Journal of Sciences, Islamic Republic of Iran 18(1): 41-48 (2007) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

The Topological Center of the Banach Algebra $UC_l(K)^*$

R.A. Kamyabi-Gol*

Department of Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Islamic Republic of Iran

Abstract

Let K be a (commutative) locally compact hypergroup with a left Haar measure. Let $L^1(K)$ be the hypergroup algebra of K and $UC_l(K)$ be the Banach space of bounded left uniformly continuous complex-valued functions on K. In this paper we show, among other things, that the topological (algebraic) center of the Banach algebra $UC_l(K)^*$ is $M(K)$, the measure algebra of K.

Keywords: Hypergroup; Hypergroup algebra; Measure algebra; Second conjugate algebra; Algebraic center

1. Introduction

The theory of hypergroups was initiated by Dunkl [4], Jewett [8] and Spector [21] in the early 1970's and has received a good deal of attention from harmonic analysts (note that Jewett calls hypergroups "convos" in his paper [8]). In [16], Pym also considers convolution structures which are close to hypergroups. A fairly complete history is given in Ross's survey article [17,18]. Hypergroups arise in a natural way as a double coset space, and the space of conjugacy classes of a compact group [17,1]. In particular, locally compact groups are hypergroups. Here we follow the method of Jewett [8]. It is still unknown if an arbitrary hypergroup admits a left Haar measure but all the known examples do [8, §5]. In particular, discrete, compact and commutative hypergroups possess Haar measures [10].

Throughout, K will denote a hypergroup with a left Haar measure λ. Let $L^1(K)$ denote the hypergroup algebra of K, *i.e.* all Borel measurable functions ϕ on K with $\|\phi\| = \int_K |\phi(x)| d\lambda(x) < \infty$ (with functions equal almost everywhere identified), and the multiplication defined by

$$\phi * \psi(x) = \int_K \phi(x * y)\psi(y) d\lambda(y) \qquad \text{(see [8, §5.5])}.$$

Let the second dual $L^1(K)^{**}$ $(= L^\infty(K)^*)$ of $L^1(K)$ be equipped with the first Arens product [3]. Then $L^1(K)^{**}$ is a Banach algebra with this product. The *topological center* of $L^1(K)^{**}$ is defined by

$Z(L^1(K)^{**}) = \{m \in L^1(K)^{**}$: the mapping $n \mapsto mn$ is w^*-continuous on $L^1(K)^{**}\}$. We have shown [9] that the topological center of $L^1(K)^{**}$ is $L^1(K)$. This fact has been shown by Lau and Losert in [13] for locally compact groups (see also [14] and [2]).

Let $UC_l(K)$ be the Banach space of all bounded left uniformly continuous complex-valued functions on K (see Section 2 for definition) and $UC_l(K)^*$ be its dual Banach space. Then there is a natural multiplication on $UC_l(K)^*$ under which it is a Banach algebra. More

2000 *Mathematics Subject Classification.* Primary 43A20, 43A10, 43A22 Secondary 46H05, 46H10
*E-mail: kamyabi@ferdowsi.um.ac.ir

specifically, for $m, n \in UC_l(K)^*$, $f \in UC_l(K)$, and $x \in K$,

$\langle mn, f \rangle = \langle m, nf \rangle$ where $nf(x) = \langle n, {}_xf \rangle$.

This product is, in fact, the restriction of the first Arens product on $L^1(K)^{**}$ to $UC_l(K)^*$, which will be proved in Lemma 3.1. The *topological center* of $UC_l(K)^*$ is defined by

$Z(UC_l(K)^*) = \{m \in UC_l(K)^*:$ the mapping $n \mapsto mn$ is w^*-continuous on $UC_l(K)^*\}$. Note that when K is commutative, then $Z(UC_l(K)^*)$ is precisely the algebraic center of $UC_l(K)^*$. For a locally compact group G, Lau in [12] has shown that $Z(UC_l(G)^*)$ is $M(G)$, the algebra of bounded regular Borel measures on G. However the method of his proof cannot be applied to hypergroups in general. The purpose of this paper is to establish these results for hypergroups. Our proof also provides a new proof of Lau's result [12, Theorem 1] in the group case.

This paper is organized as follows:

Section 2 consists of some notations and preliminary results that we need in the sequel. The technical Lemma 2.7 in this section plays a key role in proving our main result (Theorem 3.11). In Section 3, we shall prove that the topological center of $UC_l(K)^*$ is $M(K)$. The results of this section generalize the corresponding ones for locally compact groups [12].

