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Analytical study for hypersonic flow past a triangular cone via perturbation method

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Abstract: In this paper flow's parameters are studied analytically over a conical body with triangular cross section. This cross section is derived by means of perturbing a circular cone as a basic cone. In analytic study of hypersonic flow the boundary layer is very thin and viscous effects are negligible and we assume the flow, outside viscous boundary layer, is invisid, adiabatic, and steady. Perturbation expansion is considered for flow variables and is used in ε and α terms. Substituting the expansions into the governing equations, eliminating terms of higher order than two, with respect to ε and α , and separating ε^0 , ε^1 and α^1 coefficients three system of equations are obtained. Solving ε^0 system, basic cone relations can be found. As a major parameter in design of aircrafts and space vehicles the lift to drag ratio is calculated and results show that changing cross section to a triangle in comparison with a circle increases the lift to drag ratio.

Key words: Hypersonic flow, Perturbation method, Lift-to-drag ratio, Conical body, Triangular cone.

1 Introduction

Various lifting body configurations are of current interest as a means for supplying high-performance hypersonic aircraft, missile and rocket characteristics. One way for studying the aerodynamics of these configurations is by means of conical bodies. These shapes provide suitable flow fields and aerodynamic properties. The axisymmetric supersonic flow past a circular cone is a suitable basis for constructing conical bodies. The flow past these shapes has been studied for many different cases. Perturbation method is mostly applied to study of flow on conical bodies. Stone [1,2] applied the power series expansion for a small angle of attack and provided first and second order of perturbation solution. Hypersonic flow over slender pointed nose elliptic cone at zero incidence is studied [3]. The analysis is similar to that of Doty and Rasmussen [4] and Rasmussen [5] for obtaining solutions for flow past circular cones at small angle of attack. The perturbation expansions which are used are not uniformly valid near to the body in the vortical layer that is thin. Tsai and Chou [6] with applying perturbation expansion in two inner and outer regions and corresponding them in vortical layer have obtained uniform solution on cone with longitudinal curvature.
In this paper considering the Stone's perturbation expansions and applying them for two conical bodies with different cross sections as circle and triangle at small angle of attack, the solution is obtained analytically. The purpose of the present work is to compare the lift to drag ratio for this two different cross sections analytically and numerically. So by calculating the dot product of the pressure force in z-direction and x-direction drag and lift forces are obtained respectively. The results will be useful in increasing the lift to drag ratio for aircrafts, satellites, missiles and space vehicles by changing the shape of the cross section or the angle of attack.

2 Problem Formulation

Spherical coordinate system r, θ and φ are used in this study, θ is the polar angle and φ is the azimuthal angle. In this flow, the following statement denotes the velocity vector

$$\vec{V} = u\hat{e}_r + v\hat{e}_\theta + w\hat{e}_\Phi \qquad (1)$$

As shown in Fig. 1, the triangular cone is represented

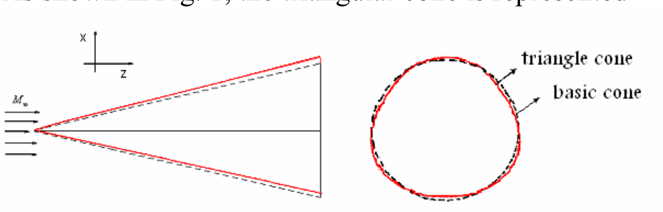

Fig.1. body produced by perturbing cone

The basic cone body is perturbed by the relation:

$$\theta_c = \delta[1 - \varepsilon \cos 3\varphi + o(\varepsilon^2)] \qquad (2)$$

Where δ is the semi-vertex angle of basic cone, ε is a small perturbation parameter and $\cos(3\varphi)$ represents the triangular shape of cross section. The shape of the corresponding shock wave is expressed in a similar way,

$$\theta_s = \delta[1-\varepsilon G\cos 3\varphi + o(\varepsilon^2)] \quad (3)$$

Where G represents the shock-perturbation factor, and $\sigma = \beta/\delta$ is the shock relation of basic conical flow.

