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Robust Stability of the Fractional Order LQG Controllers

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Abstract: *This paper deals with the novel fractional order Linear Quadratic Gaussian (FLQG) controller and study of the robustness of this proposed controller in comparison with classical LQG. The significance of fractional order control is that it is a generalization and "interpolation" of the classical integer order control theory, which can achieve more adequate modeling and clear-cut design of robust control system. In this paper, LQR controller with fractional derivatives and Fractional Kalman filters are proposed. In addition fractional LQG is used to control the aircraft system. To demonstrate the enhancement in using fractional LQG, robustness of the control design is compared with the integer order LQG in the presence of coprime factor uncertainty. Simulations confirm much more robustness of the fractional order LQG than classical LQG.*

Keywords: Fractional order derivatives, LQG controller, Kalman filters, coprime factor uncertainty, robustness.

I. Introduction

Recently, several authors have considered mechanical systems described by fractional-order state equations [1], [2], [3], which mean equations involving so-called fractional derivatives and integrals (for the introduction to this theory see [4]).
Fractional derivative-based models are more adequate than the previously used integer-order models. This has been demonstrated, for instance, by Caputo [7], [8], Friedrich [6] and [5].
Important fundamental physical considerations in favor of the use of fractional-derivative-based models were given in [8] and [1]. Fractional order derivatives and integrals provide a powerful instrument for the description of memory and hereditary effects in various substances, as well as for modeling dynamical processes in fractal (as defined by in [9]) media. This is the most significant advantage of the fractional-order models in comparison with integer-order models, in which, in fact, such effects or geometry are neglected.
However, because of the absence of appropriate mathematical methods, fractional-order dynamic systems were studied only marginally in the theory and practice of control systems. Works in [10], [11], [12], [13], and [14] in frequency domain must be mentioned, but the study in the time domain has been almost avoided.
Fractional Order Control (FOC) means controlled systems and/or controllers described by fractional order differential equations. Expanding calculus to fractional orders is by no means new and actually had a firm and long standing theoretical foundation.
Kiani and his colleagues studied several aspects of fractional order systems, such as application of Kalman filters in secure communication, the novel methodology in designing fractional PID and also fractional optimal control [1, 2, and 3].
In this paper, a fractional order controller is implemented in conjunction with the LQR algorithm. Based on the Separation Theorem LQG problem consists of first determining the optimal fractional LQR and finding an optimal

estimate $\hat{x}$ of the state x, so that $E\{[x-\hat{x}]^T[x-\hat{x}]\}$ is minimized. The optimal state estimate is given by a fractional order Kalman filter.

An example is provided to demonstrate the necessity of such controllers for the more efficient control of systems. Also better performance of fractional LQG when used for the control of aircraft systems than the classical LQG will be shown. This paper is organized as follow: fractional introduction and fractional LQR is described in section 2. Fractional Kalman filter is presented in section 3. Fractional LQG is discussed in section 4, and finally simulation results are presented in section 5 and in section 6 robustness of the proposed method is compared with classical LQG.

II. Fractional Kalman Filter

The generalization of the discrete state space model for fractional order derivatives, which will be used later, is presented first.

Let us assume a traditional (integer order) discrete linear stochastic state-space system,

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + w_k \\ y_k = Cx_k + v_k \end{cases} \tag{1}$$

where x_k is a state vector, u_k is a system input, y_k is a system output, w_k is a system noise and v_k is an output noise at time instant k.

Equation (1) could be rewritten as follows:

$$\Delta^1 x_{k+1} + x_k = x_{k+1} = Ax_k + Bu_k + w_k \rightarrow \Delta^1 x_{k+1} = (A-I)x_k + Bu_k + w_k$$
$$\xrightarrow{A-I=A_d} \Delta^1 x_{k+1} = A_d x_k + Bu_k + w_k$$

where $A - I = A_d$ and I is an identity matrix and $\Delta^1 x_k$ is the first order difference for x_k sample, , so that $\Delta^1 x_{k+1} = x_{k+1} - x_k$.