2. Preliminaries and Some Technical Lemmas

The notations used in this paper are those of [8] with the following exceptions:

The mapping $x \to \breve{x}$ denotes the involution on the hypergroup K, δ_x the Dirac measure concentrated at x ($x \in K$), and 1_X the characteristic function of the non-empty set $X \subseteq K$. For $C \subseteq K$ and $y \in K$, $C * y$ denotes the subset $C * \{y\}$ of K.

Lemma 2.1. *Let K be a locally compact non-compact hypergroup. Then there exists a family $\{C_i : i \in I\}$ of compact subsets of K, and $y_i, z_i \in K$, for each $i \in I$ such that C_i° (the interior of C_i) is non-empty, $\cup_{i \in I} C_i^\circ = K$, $\{C_i : i \in I\}$ is closed under finite unions, and*

(a) the families $\{C_i * y_i : i \in I\}$ and $\{C_i * z_i : i \in I\}$ are pairwise disjoint.

(b) $C_i * y_i * \breve{y}_j \cap C_p * z_p * \breve{z}_q = \varnothing$, $i \neq j$ and $p \neq q$, $i, j, p, q \in I$.

Proof. See [9, Lemma 2.1]. □

For a Borel function f on K and $x \in K$, ${}_xf$ denotes the left translation

$${}_xf(y) = f(x * y) = \int_K f(t)\, d(\delta_x * \delta_y)(t),$$

and f_x is the right translation

$$f_x(y) = f(y * x) = \int_K f(t)\, d(\delta_y * \delta_x)(t),$$

if the integrals exist. We write ${}_{x*y}f$ and f_{x*y} for ${}_y({}_xf)$ and $(f_y)_x$, respectively.

The function $\breve{f}$ is given by $\breve{f}(x) = f(\breve{x})$. The integral $\int \ldots d\lambda(x)$ is often denoted by $\int \ldots dx$.

Let $(L^p(K), \|.\|_p)$, $1 \le p \le \infty$, denote the usual L^p spaces on K [8, §6.2]. Then $L^\infty(K)$ is a commutative Banach algebra with pointwise multiplication and the essential supremum norm $\|.\|_\infty$, and moreover, $L^\infty(K) = L^1(K)^*$ [8, §6.2]. We say that $X \subseteq L^\infty(K)$ is *translation invariant* if ${}_xf \in X$ and $f_x \in X$ for all $f \in X$, $x \in K$; also X is *topologically translation invariant* if $\phi * f \in X$ and $f * \breve{\phi} \in X$ for all $f \in X$, $\phi \in P^1(K) = \{\phi \in L^1(K) : \phi \ge 0, \|\phi\|_1 = 1\}$.

In addition, we use the following notations:

$C_{00}(K)$: the set of continuous functions with compact supports on K.

$C(K)$: the set of bounded continuous functions on K.

$UC_l(K) = \{f \in C(K) : x \mapsto {}_xf$ is continuous from K into $(C(K), \|\cdot\|_\infty)\}$.

$UC_r(K) = \{f \in C(K) : x \mapsto f_x$ is continuous from K into $(C(K), \|\cdot\|_\infty)\}$.

It is known that $UC_l(K) = \{f \in C(K) : x \mapsto {}_xf$ is continuous from K into $C(K)$ with the weak-topology$\}$ [20, Theorem 4.2.2, p. 88].

Each of the spaces $UC_l(K)$ and $UC_r(K)$ is a normed closed, conjugate closed, translation invariant and topologically translation invariant subspace of $C(K)$ containing the constant functions and $C_0(K)$

[19, Lemma 2.2]. Furthermore

(i) $UC_l(K) = L^1(K) * UC_l(K) = L^1(K) * L^\infty(K)$ [19, Lemma 2.2]

(ii) $UC_r(K) = UC_l(K) * L^1(K)\tilde{} = L^\infty(K) * L^1(K)\tilde{}$ [19, Lemma 2.2].

Note that $UC_l(K)$ is not an algebra in general [19, Remark 2.3(b)].

For $\phi \in L^1(K)$, we write $\tilde{\phi}(x) = \Delta(\check{x})\phi(\check{x})$ where Δ is the modular function on K; then $\|\tilde{\phi}\| = \|\phi\|_1$. If $f \in L^p(K)$, $1 \le p \le \infty$, $x \in K$, then $\|_x f\|_p \le \|f\|$, and this is in general not an isometry [8, §3.3]. The mapping $x \mapsto {}_x f$ is continuous from K to $(L^p(K), \|.\|_p)$, $1 \le p < \infty$, [8, 2.2B and 5.4H].

It is easy to show that $L^1(K)$ has a bounded approximate identity (B.A.I) $\{e_i : i \in I\} \subseteq C_{00}^+(K)$ such that $\|e_i\| = 1$ (see [19, Lemma 2.1]).

