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NON-ARCHIMEDEAN STABILITY OF QUADRATIC EQUATIONS

ALIREZA KAMEL MIRMOSTAFAEE

Department of Pure Mathematics
Centre of Excellence in Analysis on Algebraic Structures
Ferdowsi University of Mashhad
P.O. Box 1159, Mashhad, Iran
E-mail: mirmostafaei@ferdowsi.um.ac.ir

Abstract. We use the fixed point method to study the Hyers-Ulam-Rassias stability of the quadratic equation in non-Archimedean normed spaces. Some applications of our result will be illustrated. We will also give an example to show that some results in stability of quadratic mappings in real normed spaces are not valid in non-Archimedean normed spaces.
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1. Introduction

In 1940, S.M. Ulam [38] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:
Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta_\varepsilon > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality
$$d(h(x * y), h(x) \diamond h(y)) < \delta_\varepsilon \quad (x, y \in G_1),$$
then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?
The first partial solution to Ulam's question was provided by D.H. Hyers [19] for Banach spaces. The theorem of Hyers was generalized by T. Aoki [1] and D.G. Bourgin [3]. Th.M. Rassias [35] employed Hyers' ideas to linear mappings. In 1994, a generalization of Rassias' theorem was obtained by Găvruta [17] by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. On the other hand, Forti [15] proved a general stability result for a large class of functional equations of the form $f(G(x, y)) = H(f(x), f(y))$. Several stability results have been recently obtained for various equations, also for mappings with more general domains and ranges (see [12, 16, 20, 21]).

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The concept of the Hyers-Ulam-Rassias stability was originated from Th.M. Rassias paper [35] for the stability of the linear mappings and its importance in the proof of further results in functional equations.

The first stability theorem for the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

was proved F. Skof [37] for a mapping from a normed space X into a Banach space Y satisfying the inequality $\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leq \epsilon$ for some $\epsilon > 0$. P.W. Cholewa [9] extended Skof's theorem by replacing X by an abelian group G. Skof's result was later generalized by S. Czerwik [10] in the spirit of Hyers-Ulam-Rassias. He also proved the stability of quadratic equation of Pexider type [11]. In 2001, G.-H Kim [28] proved the stability of the quadratic functional equation by considering a general control function $\varphi(x,y)$ with suitable conditions. Recently, the stability problem of the quadratic equation has been investigated by a number of mathematicians, see [22, 23, 24, 29, 30] and references therein.

In 2003, V. Radu [33], noticed that a fixed point alternative method is very important for the solution of the Hyers-Ulam stability problem. Subsequently, this method was applied to investigate the generalized Ulam-Hyers stability for Jensen functional equation [5], as well as for the additive Cauchy functional equation [4], by considering a general control function $\varphi(x,y)$, with suitable properties. Using such an elegant idea, several authors applied the method to investigate the stability of some functional equations, see [6, 7, 8, 30, 32, 33, 34].

L. M. Arriola and W. A. Beyer in [2] initiated the stability of the Cauchy functional equation over p-adic fields. In [36], the authors investigated stability of some functional equations in non-Archimedean normed spaces. Z. Kaiser [25] proved the stability of monomial functional equation in normed spaces over files with valuation. In this paper, by using the fixed point method, we give a new approach to the stability of quadratic functional equations in non-Archimedean normed spaces.

In section 2, we introduce non-Archimedean generalized metric space and non-Archimedean version of the alternative fixed point Theorem. Later, in section 3, we use this Theorem to prove stability of the quadratic equation in non-Archimedean normed spaces. We give a counter example in section 4 to show that the exact form of some results in real normed spaces may fail in some non-Archimedean normed spaces. Then we give non-Archimedean versions of Czerwik's Theorem for non-Archimedean normed spaces.

2. Preliminaries

In this section we introduce some notions which we be used in the sequel.

In 1897, Hensel [18] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [13, 26, 27, 31].