For the case when orders of equations are not integer, the following generalized definition is introduced:

<u>*Definition1:*</u> The generalized linear fractional order stochastic discrete state space system is given by the following:

$$\begin{cases} \Delta^{\Upsilon} x_{k+1} = A_d x_k + Bu_k + w_k \\ x_{k+1} = \Delta^{\Upsilon} x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j x_{k-j+1} \\ y_k = C_k x_k + v_k \end{cases} \tag{2}$$

Where

$$\Upsilon_k = diag\left[\binom{n_1}{k}, \cdots, \binom{n_N}{k}\right] \tag{3}$$

$$\Delta^{\Upsilon} x_{k+1} = \begin{bmatrix} \Delta^{n_1} x_{1,k+1} \\ \vdots \\ \Delta^{n_N} x_{N,k+1} \end{bmatrix} \tag{4}$$

and $n_1, \cdots, n_N$ are orders of system equations.

Results of estimation by fractional Kalman filter are obtained by minimizing in each step the following cost function:

$$\hat{x}_k = \arg\min_x \left[(\tilde{x}_k - x)\tilde{P}_k^{-1}(\tilde{x}_k - x)^T + (y_k - Cx)R_k^{-1}(y_k - Cx)^T \right]$$

Where $\tilde{x}_k = E\left[x_k \mid z_{k-1}^*\right]$ is a state vector prediction at time instant k ,defined as the random variable x_k conditioned on the measurement stream z_{k-1}^*.

$\tilde{P}_k = E\left[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^T\right]$ is a prediction of an estimation error covariance matrix. $R_k = E\left[v_k v_k^T\right]$ is a covariance matrix of an output noise v_k. $Q_k = E\left[w_k w_k^T\right]$ is a covariance matrix of a system noise w_k.

Theorem 1 [4]: *For the fractional order stochastic discrete state-space system defined by Definition & the simplified Kalman Filter (called fractional Kalman Filter) is given by the set of following equations*

$$\Delta^{\Upsilon} \tilde{x}_{k+1} = A_d \tilde{x}_k + Bu_k$$

$$\tilde{x}_{k+1} = \Delta^{\Upsilon} \tilde{x}_{k+1} - \sum_{j=0}^{k+1} (-1)^j \Upsilon_j \tilde{x}_{k+1-j}$$

$$\tilde{P}_k = (A_d + \Upsilon_1) P_{k-1} (A_d + \Upsilon_1)^T + Q_{k-1} + \sum_{j=2}^{k} \Upsilon_j P_{k-j} \Upsilon_j^T$$

$$\hat{x}_k = \tilde{x}_k + K_k (y_k - C\tilde{x}_k)$$

$$P_k = (I - K_k C)\tilde{P}_k$$

$$K_k = \tilde{P}_k C^T (C\tilde{P}_k C^T + R_k)^{-1}$$

with initial conditions

$x(0) = x_0$

$$P_0 = E\left[\left(\tilde{x}_0 - x_0\right)\left(\tilde{x}_0 - x_0\right)^T\right]$$

And v_k *and* w_k *are assumed to be independent and with zero expected value.*

Based on the mentioned theorem, optimal state estimation of the fractional model is obtained. In the next section fractional LQR is discussed.

III. Fractional LQR:

In this paper, we propose to include fractional derivative or integral of the state x in the feedback control law as follow :

$$u(t) = -K_{LQR}x + K_{FOC}\frac{d^\alpha x}{dt^\alpha} \tag{5}$$

Where K_{FOC} is the gain matrix to be found using optimization procedures and $\frac{d^\alpha x}{dt^\alpha}$ is defined as follows (Caputo definition, [9,10]):

$$\frac{d^\alpha x}{dt^\alpha} = \frac{1}{\Gamma(\alpha - n)}\int_0^t \frac{x^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}} \tag{6}$$

Where n is an integer satisfying $n-1 \prec \alpha \le n$ and Γ is the Euler's Gamma function. The fractional order α , a real number such that $\alpha \in (-1,1)$.