For any Banach space X, we denote its first and second dual by X^* and X^{**}. Let A be a Banach algebra. For any $f \in A^*$ and $a \in A$, we may define a linear functional fa on A by $\langle fa, b\rangle = \langle f, ab\rangle, (b \in A)$. One can check that $fa \in A^*$ and $\|fa\| \le \|f\| \|a\|$. Now for $n \in A^{**}$, we may define $nf \in A^*$ by $\langle nf, a\rangle = \langle n, fa\rangle$; clearly we have $\|nf\| \le \|n\| \|f\|$. Next for $m \in A^{**}$, define $mn \in A^{**}$ by $\langle mn, f\rangle = \langle m, nf\rangle$. We have $\|mn\| \le \|m\| \|n\|$, and A^{**} becomes a Banach algebra with the multiplication mn, just defined, referred to as the first *Arens product*. There is another multiplication on A^{**}, called the second Arens product, which is denoted by $m \circ n$ and defined successively as follows:

$\langle m \circ n, f\rangle = \langle n, fm\rangle$, where $\langle fm, a\rangle = \langle m, af\rangle$, $\langle af, b\rangle = \langle f, ba\rangle$, and m, n, f, a, b are taken as above.

From now on A^{**} will always be regarded as a Banach algebra with the first Arens product.

Let $Z(A^{**})$ denote the set of all $m \in A^{**}$ such that $mn = m \circ n$ for all $n \in A^{**}$. We call $Z(A^{**})$ the *topological center* of A^{**}.

Lemma 2.2. *$Z(A^{**})$ is a closed subalgebra of A^{**} containing A.*

Proof. [3, p. 310] or [13, Lemma 1]. □

Lemma 2.3 *For any $m \in A^{**}$, the following are equivalent:*

(a) $m \in Z(A^{**})$;

(b) the map $n \to mn$ from A^{**} into A^{**} is w^*-w^* continuous;

(c) the map $n \to mn$ from A^{**} into A^{**} is w^*-w^* continuous on norm bounded subsets of A^{**}.

Proof. [3, p. 313]. □

Note that for n fixed in A^{**}, the mapping $m \mapsto mn$ is always w^*-w^* continuous.

We collect here some facts about the Arens product on $L^1(K)^{**}$ that we shall need later.

Lemma 2.4. *Let $\phi, \psi \in L^1(K)$, $f \in L^\infty(K)$. Then*

(i) $\langle \psi f, \phi\rangle = \langle f\phi, \psi\rangle$.

(ii) $\psi f = f * \tilde{\psi} \in UC_r(K)$, $f\phi = \tilde{\phi} * f \in UC_l(K)$.

(iii) ${}_a(\psi f) = \psi({}_a f)$, $(f\phi)_a = (f_a)\phi$ for $a \in K$.

Proof. immediate. □

Lemma 2.5. *Let $0 \ne m \in L^\infty(K)^*$. Then there is a net $\{u_\alpha\}$ in $L^1(K)$ such that $\|u_\alpha\| \le \|m\|$, all u_α have compact support and $u_\alpha \to m$ in the w^*-topology of $L^\infty(K)^*$.*

Proof. This follows from Goldstine's theorem and the density of $C_{00}(K)$ in $L^1(K)$. □

Lemma 2.6. *If $m \in Z(L^1(K)^{**})$ and $f \in L^\infty(K)$, then $fm \in UC_l(K)$ and $(fm)(x * y) = \langle m, f_{x*y}\rangle$.*

Proof. See [9, Lemma 2.6]. □

Lemma 2.7. *If $n \in Z(L^1(K)^{**})$ and $u \in L^1(K)$ are such that $(n - u)(f) = 0$ for all $f \in C_0(K)$, then $n = u$.*

Proof. See [9, Lemma 2.7]. □

3. Topological Center of $UC_l(K)^*$

In this section we show that the topological center of $UC_l(K)^*$ is $M(K)$. Let $f \in UC_l(K)$ and $m \in UC_l(K)^*$. Define the function mf on K by $mf(x)$

$=\langle m, {}_x f\rangle$. Then $mf \in UC_l(K)$. Indeed, it is easy to see that $mf \in C(K)$. Also

$$\begin{aligned}
{}_x(mf)(y) &= mf(x*y)\\
&= \int_{x*y} mf(t)\, d(\delta_x*\delta_y)(t)\\
&= \int_{x*y} \langle m, {}_t f\rangle\, d(\delta_x*\delta_y)(t)\\
&= \langle m, \int_{x*y} {}_t f\, d(\delta_x*\delta_y)(t)\rangle \qquad (*)
\end{aligned}$$

