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SOME CONCEPTS OF NEGATIVE DEPENDENCE FOR BIVARIATE DISTRIBUTIONS WITH APPLICATIONS

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ABSTRACT. In this paper, some concepts of negative dependence for bivariate distributions, especially hazard and local negative dependence (HND,LND) are studied. The Clayton-Oakes, φ and γ measures of association and relationship of HND with this measures is studied. In addition, various examples illustrate the usefulness of these notions in some family of distributions.

1. INTRODUCTION

Let X and Y be absolutely continuous random variables having joint density $f(x,y)$ and survival function $\bar{F}(x,y)$. Basu [2] introduced bivariate hazard function, $r(x,y) = f(x,y)/\bar{F}(x,y)$. In the independent case the bivariate hazard function is equal to product of conditional hazard functions, $\frac{\partial}{\partial x}[-\log \bar{F}(x,y)]$ and $\frac{\partial}{\partial y}[-\log \bar{F}(x,y)]$. If equality failed we deal with dependent (positive or negative) random variables. In this paper we used notions of negatively hazard and local dependence, say HND, LND, and have investigated relationship between this concepts with some other concepts of dependence. More details about notions of dependence are in Lehmann[13] ,Karlin[12], Esary and Proschan[4], Joe[9] and Shaked and Shanthikumar[17]. Oluyede[14], [15] has obtained some properties and inequalities for positively hazard and local dependence. We have obtained some measures of association, like θ-measure (known as Clayton-Oakes measure), φ-measure and γ-measure, and have connected these measures with HND and LND.

Let (X,Y) be an absolutely continuous random vector having distribution (survival) function $F(\bar{F})$. In the next sections we need the following definitions.

DEFINITION 1.1. Absolutely continuous random variables X and Y having a joint density function $f(x,y)$ are hazard negative (positive)dependence, HND(HPD), if and only if

$$\frac{f(x,y)}{\bar{F}(x,y)} \leq (\geq) \int_x^\infty \frac{f(u,y)du}{\bar{F}(x,y)} \int_y^\infty \frac{f(x,v)dv}{\bar{F}(x,y)} \qquad (1)$$

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where $\frac{f(x,y)}{\bar{F}(x,y)}$ is the bivariate hazard rate function, $\bar{F}(x,y)$ is bivariate reliability function and

$$\int_x^\infty \frac{f(u,y)du}{\bar{F}(x,y)} = \frac{\partial}{\partial y}[-\log \bar{F}(x,y)], \quad and \quad \int_y^\infty \frac{f(x,v)dv}{\bar{F}(x,y)} = \frac{\partial}{\partial x}[-\log \bar{F}(x,y)]$$

are conditional hazard functions.

Definition 1.2. Absolutely continuous random variables X and Y having a joint density function $f(x,y)$ are locally negative (positive) dependence, LND(LPD), if and only if

$$F(x,y)f(x,y) \le (\ge) \int_{-\infty}^x f(u,y)du \int_{-\infty}^y f(x,v)dv, \tag{2}$$

where $F(x,y)$ is joint cumulative distribution of X and Y.

Definition 1.3. A non-negative function h on A^2 , where $A \subseteq I\!R$, is reverse rule of order 2 (RR_2) if for all $x_1 < x_2$ and $y_1 < y_2$, with $x_i, y_j \in A$ $i = 1,2$ $j = 1,2$

$$h(x_1,y_1)h(x_2,y_2) \le \ h(x_1,y_2)h(x_2,y_1). \tag{3}$$

Definition 1.4. Let X and Y be continuous random variables.Then X and Y are right corner set decreasing, $RCSD$, if

$$P(X > x, Y > y | X > x', Y > y') \tag{4}$$

is decreasing (non-increasing) in x' and in y' , for all x and y .

Definition 1.5. Let X and Y be continuous random variables.Then X and Y are left corner set increasing, $LCSI$, if

$$P(X \le x, Y \le y | X \le x', Y \le y') \tag{5}$$

is increasing (non-decreasing) in x' and in y' , for all x and y .

Definition 1.6. Let $F_\theta(x)$ be a family of distribution functions. This family is called monotone decreasing likelihood ratio, (MDLR)(monotone increasing likelihood ratio, (MILR)) if for all $\eta > \theta$, $\frac{F_\eta(x)}{F_\theta(x)}$ is decreasing (increasing) in x.