Definition 2.1. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\ |:\mathbb{K}\to\mathbb{R}$ such that for any $a,b\in\mathbb{K}$ we have

(i) $|a|\geq 0$ and equality holds if and only if $a=0$,

(ii) $|ab| = |a||b|$,
(iii) $|a + b| \leq \max\{|a|, |b|\}$.

The condition (iii) is called the strong triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume in addition that $|\ |$ is non trivial, i.e.,

(iv) there is an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

Definition 2.2. Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\ .\ |$. A function $||\ .\ || : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely,

$$||x + y|| \leq \max\{||x||, ||y||\} \quad (x, y \in X).$$

Then $(X, ||\ .\ ||)$ is called a non-Archimedean normed space.
By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Remark 2.3. Thanks to the inequality

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n - 1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all x and $y > 0$, there exists an integer n such that $x < ny$.

Example 2.4. Let p be a prime number. For any nonzero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p = p^{-r}$. Then $|\ |$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\ |$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.
Note that if $p \geq 3$, then $|2^n| = 1$ in for each integer n.

Definition 2.5. Let X be a nonempty set and $d : X \times X \to [0, \infty]$ satisfy the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$(symmetry),
(iii) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$(strong triangle inequality),

for all $x, y, z \in X$. Then (X, d) is called a non-Archimedean generalized metric space. (X, d) is called complete if every d-Cauchy sequence in X is d-convergent.

Example 2.6. For each nonempty set X, define

$$d(x, x') = \begin{cases} 0 & \text{if } x = x' \\ \infty & \text{if } x \neq x' \end{cases}$$

Then d is a generalized non-Archimedean metric on X.

Example 2.7. Let X and Y be two non-Archimedean spaces over a non-Archimedean field $\mathbb{K}$. If Y has a complete non-Archimedean norm over $\mathbb{K}$ and $\psi : X \to [0, \infty)$, for each $f, g : X \to Y$, define

$$d(f, g) = \inf\{\alpha > 0 : ||f(x) - g(x)|| \leq \alpha\psi(x) \ \forall x \in X\}.$$

Clearly, $d(f, g) = \infty$ if the set of the right hand side is empty. Then an easy computation, similar to Theorem 2.5 of [4], shows that d defines a generalized non-Archimedean complete metric on

$$\mathcal{F} = \{f | f : X \to Y\}.$$

Using the strong triangle inequality in the proof of the main result of [14], we get to the following result:

Theorem 2.8. (Non-Archimedean Alternative Contraction Principle) *If (X, d) is a non-Archimedean generalized complete metric space and $J : X \to X$ a strictly contractive mapping (that is $d(J(x), J(y)) \leq Ld(y, x)$, for all $x, y \in X$ and a Lipschitz constant $L < 1$). Let $x \in X$, then either*

(i) *$d(J^n(x), J^{n+1}(x)) = \infty$ for all $n \geq 0$, or*
(ii) *there exists some $n_0 \geq 0$ such that $d(J^n(x), J^{n+1}(x)) < \infty$ for all $n \geq n_0$;*

the sequence $\{J^n(x)\}$ is convergent to a fixed point x^* of J; x^* is the unique fixed point of J in the set

$$\mathcal{Y} = \{y \in X : d(J^{n_0}(x), y) < \infty\}$$

and $d(y, x^*) \leq d(y, J(y))$ for all y in this set.

