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On shadowing: Ordinary and ergodic ☆

Abbas Fakhari [a,*], F. Helen Ghane [b]

[a] School of Mathematics and Computer Science, DUBS, Damghan, Iran
[b] Department of Mathematics, Ferdowsi University, Mashhad, Iran

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ABSTRACT

The aim of this paper is to introduce the notion of ergodic shadowing for a continuous onto map which is equivalent to the map being topologically mixing and has the ordinary shadowing property. In particular, we deduce the chaotic behavior of a map with ergodic shadowing property. Moreover, we define some kind of specification property and investigate its relation to the ergodic shadowing property.

1. Introduction

The notion of shadowing plays an important role in the general qualitative theory of dynamical systems. It was developed intensively in recent years and became a significant part of dynamical systems containing a lot of deep connections to the notions of stability and chaotic behavior (e.g., [8,10,12,14]). Topological mixing and specification are two another formal ways to get some special dynamical properties and also chaos in global sense [2,11]. Here, we investigate the dynamical property which imply the mentioned known conditions at the same time.

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous map. For any two open subsets U and V of X, put

$$N(U, V, f) = \{m \in \mathbb{N};\ f^m(U) \cap V \neq \emptyset\}.$$

When there is no ambiguity, we denote it by $N(U, V)$. We say that f is *topologically transitive* if for any two open subsets U and V of X, $N(U, V) \neq \emptyset$. A mapping f is *weakly mixing* if $f \times f$ is transitive on $X \times X$. *Topological mixing* means that for any two open subsets U and V, the set $N(U, V)$ contains any natural number $n \geqslant n_0$, for some fixed n_0.

For $\delta > 0$, a sequence $\{x_i\}_{1 \leqslant i \leqslant b}$ is called a *δ-pseudo orbit* of f for any $1 \leqslant i \leqslant b$, $d(f(x_i), x_{i+1}) < \delta$. If $b < \infty$, then we say that the finite δ-pseudo orbit $\{x_i\}_{1 \leqslant i \leqslant b}$ of f is a δ-chain of f from x_1 to x_b of length b. A point $x \in X$ is called a *chain recurrent point* of f if for every $\delta > 0$, there is a δ-chain from x to x. The set of all chain recurrent points of f is denoted by $CR(f)$. A sequence $\{x_i\}_{1 \leqslant i \leqslant b}$ is said to be ϵ-shadowed by a point x in X if $d(f^i(x), x_i) < \epsilon$ for each $1 \leqslant i \leqslant b$. A mapping f is said to have *shadowing property* if for any $\epsilon > 0$, there is a $\delta > 0$ such that every δ-pseudo orbit of f can be ϵ-shadowed by some point in X.

A mapping f is called *chain transitive* if for any two points $x, y \in X$ and any $\delta > 0$ there exists a δ-chain from x to y. The mapping f is called *chain mixing* if for any two points $x, y \in X$ and any $\delta > 0$, there is a positive integer N such that for any integer $n \geqslant N$ there is a δ-chain from x to y of length n.

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* Corresponding author.
E-mail addresses: fakhari@dubs.ac.ir (A. Fakhari), ghane@math.um.ac.ir (F.H. Ghane).

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The term chaos in a connection with a map was first used by Li and Yorke [13]. A continuous onto mapping f is said to be *chaotic in the sense of Li–Yorke* if X contains an uncountable set such that for any two points x and y belong to it we have

$$\liminf_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0, \qquad \limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) > 0.$$

A point x is called *Lyapunov stable* point if for any $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(f^n(x), f^n(y)) < \epsilon$ for any natural number n. A mapping f is called *chaotic in the sense of Auslander–Yorke* if it is transitive and has no Lyapunov stable points [5]. Devaney's definition of chaos [7] is Auslander–Yorke's with an extra assumption which is the existence of a dense set of periodic points (see also [9]).

