right)^2 + \frac{2mc^2}{r^2} \left(1 - \frac{2m}{r}\right)^{-1} \times \left( 1 - \frac{\left( V(r) \right)^2}{c^2} \right) - \frac{2L^2}{r^3} \left(1 - \frac{2m}{r}\right) \right] = -\frac{d \ln \rho}{dr}. \quad (15)$$

Integrating eq. (14) gives

$$V(r) = n \frac{m^2}{r^2} c,$$

wherein  $n$  is a free parameter. The radial inflow velocity (dotted lines in Figure 1b) becomes faster inward similar to the previous model (Section 3.1). However, the descending slope of its profile is so steeper that it falls off rapidly and reaches to zero whereas for  $\bar{p} = K$  model it remains constant after an initial gentle infall. The radial velocity in  $\bar{p} = \rho c^2$  model is both



**Figure 2.** - Density and pressure profiles for different values of  $\eta_b$  for a)  $\bar{p} = K$  and b)  $\bar{p} = \rho c^2$  state equation. In the case of  $R = 6m$ ,  $l = 0.5$ ,  $n = 10$  and  $\rho_0 = 10^{-23} \text{ kg / m}^3$ . The values of  $\eta_b$  are represented in legend.

independent of  $l$  and slower in comparison with the other model. Furthermore, the upper bound for  $l$  is lower. Since  $l$  exceeds 0.5, the azimuthal velocity becomes higher than the radial velocity that it disturbs the accretion flow condition in last stages. Moreover, beyond  $r = 20m$  there is neither radial inflow nor rotation for the fluids, thus the accretion region vastness for this state equation is more bounded. Substituting  $V^{(r)}$  (eq. (16)) in the radial equation (eq. (15)) and integrating yields

$$\rho = \rho_0 \frac{r^4 - m^2 l^2 r^2 + 2 m^3 l^2 r - n^2 m^4}{r^3 (r - 2m)}, \tag{17}$$

wherein  $\rho_0$  is the integration constant. Obtaining the density means identifying the pressure via

$$p = \bar{p} + \eta_b \Theta = \rho c^2 + \eta_b \Theta, \tag{18}$$

here  $\Theta$  has its previous definition (eq. (12)), however, the fluid total velocity is changed as

$$V^2 = \left( V^{(r)} \right)^2 + \left( V^{(\phi)} \right)^2 = n^2 \frac{m^4}{r^4} c^2 + \frac{L^2}{r^2} \left( 1 - \frac{2m}{r} \right).$$

$n$  increasing leads to diminishing the density, for this reason there exists an upper bound for  $n$  for which the

density and pressure are meaningful. Figure 2 shows the effect of  $\eta_b$  on the pressure and density distributions for two state equations. The density and pressure for both state equations are descending functions of  $r$ , however, slope of the density profile for  $\bar{p} = K$  state equation is steeper in the sense that it drops to zero quickly whereas for  $\bar{p} = \rho c^2$  model it tends to be constant after an initial falling. The behavior of the density and pressure distributions in  $\bar{p} = \rho c^2$  model are similar. However, for the other state unlike the density which falls off rapidly outwards, the pressure remains constant after an initial decrease. With respect to  $\eta_b$ , two state equations express similar behavior for the density and different ones for the pressure. The density and velocity distributions are not affected by the viscosity parameter. However, about the pressure distribution, in  $\bar{p} = K$  model (Figure 2a) the higher the value of  $\eta_b$ , the lower the value of pressure in the outer regions, while the pressure descent outward is appreciable only for high  $\eta_b$ . Also, if there is no bulk viscosity the pressure remains constant throughout the fluids' flow. Extending the marginally stable orbit beyond  $R = 6m$  (Figure 3) leads to an increase in the pressure.

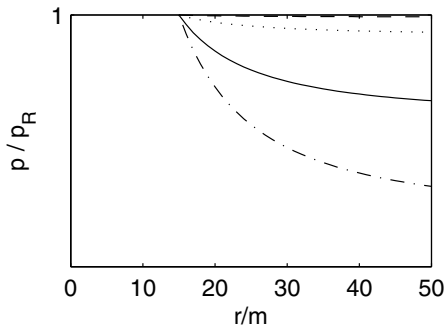


Figure 3. Same as Figure 2a but for  $R = 15 m$ .

This increase is significant only for high  $\eta_b$ . Incidentally, the descending slope of the pressure profile becomes more gentle.

But for  $\bar{p} = \rho c^2$  state equation, an increase in  $\eta_b$  results in rising the pressure in the inner regions. The pressure relation (eq. (18)) consists of two non-constant terms. Therefore, the pressure ever keeps the descending outward distribution even in the case of vanishing the bulk viscosity coefficient contrary to  $\bar{p} = K$  model. The role of viscosity term in pressure ( $\eta_b \Theta$ ) is just considerable once two terms are of the same order. Accordingly, the integration constant  $\rho_0$  is chosen to be

$\rho_0 = 10^{-23} \text{ kg/m}^3$  so that the effect of viscosity is observed on the pressure distribution. In fact, for the densities higher than this value, the importance of bulk viscosity is negligible. However for the lower densities, presence of this parameter is more noticeable.

In the allowed interval for  $l$ , the density and pressure distributions for both states to should be omitted with respect to  $l$ . Nevertheless, once  $l$  exceeds this limit, the density and pressure go up with  $l$  increasing in  $\bar{p} = K$  state equation and go down for the other state. Owing to the fact that this behavior is beyond the physical limit for  $l$ , the profiles have not been plotted.

#### 4. Conclusions

We have written the equations describing the viscous accreting fluids around a static compact object in the final stages of accretion flow. We have achieved two sets of self-consistent solutions for two different state

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equations in non-magnetofluid case. The density and pressure for both state equations ascend inward. The increasing density at the inner regions in last phases of accretion flow suggests that the inner regions could be blown up with matter moving along  $\theta$  direction on either side of the equatorial plane. Accretion flow condition in last stages demands that the fluids' radial inflow is carried out much faster than its rotation. In those instants, there is no need to any angular momentum transport mechanism, therefore one may ignore the shear viscosity. Elimination of shear viscosity coefficient  $\eta_s$  both simplifies the equations and exerts

an upper limit on the angular momentum parameter  $l$ . Because small gradient of angular velocity is incidental to vanishing the shear viscosity and since  $l$  increasing makes the slope of the azimuthal velocity steeper, there should be an upper bound on it. This upper limit for  $\bar{p} = \rho c^2$  state equation is lower than the other state. Thus  $\bar{p} = K$  model can support a higher value of azimuthal velocity against  $\bar{p} = \rho c^2$  model. The results that we have been attained for the non-magnetofluids surrounding a static black hole in the case of  $\bar{p} = K$  state equation from the viewpoint of qualitative behavior of physical functions are just the same as whatever have been achieved for fluids around a slowly rotating compact object where the fluids' angular velocity is essentially due to the dragging originated from the surrounding space-time (Prasanna 1989).

In conclusion, the solutions that we have examined here for the accreting fluids confined to the equatorial plane of the non-rotating compact object may be important in this respect: By incorporating the effects of other feasible parameters like shear viscosity, heat conduction and resistivity, these solutions can be used plus the terms including the relevant coefficients (i.e.,  $\eta_s$  shear viscosity or  $\chi$  heat conduction coefficient and  $\sigma$  conductivity).

To generalize the problem, it is interesting to put aside the thin disk approximation and also assume the non zero meridional flow. We will address this problem in the forthcoming paper.

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