3. Stability of quadratic functional equations

Hereafter, we will assume that X linear space over a non-Archimedean field $\mathbb{K}$ with a valuation $| \, . \, |$ and Y is a complete non-Archimedean normed space over $\mathbb{K}$.
For each mapping $f : X \to Y$, we define

$$Df(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y) \quad (x, y \in X).$$

If for some function $\varphi : X \times X \to [0, \infty)$, a mapping $f : X \to Y$ satisfies

$$||Df(x, y)|| \leq \varphi(x, y) \tag{3.1}$$

for all $x, y \in X$, then f is called a *φ-approximately quadratic function.*

Theorem 3.1. *Let $f : X \to Y$ be a φ-approximately quadratic function. If for some natural number $k \in \mathbb{K}$ and $0 < L < 1$,*

$$|k|^2\varphi(k^{-1}x, k^{-1}y) \leq L\varphi(x, y) \tag{3.2}$$

for each $x, y \in X$. Then there exists a unique quadratic mapping $q : X \to Y$ such that

$$||f(x) - f(0) - q(x)|| \leq \frac{L\psi(x)}{|k|^2} \tag{3.3}$$

for all $x \in X$, where

$$\psi(x) = \max\Big\{\varphi(0, 0), \varphi(x, x), \varphi(2x, x), \ldots, \varphi((k - 1)x, x)\Big\} \quad (x \in X). \tag{3.4}$$

Proof. Let $f_1(x) = f(x) - f(0)$ for each $x \in X$. By induction on i, we will show that for each $x \in X$ and $i \geq 2$,

$$||f_1(ix) - i^2 f_1(x)|| \leq \max\{\varphi(0,0), \varphi(x,x), \ldots, \varphi((i-1)x, x)\}. \tag{3.5}$$

Let $x = y$ in (3.1), then

$$||f_1(2x) - 4f_1(x)|| \leq \max\{\varphi(0,0), \varphi(x,x)\}, \qquad (x \in X). \tag{3.6}$$

This proves (3.5) for $i = 2$. Let (3.5) hold for $i = 2, \ldots, j$. Replacing x by jx and y by x in (3.1) for each $x \in X$, we have

$$\begin{aligned} ||Df_1(jx, x)|| &= ||f_1((j+1)x) + f_1((j-1)x) - 2f_1(jx) - 2f_1(x)|| \\ &\leq \max\{\varphi(0,0), \varphi(jx, x)\}. \end{aligned} \tag{3.7}$$

Since

$$\begin{gathered} Df_1(jx, x) = f_1\big((j+1)x\big) - (j+1)^2 f_1(x) \\ + f_1((j-1)x) - (j-1)^2 f_1(x) - 2[f_1(jx) - j^2 f_1(x)] \end{gathered}$$

for each $x \in X$, it follows from (3.7) and our induction hypothesis that for each x in X,

$$\begin{aligned} &||f_1((j+1)x) - (j+1)^2 f_1(x)|| \\ &\leq \max\{||Df_1(jx, x)||, ||f_1((j-1)x) - (j-1)^2 f_1(x)||, |2|||f_1(jx) - j^2 f_1(x)||\} \\ &\leq \max\{\varphi(0,0), \varphi(x,x), \ldots, \varphi(jx, x)\}. \end{aligned}$$

This proves (3.5) for all $i \geq 2$. In particular

$$||f_1(kx) - k^2 f_1(x)|| \leq \psi(x) \quad (x \in X). \tag{3.8}$$

Replacing x by $k^{-1}x$ in (3.8), it follows that for each $x \in X$,

$$||f_1(x) - k^2 f_1(k^{-1}x)|| \leq \psi(k^{-1}x) \quad (x \in X). \tag{3.9}$$

Let

$$\begin{gathered} \mathcal{F} = \{h : X \to Y\} \\ d(g, h) = \inf\{\alpha > 0 : ||g(x) - h(x)|| \leq \alpha\ \psi(x) \quad \forall x \in X\}. \end{gathered}$$

By Example 2.7, d defines a generalized non-Archimedean complete metric on $\mathcal{F}$. Define $J : \mathcal{F} \to \mathcal{F}$ by $J(h) = k^2 h(k^{-1}x)$. Then J is strictly contractive on E, in fact if

$$||g(x) - h(x)|| \leq \alpha\ \psi(x), \quad (x \in X),$$

then by (3.2),

$$||J(g)(x) - J(h)(x)|| = |k|^2 ||g(k^{-1}x) - h(k^{-1}x)|| \leq \alpha\ |k|^2 \psi(k^{-1}x) \leq L\alpha\ \psi(x),\ (x \in X).$$