2. Some related terminology

Given a sequence $\xi = \{x_i\}_{i\in\mathbb{N}}$, put $\mathsf{Npo}(\xi, \delta) = \{i; d(f(x_i), x_{i+1}) \geqslant \delta\}$ and $\mathsf{Npo}_n(\xi, \delta) = \mathsf{Npo}(\xi, \delta) \cap \{1, \dots, n-1\}$. For a sequence ξ and a point x of X, put $\mathsf{Ns}(\xi, x, \delta) = \{i; d(f^i(x), x_i) \geqslant \delta\}$ and $\mathsf{Ns}_n(\xi, x, \delta) = \mathsf{Ns}(\xi, x, \delta) \cap \{1, \dots, n-1\}$. A sequence ξ is a *δ-ergodic pseudo orbit* if the set of times where the δ-pseudo orbit condition fail, i.e. $\mathsf{Npo}(\xi, \delta)$, has density zero, it means that

$$\lim_{n\to\infty} \big|\mathsf{Npo}_n(\xi, \delta)\big|/n = 0,$$

where $|.|$ denotes the cardinal number. A δ-ergodic pseudo orbit ξ is said to be *ϵ-ergodic shadowed* by a point x in X if

$$\lim_{n\to\infty} \big|\mathsf{Ns}_n(\xi, x, \epsilon)\big|/n = 0.$$

In other words, in the ergodic shadowing, the set of times where shadowing fail, i.e. $\mathsf{Ns}(\xi, x, \epsilon)$, has density zero. A mapping f has *ergodic shadowing property* if for any $\epsilon > 0$ there is a $\delta > 0$ such that any δ-ergodic pseudo orbit of f can be ϵ-ergodic shadowed by some point in X.

A mapping f has specification property if one can approximates distinct pieces of orbits by a single periodic orbit. By some little modifications, we say that a mapping f has *pseudo-orbital specification property* if for any $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $K = K(\epsilon) > 0$ such that for given nonnegative integers

$$a_0 \leqslant b_0 < a_1 \leqslant b_1 < \cdots < a_n \leqslant b_n$$

with $a_{j+1} - b_j \geqslant K$ and δ-pseudo orbits $\xi_0, \dots, \xi_n$ with $\xi_j = \{x_{(j,i)}\}$, $i \in I_j = [a_j, b_j] \subseteq \mathbb{N}$ and $0 \leqslant j \leqslant n$, there is some point $x \in X$ such that $d(f^i(x), x_{j,i}) < \epsilon$, for $i \in I_j$ and $0 \leqslant j \leqslant n$.

The main goal of the paper is the following theorem which will be proved by a few lemmas.

Theorem A. *Let f be a continuous onto map of a compact metric space X. For the dynamical system (X, f), the following properties are equivalent*:

(a) *ergodic shadowing,*
(b) *shadowing and chain mixing,*
(c) *shadowing and topological mixing,*
(d) *pseudo-orbital specification.*

In this paper, any mapping assumed to be surjective.

3. Topological mixing

Lemma 3.1. *Any mapping with the ergodic shadowing property is chain transitive.*

Proof. Suppose that a mapping f has the ergodic shadowing property and $x, y \in X$. Given $\epsilon > 0$, let $\delta > 0$ be an ϵ modulus of ergodic shadowing. Let $a_0 = 0$ and $b_0 = 1$ and for any $k \geqslant 1$, put $a_k = b_{k-1} + k$ and $b_k = a_k + k + 1$. Choose some point $u_k \in f^{-k}(y)$. For any natural number i, if $a_k \leqslant i < b_k$ then put $x_i = f^j(x)$ where $j = i - a_k$ and if $b_k \leqslant i < a_{k+1}$ then put $x_i = f^j(u_k)$ where $j = i - b_k$. By the choice, on $[a_k\ b_k)$ we use the piece of the orbit of x beginning at x and of length k and on $[b_k\ a_{k+1})$ we use a piece of an orbit of length k which ends at y. The set $\mathsf{Npo}(\xi, \delta)$ is contained in the set $\{a_k, b_k;\ k = 0, 1, 2, \dots\}$ which has density zero. Thus, $\xi = \{x_i\}_{i\in\mathbb{N}}$ is a δ-ergodic pseudo orbit and can be ϵ-ergodic shadowed by some point z. Since $\mathsf{Ns}(\xi, z, \epsilon)$ has density zero, its complement must intersects infinitely many intervals $[a_k\ b_k)$ and infinitely many intervals $[b_k\ a_{k+1})$. So, we can find $i, j, l, s \in \mathbb{N}$ with $j < l$ such that

$$d\big(f^j(x), f^i(z)\big) < \epsilon \quad \text{and} \quad d\big(f^l(z), w\big) < \epsilon, \quad w \in f^{-s}(y).$$

Therefore, $\{x, f(x), \dots, f^{j-1}(x), f^i(z), \dots, f^{l-1}(z), w, f(w), \dots, f^{s-1}(w), y\}$ is an ϵ-pseudo orbit form x to y. □

There are examples of subshifts of finite type with the shadowing property which are not chain transitive [3]. By Lemma 3.1, such mappings do not satisfy the ergodic shadowing property.