But the Bochner integral $\int_{x*y} {}_t f\, d(\delta_x*\delta_y)(t)$ is ${}_y({}_x f)$ since

$$\begin{aligned}
\int_{x*y} {}_t f\, d(\delta_x*\delta_y)(t)(\xi) &= \langle \delta_\xi, \int_{x*y} {}_t f\, d(\delta_x*\delta_y)(t)\rangle\\
&= \int_{x*y} \langle \delta_\xi, {}_t f\rangle d(\delta_x*\delta_y)(t)\\
&= \int_{x*y} {}_t f(\xi) d(\delta_x*\delta_y)(t)\\
&= \int_{x*y} f_\xi(t) d(\delta_x*\delta_y)(t)\\
&= f_\xi(x*y) = {}_y({}_x f)(\xi).
\end{aligned}$$

So (*) implies that

$${}_x(mf)(y) = \langle m, {}_y({}_x f)\rangle = m({}_x f)(y),$$

that is,

$${}_x(mf) = m({}_x f). \qquad (1)$$

Hence

$$\begin{aligned}
\| {}_x(mf) - {}_y(mf) \| &\le \| m({}_x f) - m({}_y f) \|\\
&\le \| m \| \| {}_x f - {}_y f \|.
\end{aligned}$$

Note that if $m = \delta_a$ for some $a \in K$, then $\delta_a f = f_a$.

Now we may define a product on $UC_l(K)^*$ by $\langle nm, f\rangle = \langle n, mf\rangle$ for $m, n \in UC_l(K)^*$ and $f \in UC_l(K)$. With this product, one can see that $UC_l(K)^*$ is a Banach algebra. **Lemma 3.1** *The product on $UC_l(K)^*$ is the restriction of the first Arens product on $L^\infty(K)^*$ to $UC_l(K)^*$*

Proof. See [15, Theorem 7]. □

Note that we can even identify $UC_l(K)^*$ as a closed right ideal of the Banach algebra $L^\infty(K)^*$ with the first Arens product (see [14, p. 13]).

Lemma 3.2. *If we take $C_0(K)^\perp = \{m \in UC_l(K)^* : m|_{C_0(K)} = 0\}$, then $UC_l(K)^* = C_0(K)^\perp \oplus M(K)$. If $m \in UC_l(K)^*$ and $m = m_1 + \mu$ for $m_1 \in C_0(K)^\perp$ and $\mu \in M(K)$, then $\|m\| = \|m_1\| + \|\mu\|$ and $C_0(K)^\perp$ is a closed ideal in $UC_l(K)^*$.*

Proof. See [15, Theorem 4]. □

Remark 3.3. For $m \in UC_l(K)^*$ and $f \in UC_l(K)$, we may define a bounded complex function fm on K by $fm(x) = \langle m, f_x\rangle$. Generally, fm is not in $UC_l(K)$ but for $m = \delta_a$ $(a \in K)$ $fm = f\delta_a = {}_a f \in UC_l(K)$. If $n \in UC_l(K)^*$ and $fm \in UC_l(K)$, for all $f \in UC_l(K)$, then we may define another product on $UC_l(K)^*$ by $\langle m \circ n, f\rangle = \langle n, fm\rangle$.

Let $Z(UC_l(K)^*)$ denote the set of all $m \in UC_l(K)^*$ such that $fm \in UC_l(K)$ for all $f \in UC_l(K)$ and $mn = m \circ n$ for all $n \in UC_l(K)^*$. One can check that $Z(UC_l(K)^*)$ contains all point evaluation functionals δ_x, $x \in K$.

Note 3.4. For $m \in UC_l(K)^*$, define the linear operator L_m from $UC_l(K)^*$ into itself by

$$L_m(n) = mn, \quad n \in UC_l(K)^*.$$

Put

$C = \{m \in UC_l(K)^* : L_m$ is w^*-w^* continuous on norm bounded subset of $UC_l(K)^*\}$.

Lemma 3.5. $M(K) \subseteq C$.

Proof. For $\mu \in M(K)$, we need to show that the map $m \to \mu m$ is w^*-w^* continuous on any norm bounded subset of $UC_l(K)^*$. Let $\{m_\alpha\}$ be a net in $UC_l(K)^*$ with $\|m_\alpha\| \le c$, for some constant c, converging to $m \in UC_l(K)^*$ in the w^*-topology of $UC_l(K)^*$. Then for any $f \in UC_l(K)$ and $s, t \in K$, we have

