Workshops of Inequalities, Harmonic, Wavelet and Fredholm

STRONG IRREGULARITY OF BOUNDED BILINEAR MAPPINGS

H. R. E. VISHKI

ABSTRACT. In this paper first we give a lower bound for the topological centres of a bounded bilinear map, and then we characterize the topological centres of certain bilinear mappings, say Banach module actions.

1. PRELIMINARIES

Suppose that $f : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{Z}$ is a bounded bilinear mapping on the normed spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ and let $\mathcal{X}^*$ and $\mathcal{X}^{**}$ be the first and second dual of $\mathcal{X}$, respectively. The adjoint of f is the bounded bilinear mapping $f^* : \mathcal{Z}^* \times \mathcal{X} \longrightarrow \mathcal{Y}^*$ defined by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle \quad (x \in \mathcal{X}, y \in \mathcal{Y}, z^* \in \mathcal{Z}^*).$$

Continuing this method, the higher rank adjoints of f can be verified by setting $f^{**} = (f^*)^*$ and so on.

The mapping f^r will be considered as the bounded bilinear mapping from $\mathcal{Y} \times \mathcal{X}$ into $\mathcal{Z}$ defined by $f^r(y, x) = f(x, y)$.

The first and second topological centers of f are defined as follows, respectively:

$$Z(f) : = \{x^{**} \in \mathcal{X}^{**}; y^{**} \longrightarrow f^{***}(x^{**}, y^{**}) : \mathcal{Y}^{**} \longrightarrow \mathcal{Z}^{**} \text{ is } w^* - \text{continuous}\}$$

$$Z^t(f) : = \{y^{**} \in \mathcal{Y}^{**}; x^{**} \longrightarrow f^{r***r}(x^{**}, y^{**}) : \mathcal{X}^{**} \longrightarrow \mathcal{Z}^{**} \text{ is } w^* - \text{continuous}\}.$$

When f is the product π of a Banach algebra $\mathcal{A}$, we usually show its topological centers by $Z(\mathcal{A}^{**})$ and $Z^t(\mathcal{A}^{**})$. It can be shown that π^{***} and π^{r***r} are really the first and second Arens products of $\mathcal{A}^{**}$ which will be denoted by $\Box$ and $\Diamond$, respectively.

The mapping f is called (Arens) regular when $f^{***} = f^{r***r}$. The Banach algebra $\mathcal{A}$ is said to be Arens regular if its product mapping is regular.

The bilinear mapping f is said to be strongly left (resp. right) irregular if $Z(f) = \mathcal{X}$ (resp. $Z^t(f) = \mathcal{Y}$). The subject of Arens regularity of bilinear mappings are investigated in [1, 2, 4, 5, 6, 8].

Key words and phrases. Arens product, bounded bilinear map, Banach module action, topological centre, second dual.

2. Main results

A bounded bilinear mapping $f : \mathcal{X} \times \mathcal{A} \longrightarrow \mathcal{X}$ is said to be approximately unital if there exist a bounded net $\{e_\alpha\}$ in $\mathcal{A}$ such that $\lim_\alpha g(x, e_\alpha) = x$, for all $x \in \mathcal{X}$. We commence with the following result which describes the topological centres of such a mapping.

Theorem 2.1. *For every approximately unital bounded bilinear mapping $f : \mathcal{X} \times \mathcal{A} \longrightarrow \mathcal{X}$ on normed spaces $\mathcal{A}$ and $\mathcal{X}$, $Z(f^*) = \mathcal{X}^*$ and $Z^t(f^*) = \mathfrak{M}_\mathcal{X}$; In which, $\mathfrak{M}_\mathcal{X} := \{x^{**} \in \mathcal{X}^{**} : J_{\mathcal{X}^{**}}(x^{**}) = (J_\mathcal{X})^{**}(x^{**})\}$.*

As an immeiate consequence we have:

Corollary 2.2. *Let $g : \mathcal{X} \times \mathcal{A} \longrightarrow \mathcal{X}$ be an approximately unital bounded bilinear mapping on normed spaces $\mathcal{A}$ and $\mathcal{X}$, then g^* is regular if and only if $\mathcal{X}$ is reflexive.*

The next result studies the strong irregularity of π_1^{r*} and π_2^*, in which π_1 and π_2 are Banach module actions.

Theorem 2.3. *Let $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ be approximately unital left and right Banach $\mathcal{A}-$modules, respectively. Then*

$$Z({\pi_1}^{r*}) = \mathcal{X}^* = Z({\pi_2}^*), \quad \text{and} \quad Z^t(\pi_1^{r*}) = \mathfrak{M}_\mathcal{X} = Z^t(\pi_2^*);$$

in particular, π_1^{r*} and π_2^* are left strongly irregular.

As an straightforward application of the latter theorem we have the next one which is a generalization of a result of [6].

Corollary 2.4 (See [6, Corollary 2.4]). *For the multiplication π of a Banach algebra $\mathcal{A}$ having a right (respectively, left) bounded approximate identity, π^* (respectively, π^{r*}) is left strongly regular; that is, $Z(\pi^*) = \mathcal{A}^*$ (respectively, $Z(\pi^{r*}) = \mathcal{A}^*$.)*

As another application of Theorem 2.3 we deduce the next result of [8], (which in turn is a generalization of [5, Proposition 4.5])

Corollary 2.5 ([8, Proposition 3.6]). *Let $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ be approximately unital left and right Banach $\mathcal{A}-$modules, respectively. Then the following assertions are equivalent:*

(i) π_1^{r*} is regular;

(ii) π_2^* is regular;

(iii) $\mathcal{X}$ is reflexive.

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[1] Department of Pure Mathematics and Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran.

E-mail address: vishki@ferdowsi.um.ac.ir