Due to high Mach number, thin boundary layer and decrease of viscous effects, the flow field outside the viscose boundary layer is governed by mass, momentum and energy equations.

$$div(\rho\vec{V}) = 0 \quad (4)$$

$$\rho[\nabla(\frac{V^2}{2}) - \vec{V}\times curl\vec{V}] = -\nabla p \quad (5)$$

$$\vec{V}.\nabla s = 0 \quad (6)$$

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \frac{u^2+v^2+w^2}{2} = cons. \quad (7)$$

$$\frac{s-s_r}{C_v} = Ln(\frac{p}{p_r}) - \gamma Ln(\frac{\rho}{\rho_r}) \quad (8)$$

Where p is static pressure, ρ density of the fluid and s is the entropy.

It can be used of superposition effect for writing the perturbation expansion for flow variables [4, 5].

Substituting the perturbation expansions in the governing equations and separating zero-order and first-order terms in ε and α, three systems of equations are obtained. Zeroth-order functions u_0, v_0, p_0, ρ_0, and s_0 are the solutions of basic cone problem. In this investigation, the zeroth-order solution from [7] is opted as the basic solution.

The cone surface $\theta = \delta$ boundary condition can be yielded by the tangency of inviscid flow across the cone surface as

$$\vec{V}.\hat{n}_C = 0 \quad (9)$$

$$v_0(\delta) = 0 \text{ and } v_{mn}(\delta) = -mV_\infty\delta \quad (10)$$

Where n_C is normal unit vector from the cone surface. With equations of mass conservation and conservation of tangential velocity, two velocity components can be calculated at the shock, $\theta = \beta$.

$$\rho_\infty V_\infty.\hat{n}_s = \rho V_s.\hat{n}_s \quad (11)$$

$$V_\infty \times \hat{n}_s = V_s \times \hat{n}_s \quad (12)$$

$$u_1(\beta) = \delta\sin\beta\left(1 - G(1-\xi_0)\right) \quad (13)$$

$$v_1(\beta) = \delta G\cos\beta\left(-\xi_0 + 2\frac{\gamma-1}{\gamma+1}\right) + \delta G v_0'(\beta) \quad (14)$$

The second-order perturbation expansion of α system yields:

$$2(\rho_0 u_2 + u_0 \rho_2) + (\rho_2 v_0 + v_2 \rho_0)' + \cot\theta(\rho_0 v_2 + v_0 \rho_2) + \frac{\rho_0 w_2}{\sin\theta} = 0 \tag{16}$$

$$v_n u_0' + v_0 u_n' - 2 v_0 v_n = 0 \tag{17}$$

$$\rho_0 (v_0 v_2)' + \rho_2 v_0 v_0' + v_0(\rho_2 u_0 + u_2 \rho_0) + \rho_0 u_0 v_2 + p_2' = 0 \tag{18}$$

$$w_2' + \frac{u_0}{v_0} w_2 + w_2 \cot\theta - \frac{1}{\sin\theta} \frac{p_2}{\rho_0 v_0} = 0 \tag{19}$$

$$v_0 s_2' + v_2 s_0' = 0 \tag{20}$$

$$s_n = \frac{p_2}{p_0} - \gamma \frac{\rho_2}{\rho_0} \tag{21}$$

$$\frac{1}{2}(u_0^2 + v_0^2) + (u_2 u_0 + v_2 v_0)\frac{\rho_0}{\rho_2} + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} - \frac{1}{2} - \frac{1}{(\gamma - 1) M_\infty^2} = 0 \tag{22}$$

To find pressure coefficient it can be shown that

$$C_p = C_{p_0} + \varepsilon C_{p1} \cos 3\varphi + \alpha C_{p2} \cos\varphi \tag{23}$$

