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CONTINUOUS WAVELET TRANSFORM FOR LOCALLY COMPACT COMMUTATIVE HYPERGROUPS

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ABSTRACT. We define the continuous Wavelet transform for locally compact commutative hypergroups using the representation theory of locally compact commutative hypergroup and theory of coherent state systems for a suitable defined coherent state system. Then we investigate the operator sense properties of the coefficient operators in this settings. Also we prove the existence of generalized Duflo-Moor operators. As an example we show how these techniques apply to the Bessel-kingman hypergroups.

1. Introduction and Preliminaries

In one of the papers initiating the study of the continuous wavelet transform on the real line, Grossmann, Morlet and Paul [5] considered systems $\{\psi_{b,a} : (b,a) \in \mathbb{R}\times\mathbb{R}^*\}$ arising from a single function ψ in $L^2(\mathbb{R})$ via $\psi_{b,a}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right)$. Also they proved that every function ψ fulfilling the admissibly condition $\int_{\mathbb{R}^*} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega = 1$, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. An equivalent formulation of this fact is that the wavelet transform $f \mapsto \mathcal{W}_\psi f$ by $\mathcal{W}_\psi f(b,a) = \langle f, \psi_{b,a}\rangle$ is an isometry of $L^2(\mathbb{R})$ in to

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$L^2(\mathbb{R}\times\mathbb{R}^+,\frac{db\,da}{|a|^2})$. However the admissibility condition as well as the choice of the measure used in the reconstruction appear to be somewhat obscure until read in grouptheoretical terms. The relation to groups was pionted out in [5] and in fact earlier in [7], where it was noted that $\psi_{b,a}=\pi(b,a)\psi$, for certain representation π of the affine and in fact hypergroup K of the real line. Moreover the above admissibility condition have natural group theoretic interpretion as well. For istance the measure used for reconstruction is just the left haar measure on K. Hence the wavelet transform is seen to be a spacial instance of the following construction, Given a continuous representation $(\pi,\mathcal{H}_\pi)$ of a locally compact commutative hypergroup K and a vector $\eta\in\mathcal{H}_\pi$, we define the coefficient operator $\mathcal{W}_\eta:\mathcal{H}_\pi\to C_b(K)$, which maps $\phi\to\mathcal{W}_\eta\phi$ by $\mathcal{W}_\eta\phi(x)=\langle\phi,\pi(x)\eta\rangle$, where $C_b(K)$ is the space of bounded continuous functions on K. We considr $\mathcal{W}_\eta$ as an operator on $\mathcal{H}_\pi$ into $L^2(K)$, with the obvious domain $\operatorname{dom}(\mathcal{W}_\eta)=\{\psi\in L^2(K):\mathcal{W}_\eta\psi\in L^2(K)\}$. We call η an admissible vector whenever $\mathcal{W}_\eta:\mathcal{H}_\pi\to L^2(K)$ is an isometric embedding and in this case $\mathcal{W}_\eta$ is called Wavelet transform for locally compact commutative hypergroup K.

While our definition it self is rather simple, the problem of identifying admissible vectors is highly nontrivial and the problem whether these vectors exists for a given representation does not have simple general answer. it is the main purpose of this article to develop in a systematical fashion criteria to deal with both problems. A point out in [5], the construction principal for wavelet transforms had also been studied in mathematics physics, where admissible vectors η are called fiducial vectors, system of type $\{\pi(x)\eta : x\in K\}$ coherent state systems, for a continuous representation of a locally compact commutative hypergroup K. And Our claim is to shift the focus from general representations of a locally compact commutative hypergroup K to subrepresentations of ϱ_K, where The left regular representation ϱ_K acts on $L^2(K)$ by

$$(\varrho_K(\mu)f)(x):=\mu*f(x)=\int_K T_z f(z^-)d\mu(z).$$

Then we investigate the operator sense properties of the coefficient operators in this settings. At the end we prove that for an irreducible representation π of a locally compact commutative hypergroup K, there is a unique, densely defined positive operator $\Lambda_\pi:L^2(K)\to L^2(K)$ with densely defined inverse such that it posses the following orthogonal relation $\langle\Lambda_\pi\zeta',\Lambda_\pi\zeta\rangle\langle\psi,\varphi\rangle=\langle\mathcal{W}_\zeta\psi,\mathcal{W}_{\zeta'}\varphi\rangle$, for each $\psi,\varphi\in\mathcal{H}_\pi$ and $\zeta,\zeta'\in\operatorname{dom}(\Lambda_\pi)$, and we call the operators Λ_π the generalized Duflo-Moore operators.