The classical objective function in LQR problem can be defined as:

$$J = \int_0^t (x^T Q x + u^T R u)d\tau \tag{7}$$

However one of the biggest issues in implementing optimal controllers is selecting the best weight parameters. The control gain obtained from the LQR algorithm is completely dependent on the objective function defined in (7). Through this index, designers can emphasize attenuating the structural responses that are of greatest concern. While this index provides intuition to select the pattern for weight matrices, it definitely will not result in an optimal design. Furthermore, the force capacities of system actuators (to apply force) and connections (to which force is exerted) are limited, and as a consequence, the calculated input force should be bounded. This issue also increases the complexity of choosing weight matrices. To solve this problem, a performance criterion different from the one introduced in classical LQR is proposed:

$$PI = \beta_1 \sum_i \frac{RMS(z_c)}{RMS(z_0)} + \beta_2 \sum_i \frac{\max|z_c|}{\max|z_0|} \tag{9}$$

where z_c and z_0 are the output of the controlled and uncontrolled cases, respectively. The first component emphasizes the mitigation of the root mean square response and the second component the peak response. The parameters β_1 and β_2 in the function give designers the ability to specialize the performance index for specific purposes. For instance, if the aim is to resist against extreme events peak response rather than RMS response should be reduced or minimized. However, in windy zones where the passengers comfort level is of greater concern, RMS response would govern design requirements and emphasis can be placed on the first component of the performance index. In this paper β_1 and β_2 are assumed to be 1 and 2 respectively.

To avoid unfeasible control input, $u(t)$ can be bounded in constraints.

IV. Fractional LQG: Combined optimal state estimation and optimal state feedback:

In fractional LQG control, it is assumed that the plant dynamics are linear and known, and that the measurement noise and disturbance signals are stochastic with known statistical properties.

Clearly, for closed-loop control systems, there are four situations:

1) IO (integer order) plant with IO controller;
2) IO plant with FO (fractional order) controller;
3) FO plant with IO controller and
4) FO plant with FO controller. In control practice, the fractional-order controller is more common, because the plant model may have already been obtained as an integer order model in the classical sense. From an engineering point of view, improving or optimizing performance is the major concern [28]. Hence, our objective is to apply the fractional-order control (FOC) to enhance the (integer order) dynamic system control performance [23, 28].

That is, we have a plant model

$$\begin{cases} \dot{x} = Ax + Bu + w \\ y = Cx + v \end{cases} \tag{10}$$

where w, v are the disturbance and measurement noise inputs respectively, which are usually assumed to be uncorrelated zero-mean Gaussian stochastic processes.
In The classical LQG control problem is to find the optimal control $u(t)$ which minimizes

$$J = E\{\lim \frac{1}{T}\int_0^T [x^T Qx + u^T Ru]dt\} \qquad (11)$$

Where Q and R are appropriately chosen constant weighting matrices.
The structure of the LQG controller is illustrated in Figure 1.

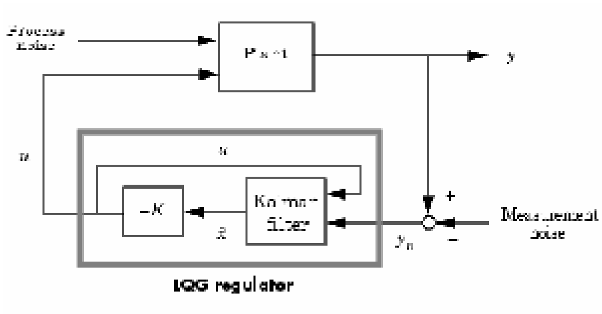

Figure 1: LQG block diagram

The solution to the LQG problem, known as the Separation Theorem consists of first determining the optimal control to a deterministic linear quadratic regulator (LQR) problem. It happens that the solution to this problem can be written in terms of the simple state feedback law:

$$u(t) = -K_{LQR}x + K_{FOC}\frac{d^{\alpha}x}{dt^{\alpha}} \qquad (12)$$

Where the computation method has been explained in section 3. The next step is to find an optimal estimate $\hat{x}$ of the state x, so that $E\{[x-\hat{x}]^T[x-\hat{x}]\}$ is minimized. The optimal state estimate is given by a fractional order Kalman filter as mentioned above. The required solution to the LQG problem is then found by replacing x by $\hat{x}$ to give

$$u(t) = -K_{LQR}\hat{x} + K_{FOC}\frac{d^{\alpha}\hat{x}}{dt^{\alpha}} \qquad (13)$$

We therefore see that the LQG problem and its solution can be separated into two distinct parts, as illustrated in Figure 1.