The following result shows that the ergodic shadowing property is stronger than the ordinary shadowing property.

Lemma 3.2. *Any mapping with the ergodic shadowing property has the ordinary shadowing property.*

Proof. Let f has the ergodic shadowing property. By [2], it is enough to show that any finite pseudo orbit of f can be shadowed by a true orbit. Given $\epsilon > 0$, let $\delta > 0$ be an ϵ modulus of ergodic shadowing. Suppose $\xi = \{x_i\}_{i=1}^n$ is a finite δ-pseudo orbit of finite length. By Lemma 3.1, we can choose a δ-pseudo orbit γ from x_n to x_1. Then $\zeta = \{\xi, \gamma, \xi, \ldots\}$ is a δ-ergodic pseudo orbit and so can be ergodic shadowed by some point x. Since x ergodic shadows ζ the set $\mathrm{Ns}(\zeta, x, \epsilon)$ can't meet every ξ interval, for then it would have positive density. Hence, at least one ξ interval is entirely ϵ shadowed by a piece of the x orbit. □

One can apply the argumentation used in the case of ordinary shadowing property to prove the following statement [2].

Proposition 3.3. *If f has the ergodic shadowing property then for any natural number k, the mapping f^k has also the ergodic shadowing property.*

Lemma 3.4. *Any mapping with the ergodic shadowing property is chain mixing.*

Proof. If f has the ergodic shadowing property then by Lemma 3.1 and Proposition 3.3, f^k is chain transitive for any natural number k, this implies chain mixing. In fact, if f is not chain mixing then there are k distinct open and closed subsets $X_1, X_2, \ldots, X_k$ of X with $k > 1$ which are remained invariant by f^k. This implies that f^k is not chain transitive [1, Chapter 8]. □

Proposition 3.5. *If f has the shadowing property then f is chain mixing if and only if it is topologically mixing.*

Proof. Suppose that a mapping f has the shadowing property. Given two open subsets U and V of X, choose $x \in U$, $y \in V$ and $\epsilon > 0$ such that $B_\epsilon(x) \subset U$ and $B_\epsilon(y) \subset V$. Let $\delta > 0$ be an ϵ modulus of shadowing. For any sufficiently large positive integer n, there is a δ-pseudo orbit of length n from x to y. By shadowing property, we can find a true orbit of length n begins in U and ends in V and so $n \in N(U, V)$. □

Corollary 3.6. *Any mapping with the ergodic shadowing property is topological mixing.*

Proof. Suppose that f has the ergodic shadowing property. By Lemma 3.2, f has shadowing property and so by Lemma 3.4 and Proposition 3.5, it is topologically mixing. □

In view of the previous corollary, one can deduce the chaos in the sense of Auslander–Yorke for a map having the ergodic shadowing property. Li–Yorke chaos can be obtain in weaker situation. In fact, any weakly mixing map is chaotic in the sense of Li–Yorke [11].

Corollary 3.7. *Any mapping with the ergodic shadowing property is chaotic in the sense of Li–Yorke and Auslander–Yorke.*

4. Pseudo-orbital specification

The notion of specification was first introduced by Bowen [6]. It has been shown that if f has the specification property, then it is topologically mixing and the set of all periodic points of f is dense in X. So, any mapping with the specification property is chaotic in the sense of Devaney [2]. Here, in a stronger statement, we prove that the pseudo-orbital specification property is equivalent to the ergodic shadowing property.

Remark 4.1. Any mapping with the shadowing and specification properties has the pseudo-orbital specification property.

Let us remark that the usual specification property do not imply the shadowing property [4, Example 3.6].