2. Main results

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space, $\eta = (\eta_x)_{x\in Y}$ denote a family of vectors indexed by elements of a measure space $(Y, \mathcal{B}, \mu)$. Then if for each $\psi \in \mathcal{H}$, the coefficient function $\mathcal{W}_\eta\psi : Y \to \mathbb{C}$, defined by $\mathcal{W}_\eta\psi(x) := \langle\psi, \eta_x\rangle$ is μ-measurable, we call η a coherent state system. And for a coherent state system $\eta = (\eta_x)_{x\in Y}$, define

$$\mathrm{dom}(\mathcal{W}_\eta) := \{\psi \in \mathcal{H} : \mathcal{W}_\eta\psi \in L^2(Y, \mu)\},$$

which may be trivial. Denote by $\mathcal{W}_\eta : \mathcal{H} \to L^2(Y, \mu)$ the coefficient operator or analysis operator with domain D_η. And moreover the coherent system $\eta = (\eta_x)_{x\in Y}$ is called admissible if the associated coefficient operator $\mathcal{W}_\eta$ is an isometry with $\mathrm{dom}(\mathcal{W}_\eta) = \mathcal{H}$.

We now define the particular class of coherent state expansions associated to locally compact commutative hypergroups representations. We first exhibit the close relation to the regular representation of the locally compact commutative hypergroups. Then we investigate the operator sense properties of the coefficient operators in this settings.

Definition 2.2. Let $(\pi, \mathcal{H}_\pi)$ be a representation of a locally compact commutative hypergroup K. Associate to the vector $\eta \in \mathcal{H}_\pi$, the orbit $(\eta_x)_{x\in K} := (\pi(x)\eta)_{x\in K}$. This is clearly a coherent state system. η is called π-admissible whenever $(\pi(x)\eta)_{x\in K}$ is admissible. And if η is π-admissible, then $\mathcal{W}_\eta : \mathcal{H}_\pi \to L^2(K)$ is called the generalized continuous wavelet transform (GCWT) for locally compact commutative hypergroups. Moreover we call the vector η, a bounded vector if $\mathcal{W}_\eta : \mathcal{H}_\pi \to L^2(K)$ is bounded on $\mathcal{H}_\pi$.

Theorem 2.3. *Let K be a locally compact commutative hypergroup and $f \in L^2(K)$ with $f^* \in L^2(K)$. Then $L^1(K) \cap L^2(K) \subset \mathrm{dom}(\mathcal{W}_f)$ and $\mathcal{W}_{f^*}$ is an extension of $\mathcal{W}_f^*$. And if one of the operators is bounded, so is the other and they coincides.*

Theorem 2.4. *Let π be an irreducible representation of a locally compact commutative hypergroup K. Then a nonzero $\eta \in \mathcal{H}_\pi$ is admissible up to normalization if and only if for some nonzero $\psi \in \mathcal{H}_\pi$ we have $\mathcal{W}_\eta\psi \in L^2(K)$.*

Theorem 2.5. *For an irreducible representation π of a locally compact commutative hypergroup K, there is a unique, densely defined positive operator $\Lambda_\pi : L^2(K) \to L^2(K)$ with densely defined inverse such that it posses the following orthogonal relation*

$$\langle\Lambda_\pi\zeta', \Lambda_\pi\zeta\rangle\langle\psi, \varphi\rangle = \langle\mathcal{W}_\zeta\psi, \mathcal{W}_{\zeta'}\varphi\rangle,$$

for each $\psi, \varphi \in \mathcal{H}_\pi$ and $\zeta, \zeta' \in \mathrm{dom}(\Lambda_\pi)$. The operators Λ_π are called generalized Duflo-Moore operators.

Corollary 2.6. *For an irreducible representation π of a locally compact commutative hypergroup K, there is a unique, densely defined positive operator $A_\pi : L^2(K) \to L^2(K)$ with densely defined inverse such that $\eta \in \mathcal{H}_\pi$ is admissible if and only if $\eta \in dom(A_\pi)$ with $\|A_\pi \eta\| = 1$.*

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