$|m_\alpha f(s) - m_\alpha f(t)| = |\langle m_\alpha, {}_s f - {}_t f\rangle| \le c\,\|{}_s f - {}_t f\|$. Hence by [11, p. 232] the family $\{m_\alpha f\}$ in $UC_l(K)$ is equicontinuous. Since $m_\alpha f \to mf$ pointwise on K, the convergence is uniform on every compact set in K [11, Theorem 7.15]. Let $\mu \in M(K)$ be with compact support, then $\langle \mu m_\alpha - \mu m, f\rangle = \langle \mu, m_\alpha f - mf\rangle = \int_K (m_\alpha f - mf)(x)\,d\mu(x) \to 0$. Since measures with compact supports are norm dense in $M(K)$ and $\|m_\alpha f\| \le c\|f\|$, it follows that $\mu m_\alpha \to \mu m$ in the w^*-topology of $UC_l(K)^*$ and we are done. □

Lemma 3.6. *If* $m \in C$ *and* $f \in UC_l(K)$, *then* $fm \in C(K)$ *and* $fm(x*y) = \langle m, f_{x*y}\rangle$ *for all* $x, y \in K$.

Proof. If $\{x_\alpha\}$ is a net in K converging to x, then the net $\{\delta_{x_\alpha}\}$ converges to δ_x in the w^*-topology of $UC_l(K)^*$ (see [8, Lemma 2.2B] and Lemma 3.2). Hence

$$fm(x_\alpha) = \langle m, f_{x_\alpha}\rangle = \langle m, \delta_{x_\alpha} f\rangle$$
$$= \langle m\delta_{x_\alpha}, f\rangle \to \langle m\delta_x, f\rangle$$
$$= \langle m, \delta_x f\rangle = \langle m, f_x\rangle = fm(x),$$

since $m \in C$ and $\{\delta_{x_\alpha}\}$ is bounded. Furthermore, we know that fm is also bounded. Consequently $fm \in C(K)$. Note that for every $x, y \in K$, the Bochner's integral $\int_{x*y} f_t\, d(\delta_x * \delta_y)$ exists. Indeed, the map $t \to f_t$ from the compact subset $x*y$ of K into $UC_l(K)$ is continuous in the topology $\sigma(UC_l(K), C)$ of $UC_l(K)$, and C separates the points of $UC_l(K)$ (C contains the point evaluations). Hence for any $m \in C$

$$\langle m, \int_{x*y} f_t\, d(\delta_x * \delta_y)(t)\rangle = \int_{x*y} \langle m, f_t\rangle d(\delta_x * \delta_y)(t)$$
$$= \int_{x*y} fm(t)\, d(\delta_x * \delta_y)(t) \quad (*)$$
$$= fm(x*y).$$

On the other hand, the Bochner's integral $\int_{x*y} f_t\, d(\delta_x * \delta_y)(t)$ is equal to f_{x*y}. By using Lemma 2.4(iii), for every $\phi \in L^1(K) \subseteq C$ (Lemma 3.5), (*) implies that

$$\langle \phi, \int_{x*y} f_t\, d\delta_x * \delta_y(t)\rangle = f\phi(x*y) = (f\phi)_y(x)$$
$$= ((f_y)\phi)(x) = ((f_y)\phi)_x(e)$$
$$= (f_y)_x\phi(e) = \langle \phi, (f_y)_x\rangle.$$

Hence from (*) we have $\langle m, f_{x*y}\rangle = fm(x*y)$. □

Lemma 3.7. *For each* $m \in UC_l(K)^*$ *the following are equivalent:*

(a) $m \in Z(UC_l(K)^*)$,

(b) The operator L_m *is* w^*-w^* *continuous,*

(c) $m \in C$.

Proof. First we show that (a) implies (b). Let $\{n_\alpha\}$ be a net in $UC_l(K)^*$ converging to $n \in UC_l(K)^*$ in the w^*-topology of $UC_l(K)^*$. Then for every $f \in UC_l(K)$,

$$\lim_\alpha mn_\alpha(f) = \lim_\alpha \langle mn_\alpha, f\rangle$$
$$= \lim_\alpha \langle m \circ n_\alpha, f\rangle = \lim_\alpha \langle n_\alpha, fm\rangle$$
$$= \langle m \circ n, f\rangle = mn(f).$$

(b) clearly implies (c).

To show that (c) implies (a), let $m \subset C$ and $f \in UC_l(K)$, then by Lemma 3.6, $fm \in C(K)$. To see that $fm \in UC_l(K)$, we first show that if $\theta \in C(K)^*$ and $a \in K$, then

$$\langle \theta, {}_a(fm)\rangle = \langle m\delta_a\theta, f\rangle. \qquad (**)$$

Indeed, for $\theta = \delta_x$ ($x \in K$), by Lemma 3.6, we have

$$\langle \delta_x, {}_a(fm)\rangle = {}_a(fm)(x) = fm(a*x)$$
$$= \langle m, f_{a*x}\rangle = \langle m, (f_x)_a\rangle$$
$$= \langle m, \delta_a(f_x)\rangle = \langle m\delta_a, f_x\rangle$$
$$= \langle m\delta_a, \delta_x f\rangle = \langle m\delta_a\delta_x, f\rangle.$$