And it can be calculated that

$$\frac{c_{p_0}}{\delta^2} = 1 + \frac{\sigma^2 \ln\sigma^2}{\sigma^2 - 1} \tag{24}$$

$$\frac{c_{p2}}{\delta} = \frac{2N}{k_\delta^2}\left(1 + \frac{\gamma}{2} k_\delta^2 \left(1 + \frac{\sigma^2 \ln\sigma^2}{\sigma^2 - 1}\right)\right) \times \left(\frac{1 - g_1}{\sigma^3} - \frac{a_0^2(\beta)}{a_0^2(\delta)} \frac{u_1(\delta)}{V_\infty \delta}\right) \tag{25}$$

where (26)

$$\frac{u_1(\delta)}{V_\infty \delta} = -2 + \frac{(1 - g_1)}{\sigma^3}\left[\frac{4\sigma^2}{\gamma + 1} + \sigma^3 - \frac{\sigma^2}{2} + 1 - \frac{\ln\left(\sigma + \sqrt{\sigma^2 - 1}\right)}{2\sqrt{\sigma^2 - 1}}\right]$$

$$\frac{a_0^2(\beta)}{a_0^2(\delta)} = 1 + \frac{(\gamma - 1)\sigma^2\left[\ln\sigma^2 + \frac{1}{\sigma^2} - 1\right]}{(\sigma^2 - 1)(2\sigma^2 + \gamma - 1)} \tag{27}$$

and (28)

$$N = \frac{2\sigma^2}{(\sigma^2 - 1)(2\sigma^2 + \gamma - 1)}$$

2.1 Calculating lift and drag forces

$$\vec{F} = -\iint_s p(\theta_c)\hat{n}dS \tag{29}$$

$$\hat{n} = \hat{e}_\theta - \frac{3\varepsilon \sin 3\varphi}{\sin\delta}\hat{e}_\varphi + o(\varepsilon^2) \tag{30}$$

$$dS = R\,dr\,d\varphi \tag{31}$$

The lift and drag in differential form is written

$$dN = -(p - p_\infty)\tan\theta_C \cos\varphi\ xdx.d\varphi \tag{32}$$

$$dD = -(p - p_\infty)\tan^2\theta_C\, xdx.d\varphi \tag{33}$$

$$C_p = \frac{p - p_\infty}{1/2\rho_\infty V_\infty^2} \tag{34}$$

It can be calculated as

$$\frac{L}{D} = \frac{\alpha C_{p2}}{2C_{p_0}\tan\delta - \varepsilon^2 C_{p2}(2 + \tan^2\delta)} \tag{35}$$

3 Problem Solution

Analytical calculations were performed on the hypersonic flow for the triangular cone as shown in Fig. 1. C_{p0}/δ^2 and $-C_{p2}/\delta$ fare shown in Figs. 2 and 3 as the following. Fig. 4 represents the L/D increases with changing cone's cross section from a circle (ε=0) to a triangle (ε=0.15). In this figure with increasing of k_δ up to 1 lift-to-drag ratio increases and then it decreases. Fig. 5 shows lift-to-drag ratio for different substances, Ar, He and air. it can be seen with increasing of specific heat ratio of substances, lift-to-drag increases. Also the figure reveals difference between He and air is larger than difference between Ar and He.

Fig. 6 shows that with increasing of attack angle, the lift-to-drag ratio increases, because in Eq.35 there is direct relation between L/D and attack angle.

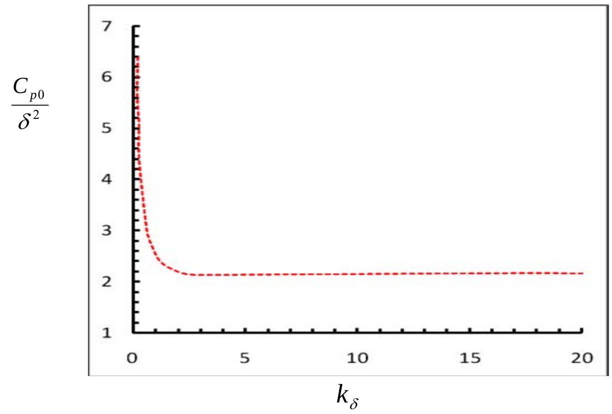

Fig. 2. Cp_0/δ^2 versus k_δ.