Remark 4.2. If f is topologically mixing, then for any $\epsilon > 0$ there exists a natural number $N(\epsilon)$, such that for any two points x and y in X

$$N\big(B_\epsilon(x), B_\epsilon(y)\big) \supseteq \big\{N(\epsilon), N(\epsilon) + 1, \ldots\big\}.$$

Indeed, given $\epsilon > 0$ assume that $X = \bigcup_{i=0}^{m} B_{\epsilon/2}(x_i)$ and put

$$N(\epsilon) = \max_{1 \leqslant i,j \leqslant m} \min\{n; k \in N(B_{\epsilon/2}(x_i), B_{\epsilon/2}(x_j)) \text{ for all } k \geqslant n\}.$$

Lemma 4.3. *If f has the shadowing and topological mixing properties then it has the pseudo-orbital specification property.*

Proof. Let $\epsilon > 0$ be given and δ be an ϵ modulus of shadowing. Suppose that $\eta < \delta$ be a δ modulus of uniform continuity of f. Put $K(\epsilon) = N(\eta)$ and suppose that $\{\xi_0, \dots, \xi_n\}$ be a finite subset of δ-pseudo orbits in X defined on n subinterval $I_j = [a_j, b_j]$ with $a_{j+1} - b_j > K(\epsilon)$. Let $\xi_j = \{x_{(j,i)}\}$, $i \in I_j$, $1 \leqslant j \leqslant n$. By the choice of $K(\epsilon)$, for any $k = 1, 2, \dots, n$ there exists a true orbit ζ_{j-1} of length $a_j - b_{j-1}$ which begins δ close to $f(x_{(j-1,b_{j-1})})$ and ends η close to a point of $f^{-1}(x_{(j,a_j)})$. Then $\{\xi_0, \zeta_0, \xi_1, \zeta_1, \dots, \zeta_{n-1}, \xi_n\}$ is a pseudo-orbit which can be shadowed. □

Corollary 4.4. *Any mapping with the ergodic shadowing property has the pseudo-orbital specification property.*

Lemma 4.5. *Any mapping with the pseudo-orbital specification property has the ergodic shadowing property.*

Proof. Let f has the pseudo-orbital specification property and $\epsilon > 0$ be given. Choose $\delta > 0$ and K according to the ϵ as in the definition of the pseudo-orbital specification property. Let $\xi = \{x_i\}_{i \in \mathbb{N}}$ be a δ-ergodic pseudo orbit. Put $A = \{i; d(f(x_i), x_{i+1}) < \delta\}$ and choose a sequence

$$a_1 < b_1 < a_2 < b_2 < \cdots$$

of natural numbers with the following properties:

- for any n, $[a_n, b_n] \subseteq A$,
- for any n, $a_{n+1} - b_n \geqslant K$,
- $\lim_{n \to \infty} (\sum_{k=1}^{n} a_{k+1} - b_k)/b_n \to 0$.

For any n, choose a point z_n which shadows n pieces of the ergodic pseudo orbit corresponding to the n intervals

$$[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n].$$

Without loss of generality, suppose that $z_n \to x$, then the point x ergodic shadows ξ. □

Corollary 4.6. *Any mapping with the ordinary shadowing and topological mixing properties has the ergodic shadowing property.*

Corollary 4.7. *Let X be a compact connected metric space. If f is chain transitive and has the ordinary shadowing property then it has the ergodic shadowing property.*

Proof. It is known that if X is connected then any chain transitive map on X is chain mixing [1]. Now, if f is chain transitive and has the ordinary shadowing property then by Lemma 3.2, Proposition 3.5 and Corollary 4.6, f satisfies the ergodic shadowing property. □

In the following, we give an example of a mapping with the ergodic shadowing property.

Example. Let $f : [0,1] \to [0,1]$ be the tent map which is defined by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ -2x + 2 & \text{if } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

By [15], f is chain transitive and satisfy the shadowing property. By Corollary 4.7, f has the ergodic shadowing property.

Proof of Theorem A. Nothing remains to prove Theorem A, it is sufficient to assemble the obtained results. (a) ⇒ (b) is Lemmas 3.2 and 3.4, (b) ⇒ (c) is Proposition 3.5, (c) ⇒ (d) is Lemma 4.3 and finally, (d) ⇒ (a) is Lemma 4.5. □

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