If θ is a mean on $C(K)$, then there is $\theta_\beta = \Sigma_{i=1}^{n_\beta}\lambda_i\delta_{x_i}$, a convex combinations of point evaluations, such that $\theta_\beta \to \theta$ in the w^*-topology of

$C(K)^*$. Hence

$$\langle \theta, {}_{\alpha}(fm)\rangle = \lim_{\beta}\langle \theta_{\beta}, {}_{\alpha}(fm)\rangle$$
$$= \lim_{\beta}\langle m\delta_{\alpha}\theta_{\beta}, f\rangle = \langle m\delta_{\alpha}\theta, f\rangle,$$

by w^*-w^* continuity of L_m on norm bounded subsets of $UC_l(K)^*$. Consequently (**) holds.

Now to see that $fm \in UC_l(K)$, by [20, Theorem 4.2.2, p. 88], it is enough to show that the map $x \to {}_x(fm)$ from K to $C(K)$ is weakly continuous. Let $\{x_\alpha\}$ be a net in K converging to x and $\theta \in C(K)^*$, then by (**),

$$\lim_{\alpha}\langle \theta, {}_{x_\alpha}(fm)\rangle = \lim_{\alpha}\langle m_{x_\alpha}\theta, f\rangle$$
$$= \langle m\delta_x\theta, f\rangle = \langle \theta, {}_x(fm)\rangle,$$

by w^*-w^* continuity of L_m on norm bounded subsets of $UC_l(K)^*$. Hence, $fm \in UC_l(K)$.

If n is a mean on $UC_l(K)$, there exists a net $n_\alpha = \Sigma_{i=1}^{I_\alpha}\lambda_i\delta_{x_i}$ in $Z(UC_l(K)^*)$ (see Remark 3.3) where $\lambda_i > 0$ and $\Sigma_{i=1}^{I_\alpha}\lambda_i = 1$ such that $n_\alpha \to n$ in the w^*-topology of $UC_l(K)^*$. Hence for each $f \in UC_l(K)$, considering Remark 3.3, we have

$$\langle m \circ n, f\rangle = \langle n, fm\rangle$$
$$= \lim_{\alpha}\langle n_\alpha, fm\rangle = \lim_{\alpha}\langle m \circ n_\alpha, f\rangle$$
$$= \lim_{\alpha}\langle mn_\alpha, f\rangle = \langle mn, f\rangle$$

by the continuity of L_m. Now by linearity, we have $m \circ n = mn$ for all $n \in UC_l(K)^*$, *i.e.* $m \in Z(UC_l(K)^*)$. □

Remark 3.8. For $\phi \in L^1(K)$ and $m \in UC_l(K)^*$, the product ϕm makes sense both as an element of $UC_l(K)^*$ and as an element of $L^\infty(K)^*$ (see [14, §3, p. 13]).

Lemma 3.9. *Let $\pi : L^\infty(K)^* \to UC_l(K)^*$ be the adjoint of the inclusion map of $UC_l(K)$ into $L^\infty(K)$. Then π is w^*-w^* continuous and $mn = m\pi(n)$ for each $m, n \in L^\infty(K)^*$.*

Proof. It is easy to check that π is w^*-w^* continuous. For the second part, we first define a continuous map $f \mapsto Ff$ of $L^\infty(K)$ into itself for each $F \in UC_l(K)^*$. Note that for $f \in L^\infty(K), \phi \in L^1(K)$, we know that $f\phi \in UC_l(K)$ (Lemma 2.4(ii)), so $\phi \mapsto \langle F, f\phi\rangle$ is a continuous linear functional on $L^1(K)$ and therefore corresponds to an element Ff of $L^\infty(K)$. The adjoint of $\phi \mapsto fF$ is a continuous and w^*-continuous map $m \mapsto mF$ of $L^\infty(K)^*$ into itself. Thus for $\phi \in L^1(K)$, $f \in L^\infty(K)$, $F \in UC_l(K)^*$, and $m \in L^\infty(K)^*$,

$$\langle Ff, \phi\rangle = \langle F, f\phi\rangle \quad , \quad \langle mF, f\rangle = \langle m, Ff\rangle \quad (*).$$

Let $\{\phi_i\} \subseteq L^1(K)$ be a net converging to m in the w^*-topology of $L^\infty(K)^*$ then for each $f \in L^\infty(K)$, by (*),

$$\langle mn, f\rangle = \lim_i\langle \phi_i n, f\rangle = \lim_i\langle \phi_i \circ n, f\rangle$$
$$= \lim_i\langle n, f\phi_i\rangle = \lim_i\langle \pi(n), f\phi_i\rangle$$
$$= \lim_i\langle \pi(n)f, \phi_i\rangle = \lim_i\langle \phi_i, \pi(n)f\rangle$$
$$= \langle m, \pi(n)f\rangle = \langle m\pi(n), f\rangle. \qquad □$$

Lemma 3.10. $Z(UC_l(K)^*) = \{m \in UC_l(K)^* : \phi m \in Z(L^\infty(K)^*)$ for each $\phi \in L^1(K)\}$.