It follows that

$$d(J(g), J(h)) \leq Ld(g, h) \quad (g, h \in E).$$

Hence d is strictly contractive mapping with Lipschitz constant L. By (3.9),

$$||(Jf_1)(x) - f_1(x)|| = ||k^2 f_1(k^{-1}x) - f_1(x)|| \leq \psi(k^{-1}x) \leq \frac{L}{|k|^2} \psi(x) \quad (x \in X).$$

This means that $d(J(f_1), f_1) \leq \frac{L}{|k|^2}$. By Theorem 2.8(ii), J has a unique fixed point $q : X \to Y$ in the set

$$\mathcal{J} = \{g \in \mathcal{F} : d(g, J(f_1)) < \infty\}$$

and for each $x \in X$,

$$q(x) = \lim_{n\to\infty} J^n(f_1(x)) = \lim_{n\to\infty} k^{2n} f_1(k^{-n}x).$$

Therefore for all $x, y \in X$,

$$\begin{aligned} &||q(x+y) + q(x-y) - 2q(x) - 2q(y)|| \\ &= \lim_{n\to\infty} |k|^{2n} ||f_1(k^{-n}(x+y)) + f_1(k^{-n}(x-y)) - 2f_1(k^{-n}x) - 2f_1(k^{-n}y)|| \\ &\leq \lim_{n\to\infty} |k|^{2n} \max\{\varphi(0,0), \varphi(k^{-n}x, k^{-n}y)\} \leq \lim_{n\to\infty} L^n \varphi(x,y) = 0. \end{aligned}$$

This shows that q is quadratic. By Theorem 2.8(ii)

$$d(f_1, q) \leq d(J(f_1), f_1),$$

that is

$$||f(x) - f(0) - q(x)|| \leq \frac{L\psi(x)}{|k|^2} \quad (x \in X).$$

Let $q' : X \to Y$ be a quadratic mapping which satisfies (3.3), then q' is a fixed point of J in F. However, by Theorem 2.8, J has only one fixed point in $\mathcal{J}$. This proves the uniqueness assertion of the Theorem. □

The proof of the following result is similar to that in Theorem 3.1, hence it is omitted.

Theorem 3.2. *Let $f : X \to Y$ be a φ-approximately quadratic function. If for some natural number $k \in \mathbb{K}$ and $0 < L < 1$,*

$$|k|^{-2}\varphi(kx, ky) \leq L\varphi(x,y) \tag{3.10}$$

for each $x, y \in X$. Then there exists a unique quadratic mapping $q : X \to Y$ such that

$$||f(x) - f(0) - q(x)|| \leq \frac{L\psi(x)}{|k|^2} \tag{3.11}$$

for all $x \in X$, where ψ is defined by (3.4).

4. Applications

In [10], S. Czerwik proved the following:

Theorem 4.1. *Let X and Y be a real normed space and a real Banach space, respectively and let $r \neq 2$ be a positive constant. If $f : X \to Y$ satisfies the inequality*

$$||Df(x,y)|| \leq \varepsilon(||x||^r + ||y||^r)$$

for some $\varepsilon > 0$ and for all $x, y \in X$, then

$$q(x) = \begin{cases} \lim_{n\to\infty} 4^n f(2^{-n}x) & r > 2 \\ \lim_{n\to\infty} 4^{-n} f(2^n x) & r < 2 \end{cases}$$

defines a unique quadratic function $q : X \to Y$ such that $||f(x) - q(x)|| \leq \frac{2.\varepsilon}{|4-2^p|}||x||^p$ for each $x \in X$.

The following example shows that this result is not valid in non-Archimedean normed spaces.