V. Implementation of fractional LQG in Aircraft system

A simplified configuration of an airframe and the body reference frame used in this study are shown in Fig. 2.

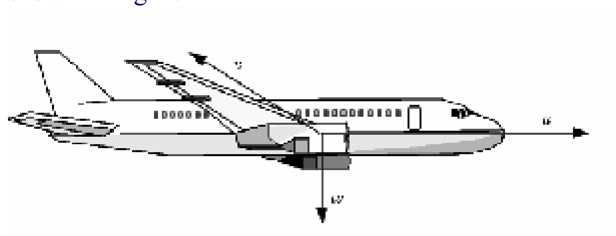

Figure 2: airframe system

We consider $x = [u, v, w, p, q, r, \theta, \phi]^T$ where the variables u, v and w are the three velocities with respect to the body frame, which is shown in the figure1. The variables θ and ϕ are roll and pitch, and p, q, and r are the roll, pitch, and yaw rates, respectively. The airframe dynamics are nonlinear. The equation below shows the nonlinear components added to the state space equation.

$$\dot{x} = Ax + Bu + \begin{bmatrix} -g\sin\theta \\ g\cos\theta\sin\phi \\ g\cos\theta\cos\phi \\ 0 \\ 0 \\ 0 \\ 0 \\ q\cos\phi - r\sin\phi \\ (q\sin\phi + r\cos\phi).\tan\theta \end{bmatrix} \qquad (14)$$

Where

$$A = \begin{bmatrix} -0.04 & 0.06 & 0.05 & 0 & 0.005 & 0 & 0 & 0 \\ -0.16 & -1.18 & 7.68 & 0 & 0.04 & 0 & 0 & 0 \\ 0.16 & -2.61 & -3.85 & 0 & 0.04 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.33 & -0.04 & -6.54 & 0 & 0 \\ 0 & 0 & 0 & -1.12 & -0.91 & -0.36 & 0 & 0 \\ 0 & 0 & 0 & 0.9931 & -0.1763 & -1.2047 & 0 & 0 \\ 0 & 0 & 0.9056 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9467 & -0.0046 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 20.39 & -0.46 & -0.24 & -0.71 \\ 0.12 & -2.7 & 0.01 & 0.03 \\ -64.69 & -75.62 & 0.6 & 3.23 \\ 0 & 0 & 0.19 & 3.66 \\ 0 & 0 & 23.6 & 5.6 \\ 0 & 0 & 3.94 & -41.41 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$y = Cx$$
$$C = I_{8\times 8}$$

For fractional LQG design purposes, the nonlinear dynamics are trimmed at $f = 15^{o}$ and p, q, r, and q set to zero. Since u, v, and w do not enter into the nonlinear term in equation 14, this amounts will become linear around with all remaining states set to zero. The goal is to control the system at steady state turn by going through a 60^{o} roll.

In the next section, both roll angle and pitch angle is controlled and robustness of the system is studied.

VI. Robustness Study of Fractional LQG

For an LQG-controlled system with a combined Kalman filter and LQR control law there are no guaranteed stability margins. This was brought starkly to the attention of the control community by Doyale (1978). He showed, by example, that there exist LQG combinations with arbitrarily small gain margins.

In this paper, also by example, we will compare the robustness of fractional order LQG with integer order LQG in the presence of coprime uncertainty. This uncertainty description is surprisingly general, it allows both zeros and poles to cross into right-half plane, and has proved to be very useful in applications [3].

One important uncertainty description can be shown as:

$$G_p = (M_l + \Delta M)^{-1}(N_l + \Delta N)$$
$$\left\| \begin{bmatrix} \Delta N & \Delta M \end{bmatrix} \right\|_\infty \le \varepsilon \qquad (15)$$

Where $G = M_l^{-1} N_l$ is a left coprime factorization of the nominal plant.

