ISSN 0940-5151

Economic Quality Control
Vol 24 (2009), No. 2, 263 – 267

A Note on Maximum Entropy in Queueing Problems

G. R. Mohtashami Borzadaran

Abstract: The main tool of information theory is the Shannon entropy introduced by C.E. Shannon in 1948. This paper, concentrates on obtaining probabilistic models for a queueing system using the principle of maximum entropy subject to certain constraints. Some properties of queueing systems following the maximum entropy approach are stated.

AMS classification : Primary 62B10; Secondary 94A17
Keywords: Shannon's entropy, maximum entropy principle, queueing theory, characterization, information theory, birth and death process.

1 Introduction

Information theory is usually considered as being initiated by C.E. Shannon [10]. During the last fifty years or so, a number of research papers, and monographs discussing and extending Shannon's original work have appeared[12]. The maximization of the entropy in a class of distributions subject to certain constraints has captured an important role in statistical theory. Kagan, Linnik & Rao [4] gave a maximum entropy characterization for various distributions. Kapur [5] and Athreya [1] used the maximum entropy principle to characterize a large number of discrete and continuous distributions under certain constraints.

Queues are formed whenever individuals have to wait for receiving service. A queue may be formed either by human beings or by machines. Waiting in queues may be very costly in time and money and a scientific study of queues is essential in order to reduce cost for given waiting time or reduce waiting time for given cost. Queuing theory is concerned with the following problems:

- Situation: The situation is described by probability distributions of the inter-arrival time and the service time, the queueing discipline, and other features like limited waiting space, persons not joining a queue if they find it too long and persons losing patience and leaving the queue after waiting in it for some time.
- Problem: The problem is to find performance distributions like, queue size distribution, waiting time distribution and busy period distribution.

[12] See for example, Mathai & Rathie [9], J. N. Kapur [6], Kullback [8], etc.

The maximum entropy principle is applicable to select appropriate probability distributions for a queueing situation. For example, some moments of the inter-arrival time or service time or queue size waiting time distribution may be given. In each of these cases, we can use the maximum entropy principle to select a corresponding probability distribution based only on the available information. Sometimes the distributions obtained by the usual methods applied in queueing theory are rather complicated. In such cases the principle of maximum entropy can be used to approximate these distributions by simpler ones of the exponential family. The contributions of Guiasu [3] and Kouvatsos [7] contain examples showing the role of maximum entropy in queueing theory.

In this paper, we will discuss some situations investigated in queueing theory by means of the entropy approach.

2 Entropy and Maximum Entropy in Queueing systems

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space and f be a measurable function from Ω to $\mathbb{R}^+$ such that $\int_\Omega f d\mu = 1$. The Shannon entropy (or simply the entropy) of f relative to μ is defined if $f \ln f$ is integrable by:

$$H(f,\mu) = -\int_\Omega f \ln f d\mu \tag{1}$$

with $f \ln f = 0$ if $f = 0$.

Let X be a random variable with density function f, then we refer to H as the entropy of X and denote it by H_X. In the case μ is a counting measure on $\mathbb{N}$, (1) leads us to the original version of entropy[13]that was introduced by Shannon [10] as:

$$H_X = -\sum_{i=1}^{n} p_i \ln p_i \tag{2}$$

where $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$.

In view of Havrda Charvat [2], we also consider the Havrda Charvat entropy entropy of f relative to μ in the case that μ is a counting measure on $\mathbb{N}$. The Havrda Charvat entropy is defined by:

$$HC_X = 1 - \int_\Omega f^2 d\mu \tag{3}$$

Kagan, Linnik & Rao [4] gave the general class of distributions for which Shannon's entropy is maximized, via the following theorem:

[13] Shannon derived the concept of entropy only for discrete probability distributions for which the entropy represents a measure of uncertainty. In case of absolute continuous probability distributions, there are some difficulties with respect to its interpretation; for details we refer to Guiasu [3] and Cover and Thomas [11].

Theorem 2.1 *Let X be a random variable with density function f (w.r.t. the measure μ) such that $f(x) > 0$ for $\{x \,|\, a < x \leq b\}$ and 0 elsewhere. Let further $h_1, h_2, ..., h_n, ..$ be integrable functions on $\{x \,|\, a < x \leq b\}$ satisfying the conditions:*

$$\int\limits_{\{x\,|\,a<x\leq b\}} h_i(x) f(x) d\mu(x) = \lambda_i \quad \textit{for } i = 1, 2, \ldots \tag{4}$$

with $\lambda_1, \lambda_2, \lambda_3, \ldots$ being constants. Then the maximum entropy is attained by the distribution with density function f of the form

$$f(x) = e^{c_0 + c_1 h_1(x) + c_2 h_2(x) + \ldots} \tag{5}$$

whenever there are $c_0, c_1, \ldots$ so that (4) is met[14].