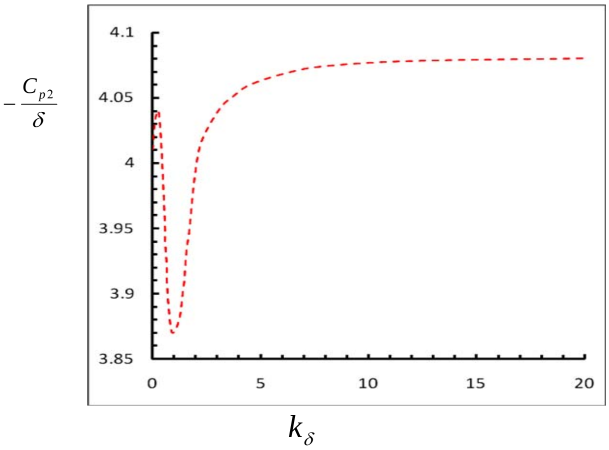

Fig.3. – Cp_2/δ versus k_δ

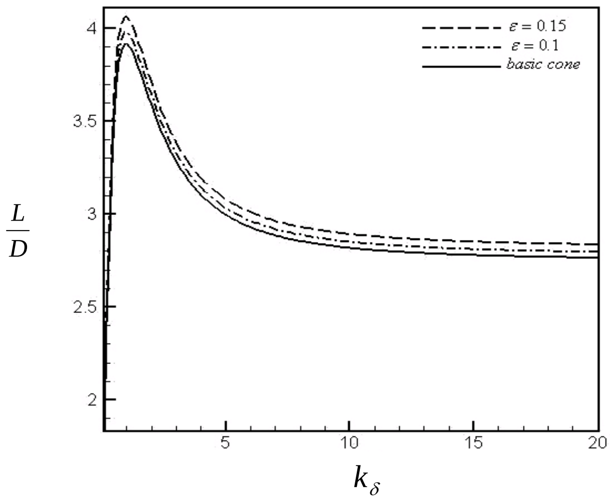

Fig. 4 . Effect of ε on L/D vs. k_δ

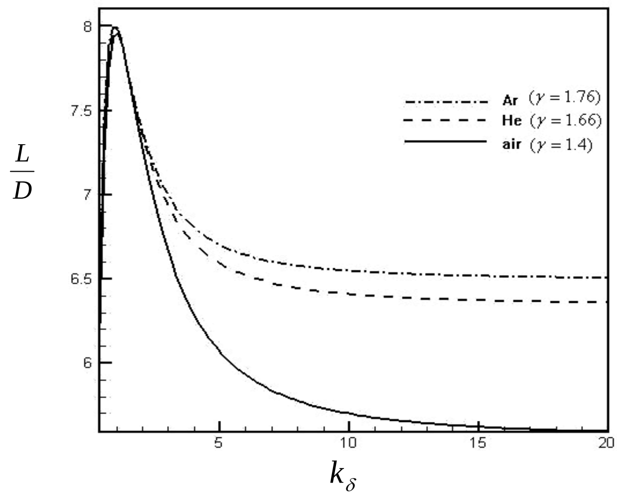

Fig. 5. Effect of specific heat ratio on L/D vs. k_δ

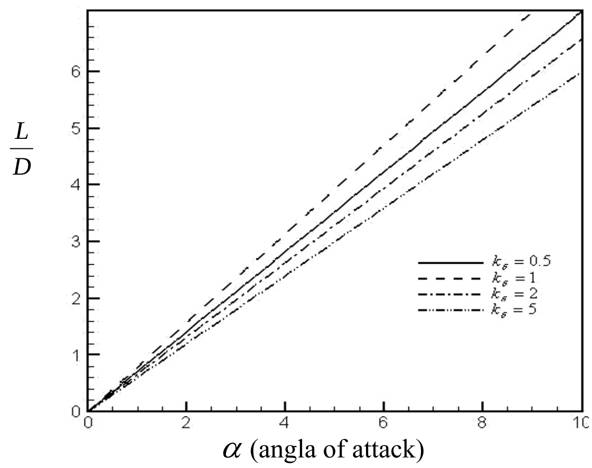

Fig. 6. Effect of k_δ on L/D vs. α in analytical solution

4 Conclusion

The perturbation method was applied to obtain analytically flow variables over conical bodies of two different cross sections, circle, and triangle. The aim of the present work is to improve lift to drag ratio by changing the cross section of the conical body. Using Fourier series a relation between δ and the shape of the cross section of the body is obtained for each case. Theses relations show that by changing the cross section from a circle to triangle the lift to drag ratio increases. L/D will also increment if the angle of attack increases.

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