Example 4.2. Let $X = Y = \mathbb{Q}_p$ for prime number $p > 2$. Define $f : X \to Y$ by $f(x) = x$. Then for $r = 1$ and $\varepsilon = 1$, we have

$$|Df(x,y)| = |-2y| = |y| \le \varepsilon(||x||^r + ||y||^r) \quad (x, y \in X).$$

But for each natural numbers n, we have

$$|4^{-(n+1)} f(2^{n+1}x) - 4^{-n} f(2^n x)| = |x| \quad (x \in X).$$

Hence for each $x \neq 0$, $\{4^{-n} f(2^n x)\}$ is not convergent.

However, we have the following result:

Corollary 4.3. *Let $(X, || \ ||_X)$ and $(Y, || \ ||_Y)$ be two non-Archimedean normed linear spaces over $\mathbb{Q}_p$, where $p > 2$ is a prime number. If $(Y, || \ ||_Y)$ is complete and $f : X \to Y$ satisfies the inequality*

$$||Df(x,y)||_Y \le \varepsilon\big(||x||_X^r + ||y||_X^r\big) \qquad (x, y \in X), \tag{4.1}$$

for some $\varepsilon > 0$ and $r < 2$. Then $q(x) = \lim_{n\to\infty} p^{2n} f(p^{-n}x)$ defines a unique quadratic mapping $q : X \to Y$ such that

$$||f(x) - q(x)||_Y \le 2\ \varepsilon p^r ||x||_X^r \quad (x \in X). \tag{4.2}$$

Proof. By (4.1), $f(0) = 0$. Let $\varphi(x, y) = \varepsilon\big(||x||_X^r + ||y||_X^r\big)$ for each $x, y \in X$, then

$$|p|^2 \varphi(p^{-1}x, p^{-1}y) = \frac{\varepsilon p^r}{p^2}\ \big(||x||_X^r + ||y||_X^r\big) = p^{r-2}\varphi(x, y) \quad (x, y \in X).$$

Moreover,

$$\psi(x) = \max\{\varphi(0, 0), \varphi(x, x), \ldots, \varphi((p-1)x, x)\} = 2\varepsilon\ ||x||_X^r \qquad (x \in X).$$

Put $L = p^{r-2}$. By Theorem 3.1, $q(x) = \lim_{n\to\infty} p^{2n} f(p^{-n}x)$ defines a unique quadratic mapping $q : X \to Y$ such that (4.2) holds. □

When $r > 2$, we can use Theorem 3.2 to obtain similar corollary. Here we give a slightly different result.

Corollary 4.4. *Let $(X, || \ ||_X)$ and $(Y, || \ ||_Y)$ be two non-Archimedean normed linear spaces over $\mathbb{Q}_p$, where $p > 2$ is a prime number. If $(Y, || \ ||_Y)$ is complete and for some $r > 2$, $f : X \to Y$ satisfies the inequality*

$$||Df(x,y)||_Y \le \max\{||x||_X^r, ||y||_X^r\} \qquad (x, y \in X), \tag{4.3}$$

then there exists a unique quadratic mapping $q : X \to Y$ such that

$$||f(x) - q(x)||_Y \le p^{4-r} ||x||_X^r \quad (x \in X). \tag{4.4}$$

Proof. By (4.3), $f(0) = 0$. Let $\varphi(x, y) = \max\{||x||_X^r, ||y||_X^r\}$ for each $x, y \in X$. Then

$$|p|^{-2}\varphi(px, py) = p^{2-r}\max\{||x||_X^r, ||y||_X^r\} = p^{2-r}\varphi(x, y) \quad (x, y \in X).$$

Moreover,

$$\psi(x) = \max\{\varphi(0, 0), \varphi(x, x), \ldots, \varphi((p-1)x, x)\} = ||x||_X^r \qquad (x \in X).$$

Put $L = p^{2-r}$. By Theorem 3.2, there exists a unique quadratic mapping $q : X \to Y$ such that (4.4) holds. □

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