Proof. Let $\phi \in L^1(K)$ and $m \in Z(UC_l(K)^*)$. By Remark 3.8, we may consider ϕm in $L^1(K)^{**}$. To prove that $\phi m \in Z(L^\infty(K)^*)$, by Lemma 2.3, it is enough to show that $n \to \phi mn$ is w^*-w^* continuous. If $n_\alpha \to n$ in the w^*-topology of $L^\infty(K)^*$, then $\pi(n_\alpha) \mapsto \pi(n)$ (since π is w^*-w^* continuous) in the w^*-topology of $UC_l(K)^*$. Hence, by Lemma 3.7, for any $f \in L^\infty(K)$,

$$\langle \phi mn_\alpha, f\rangle = \langle \phi \circ (mn_\alpha), f\rangle$$
$$= \langle mn_\alpha, f\phi\rangle$$
$$= \langle m\pi(n_\alpha), f\phi\rangle \to \langle m\pi(n), f\phi\rangle$$
$$= \langle \phi \circ (m\pi(n)), f\rangle$$
$$= \langle \phi m\pi(n), f\rangle = \langle \phi mn, f\rangle,$$

so by Lemma 2.3, $\phi m \in Z(L^\infty(K)^*)$.

Conversely, let $m \in UC_l(K)^*$, and $n_\alpha \to n$ in the w^*-topology of $UC_l(K)^*$, then for each $f \in UC_l(K)$, there exists $g \in UC_l(K)$ and $\phi \in L^1(K)$ such that $f = g\phi$ ([19, Lemma 2.2] and Lemma 2.4(ii)). Hence

$$\langle mn_\alpha, f\rangle = \langle mn_\alpha, g\phi\rangle = \langle \phi\circ(mn_\alpha), g\rangle$$
$$= \langle \phi mn_\alpha, g\rangle \to \langle \phi mn, g\rangle$$
$$= \langle mn, g\phi\rangle = \langle mn, f\rangle. \quad \square$$

Now we are ready for the main theorem of this section.

Theorem 3.11. $Z(UC_l(K)^*) = M(K)$

Proof. By Lemmas 3.5 and 3.7, it is enough to show that $Z(UC_l(K)^*) \subseteq M(K)$. Let $m \in Z(UC_l(K)^*)$, then by Lemma 3.2, $m = \mu + m_1$, for some $\mu \in M(K)$ and $m_1 \in C_0(K)^\perp$. It is enough to show that $m_1 = 0$. Let $\phi \in L^1(K)$. Since $C_0(K)^\perp$ is an ideal in $UC_l(K)^*$ (Lemma 3.2) $\phi m_1 \in C_0(K)^\perp$ and $\phi m_1 \in Z(L^1(K)^{**})$, by Lemma 3.10. Hence $\phi m_1 = 0$ (Lemma 2.7). Let $f \in UC_l(K)$, then $f = g\phi$, for some $g \in UC_l(K)$, and $\phi \in L^1(K)$ ([19, Lemma 2.2] and Lemma 2.4(ii)), and

$$\langle m_1, f\rangle = \langle m_1, g\phi\rangle = \langle \phi\circ m_1, g\rangle = \langle \phi m_1, g\rangle = 0.$$

Hence $m_1 = 0$, as desired. $\square$

Corollary 3.12. If K is commutative, then $M(K)$ is the algebraic center of $UC_l(K)^*$.

Corollary 3.13. Let $m \in UC_l(K)^*$ be such that L_m is weak*-weak* continuous on any bounded sphere of $UC_l(K)^*$, then $m \in M(K)$.

Definition 3.14. A bounded continuous function f is called weakly almost periodic if $\{{}_x f : x \in K\}$ is relatively weakly compact in the space of all bounded continuous functions on K. We denote the Banach space of all weakly almost periodic functions on K by $WAP(K)$.

The following corollary was proved by Skantharajah for hypergroups in [20, Theorem 4.2.7, p. 94], and by Granirer for groups in [7, p. 62-64]. Another version of this Corollary was proved in [15, Theorem 19]. A.T. Lau has also proved it in [12, Corollary 4].

Corollary 3.15. Let K be a locally compact hypergroup. Then K is compact if and only if $UC_l(K) = WAP(K)$.