Athreya [1] showed the unique maximizer of the entropy via the following corollary for using in stochastic processes:

Lemma 2.1 *Let (S, S) be a measurable space, $h : S \times \Omega \to \mathbb{R}$ and $\lambda : S \to \mathbb{R}$ be measurable. Let ν be a measure on (S, S) and $c : S \to \mathbb{R}$ be measurable such that*

$$\int\limits_{\Omega} e^{\int\limits_S h(s,\omega) c(s) \nu(ds)} \mu(d\omega) < \infty \tag{6}$$

$$\int\limits_{\Omega} h(s,\omega) e^{\int\limits_S h(s',\omega) c(s') \nu(ds')} \mu(d\omega) = \lambda(s) \tag{7}$$

for all $s \in S$ and

$$F_\lambda = \left\{ f : \int\limits_{\Omega} f d\mu = 1, \int\limits_{\Omega} f(\omega) h(s,\omega) d\mu = \lambda(s) \quad \textit{for } s \in S \right\} \tag{8}$$

Then, if

$$\sup\{H(f,\mu) : \; f \in F_\lambda\} = H(f_0, \mu) \tag{9}$$

(note that H is defined and finite), we have

$$f_0 = k e^{\int\limits_S h(s,\omega) c(s) \nu(ds)} \tag{10}$$

as the maximizer of the entropy where k is constant.

Let $p_n(t)$ denote the probability of the event that there are n persons in the queue at time t; moreover, let n_0 denote the number of persons in the queue at time $t = 0$.

- Let λ and μ denote the arrival and service rates in the steady state of a queueing system. Then H_n increases from 0 to ∞ as $\rho = \frac{\lambda}{\mu}$ increases from zero to 1. HC_n increases from 0 to 1 as ρ increases from zero to 1.

[14] Note that the constants λ_i cannot be selected arbitrarily, but depend on one another and must meet many very restrictive conditions.

- For a birth-death process being not in the steady state with $\lambda = \mu$ and $n_0 = 1$, the entropy H_n increases in t as long as $\lambda t < 1$ and decrease for $\lambda t > 1$. The maximum entropy is attained when $\lambda t = 1$; HC_n is also maximum when $\lambda t = 1$.
- For a birth-death process being not in the steady state with $\lambda < \mu$, the Havrda Charvat entropy HC_n is always increasing from 0 to 1.
- For a pure birth process with birth rate λ, the entropy H_n increases monotonically in t from 0 to ∞ and is a concave function of λt and t; HC_n increases with t, too.
- Even for simple processes such as Poisson processes, it is not easy to express explicitly H_n and HC_n. However, the entropies can be obtained numerically.
- When the waiting space capacity is infinite and the mean of the system size is prescribed, the entropy is maximized by the geometric distribution with the parameter value depending on the mean system size.
- When the waiting space capacity is given by N persons and the mean of the system size is prescribed, the entropy is maximized by a form of the truncated geometric distribution.

.

The usual approach is to first select appropriate arrival and service distributions and then to use the structure of birth-death processes to derive the performance distributions. The maximum entropy approach to queueing theory starts with finding some moments of the performance distributions by empirical observations. Subsequently, the performance distributions are determined by means of the maximum entropy principle on the basis of the knowledge about these moments without the need to specify the arrival and service distributions.

The number of moments that can be determined increases with increasing empirical experience. An increased number of known moments yields a maximum entropy distribution which is better adapted to the real but unknown probability distribution.

3 Conclusions and Further Works

This paper aims to show the relevance of the concept of entropy and the principle of maximum entropy to a further development of queueing theory. At present, our research focuses on two issues:

- On a comparison of different measures of entropy (Shannon entropy , Renyi entropy and Havrda Charvat entropy), which could be used for solving queueing problems.
- On developing numerical algorithms for the analysis of the behavior of the entropy of complicated queueing systems.

Acknowledgment: The partial support from the *Ordered and Spatial Data Center of Excellence* of Ferdowsi University of Mashhad is acknowledged.

References

[1] Athreya, K. B. (1994): Entropy maximization. *ISU Stat. Dept. Tech. Report, IMA preprint*, 1231 .

[2] Havrda, J. H. & Charvat, F. (1967): Quantification methods classificatory processes, concept of structural α entropy. *Kybernetica* 3, 30-35.

[3] Guiasu, S.(1986): Maximum entropy condition in queueing theory. *J. Opl. Res. Soc.*37, 293-301.

[4] Kagan, A. M., Linnik, Yu. V. & Rao, C. R. (1973): *Characterization Problems in Mathematical Statistics.* John Wiley & Sons, New York.

[5] Kapur, J. N. (1989): *Maximum Entropy Models in Sciences and Engineering*. John Wiley & Sons, New York.

[6] Kapur, J. N. (1994): *Measure of Information and their Applications*. John Wiley & Sons, New York.

[7] Kouvatsos, D. D.(1988): A Maximum entropy analysis of the $G/G/1$ queue at equilibrium. *J. Opl. Res. Soc.* 39, 183-200.

[8] Kullback, S. (1959): *Information and Statistics.* John Wiley & Sons, New York.

[9] Mathai, A. M. & Rathie, P. N. (1975): *Basic Concepts in Information Theory and Statistics.* Halsted Press, New York.

[10] Shannon, C. E. (1948): A mathematical theory of communication. *Bell System Technical Journal* 27, 379-423.

[11] Cover, T. and Thomas, J. (2006): *Elements of Information Theory.* John Wiley & Sons, New York.

G. R. Mohtashami Borzadaran
Department of Statistics and Ordered and Spatial Data Center of Excellence
Ferdowsi University of Mashhad
Mashhad-Iran
gmb1334@yahoo.com