VII. Simulation Results:

This section provides the simulation results of fractional LQG controller for the airframe system. The MATLAB SIMULINK package and Optimization Toolbox were used to simulate the system under control. In these simulations we consider:

$$E[v_k v_k^T] = 0.3 I_{8\times 8}, E[w_k w_k^T] = 0.3 I_{8\times 8}$$

Fractional Kalman filter parameters used in this simulation are:

$$P_0 = 100 I_{8\times 8}, Q = 0.3 I_{8\times 8}$$
$$R = 0.3 I_{8\times 8}$$

The resulting roll angle is shown in figure3. In this figure the step response of both classical LQG and fractional LQG is plotted. As can be seen, the response quality of fractional order controller is much better than classical LQG. Figure4 is the output signal of q with both classical LQG and fractional LQG. The most important contribution of this proposed method is enhancing the robustness in the presence of uncertainty in fractional LQG. As can be seen in figure5, classical LQG losses its robustness with $\Delta = \begin{bmatrix} \Delta N & \Delta M \end{bmatrix} = diag\{0.2\}$ which it means that system is robust stable with 20% uncertainty. On the other hand, by implementing fractional LQG system remains robust with about 45% uncertainty, as can be seen in figure 5. In this study, coprime uncertainty is used because this model of uncertainty is the most general form.

VIII. Conclusion

In this paper, fractional derivative concept is implemented to design fractional order LQG controller. The proposed controller is used to control airframe system. We briefly describe the fractional LQR design and fractional Kalman filter and LQG as a combination of optimal feedback control and optimal state estimation. In this paper, the novel objective function for designing optimal state feedback is presented which is significant in our benchmark problem both to resist against extreme events and also to concern the passengers comfort level.

Finally, the stability of the proposed method in the presence of uncertainty is studied. The robustness of classical LQG and fractional LQG are compared with the coprime factor uncertainty model. As a result, fractional LQG controller improves both transient response and also robustness of the closed-loop system.

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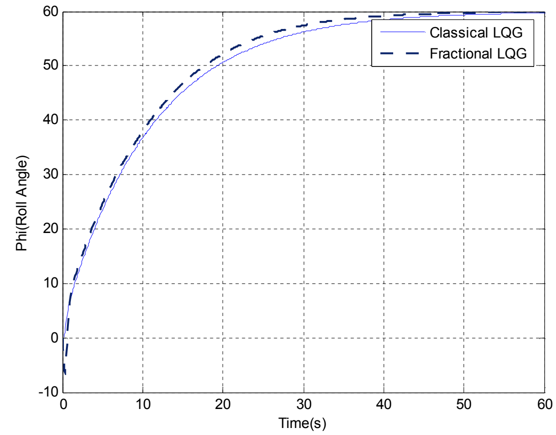

Figure 3: Roll angle output

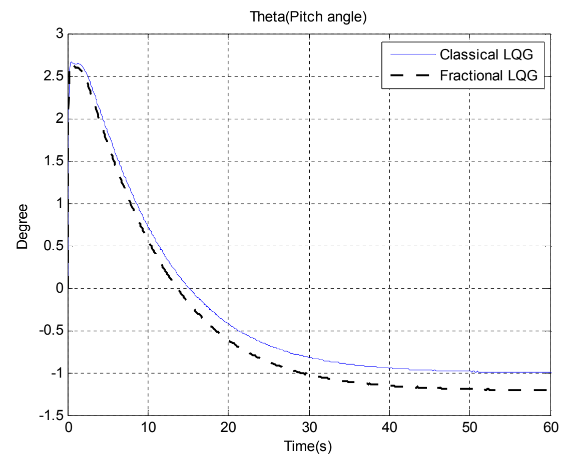

Figure4: Pitch angle output

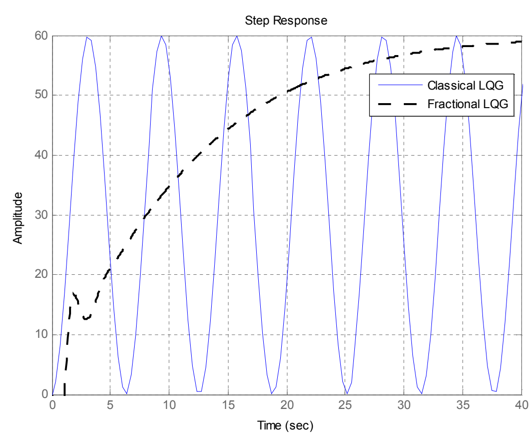

Figure 5: Classical LQG and Fractional LQG in roll angle