Proof. If K is compact, then by [8, 2.2D and 4.2F] we have $UC_l(K) = C(K) = WAP(K)$. For the converse, from $UC_l(K) = WAP(K) = Z(UC_l(K)^*) = M(K)$, it follows that K is compact. $\square$

For the following corollary in the group-case, see [12, Corollary 5].

Corollary 3.16. Let K be a locally compact hypergroup. Then K is compact if and only if $UC_l(K)$ has a unique left invariant mean.

Proof. If K is compact, then the normalized Haar measure is the unique left invariant mean on $UC_l(K) = C(K)$.

Conversely, let m be the unique left invariant mean on $UC_l(K)$, then one can check that mn is also left invariant mean on $UC_l(K)$, for each $n \in UC_l(K)^*$. Hence $mn = \lambda m$, for some complex number λ. Let $\{n_\alpha\}$ be a net in $UC_l(K)^*$ converging to n in the weak*-topology, and $mn_\alpha = \lambda_\alpha m$, $mn = \lambda m$, then $\lambda_\alpha = mn_\alpha(1) = n_\alpha(1)$ converges to $n(1) = mn(1) = \lambda$. Hence L_m is weak*-weak* continuous, and by Theorem 3.11 and Proposition 3.7, $m \in M(K)$ and by [8, 7.2B], K is compact. $\square$

Acknowledgement

The author would like to sincerely thank the referees for their valuable comments and useful suggestions.

References

1. Chilana A.K. Harmonic analysis and hypergroups. Proceedings of the Symposium on Recent Developments in Mathematics, Allahabad, pp. 93-121 (1978).
2. Dales H.G. and Lau A.T.-M. The second dual of Beurling algebras. *Mem. Amer. Math. Soc.*, **177**(836): (2005).
3. Duncan J. and Hosseiniun S.A.R. The second dual of a Banach algebra. *Proc. Roy. Soc. Edinburgh*, **A84**: 309-325 (1979).
4. Dunkl C.F. The measure algebra of a locally compact hypergroup. *Trans. Amer. Math. Soc.*, **179**: 331-348

(1973).

5. Ghaffari A. and Metghalchi A.R. *Acta Mathematica Sinica*, English series. **20**(2): 201-208 (2004).
6. Ghahramani F., Lau A.T., and Losert V. Isometric, isomorphism between Banach algebras related to locally compact groups. *Trans. Amer. Math. Soc.*, **321**: 273-283 (1990).
7. Granirer E.E. Exposed points of convex sets and weak sequential convergence. *Mem. Amer. Math. Soc.*, **123** (1972).
8. Jewett R.I. Spaces with an abstract convolution of measures. *Advances in Math.*, **18**: 1-101 (1975).
9. Kamyabi-Gol R.A. The Topological center of $L^1(K)^{**}$. *Scientiae Mathematicae Japonicae*, **62**(1): 81-89 (2005).
10. Kamyabi-Gol R.A. A short proof for the existence of Haar measure on commutative hypergroups. *Journal of Sciences, Islamic Republic of Iran*, **13**(3): 263-265 (2002).
11. Kelly J.L. *General Topology*. Van Nostrand, New York (1968).
12. Lau A.T. Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups. *Math. Proc. Camb. Phil. Soc.*, 99: 273-283 (1986).
13. Lau A.T. and Losert V. On the second conjugate algebra of $L^1(G)$ of a locally compact group. *J. London Math. Soc.*, **37**(2): 464-470 (1988).
14. Lau A.T. and Ülger A. Topological centers of certain dual algebras. *Trans. Amer. Math. Soc.*, **348**: 1191-1212 (1996).
15. Medghalchi A.R. The second dual algebra of a hypergroup. *Math. Z.*, **210**: 615-624 (1992).
16. Pym J. S. Weakly separately continuous measure algebras. *Math. Ann.*, **175**: 207-219 (1968).
17. Ross K.A. Hypergroups and centers of measure algebras. *Ist. Naz. di Alta. Mat., Symposia Math.*, **22**: 189-203 (1977).
18. Ross K.A. Signed hypergroups- a survey. Applications of Hypergroups and Related Measure Algebras. *Contemporary Math*, **22**: 319-329 (1995).
19. Skantharajah M. Amenable hypergroups. *Ill. J. Math.*, **36**: 15-46 (1992).
20. Skantharajah M. Amenable hypergroups. *Doctoral Thesis*, University of Alberta (1989).
21. Spector R. Aperçu de la theorie des hypergroupes. Analyse Harmonique sur les groupes de Lie. Lecture Notes in Math., **497**, Springer-Verlag, Berlin, Heidelberg, New York (1975).