Definition 1.7. (Holland and Wang[8]) Suppose that the mixed partial derivative of $h(x,y)$ exists and h is defined on a Cartesian product set. Then local dependence function, $\gamma_h(x,y)$, define as follows

$$\gamma_h(x,y) = \frac{\partial^2 Logh(x,y)}{\partial x \partial y} = \frac{1}{h(x,y)}\{h^{11}(x,y) - \frac{h^{10}(x,y)h^{01}(x,y)}{h(x,y)}\}, \tag{6}$$

where $h^{ij} = \frac{\partial^{i+j}h(x,y)}{\partial x^i \partial y^j}$, $\ i,j = 0,1.$

Remark 1.8. Let X and Y be continuous random variables with joint distribution (survival) function F $(\bar{F})$. Then

a: $HND(X,Y)(HPD(X,Y)) \Leftrightarrow \gamma_{_{\bar{F}}}(x,y) \leq (\geq) 0$.
b: $LND(X,Y)(LPD(X,Y)) \Leftrightarrow \gamma_{_F}(x,y) \leq (\geq) 0$.
c: X and Y are independent if and only if equality occur in (1) or (2) or equivalently $\gamma_{_F}(x,y) = 0$ or $\gamma_{_{\bar{F}}}(x,y) = 0$.

2. MAIN RESULTS

In this section we obtain some useful results about HND and LND which show relation of these concepts with some notions of dependence.

Proposition 2.1. *Let (X,Y) be absolutely continuous random vector with distribution $F(x,y)$ and reliability function $\bar{F}(x,y)$ .Then*

a: *$\bar{F}(x,y)$ is RR_2 if and only if for all $x_1 < x_2$ and $y_1 < y_2$,*

$$P(X > x_2, Y > y_2)\ P(x_1 < X \leq x_2, y_1 < Y \leq y_2) \leq P(x_1 < X \leq x_2, Y > y_2) P(X > x_2, y_1 < Y \leq y_2). \tag{7}$$

b: *$F(x,y)$ is RR_2 if and only if for all $x_1 < x_2$ and $y_1 < y_2$,*

$$P(X \leq x_1, Y \leq y_1)\ P(x_1 < X \leq x_2, y_1 < Y \leq y_2) \leq P(X \leq x_1, y_1 < Y \leq y_2) P(x_1 < X \leq x_2, Y \leq y_1). \tag{8}$$

Proof. We proof part a. The part of b is similar. We note that $\bar{F}(x,y)$ is RR_2 , *i.e.* for $x_1 < x_2$ and $y_1 < y_2$

$$\begin{vmatrix} P(X > x_1, Y > y_1) & P(X > x_1, Y > y_2) \\ P(X > x_2, Y > y_1) & P(X > x_2, Y > y_2) \end{vmatrix} \leq 0. \tag{9}$$

It is easy to show that (9) is equivalent to

$$\begin{vmatrix} P(x_1 < X \leq x_2, y_1 < Y \leq y_2) & P(x_1 < X \leq x_2, Y > y_2) \\ P(X > x_2, y_1 < Y \leq y_2) & P(X > x_2, Y > y_2) \end{vmatrix} \leq 0. \tag{10}$$

and this is equivalent to (7), so that proof is complete. □

The following Proposition give a relationship between RR_2 and HND(LND).

Proposition 2.2. *Let (X,Y) be absolutely continuous.*

a: *If $\bar{F}(x,y)$ is RR_2 , then (X,Y) is HND.*
b: *If $F(x,y)$ is RR_2 , then (X,Y) is LND.*

Proof.

a: Let $x_1 = x$, $x_2 = x + \Delta x$, $y_1 = y$, $y_2 = y + \Delta y$ where $\Delta x, \Delta y > 0$. By using (7) and dividing the result by $\Delta x \Delta y$ and letting $\Delta x \to 0, \Delta y \to 0$, the result follows.
b: Proof is similar.

□

THEOREM 2.3. *Let X , Y be continuous random variables having distribution function $F(x,y)$ and reliability function $\bar{F}(x,y)$.*

a: *X , Y are $RCSD(X,Y)$ if and only if $\bar{F}(x,y)$ is RR_2.*
b: *X , Y $LCSI(X,Y)$ if and only if $F(x,y)$ is RR_2.*

PROOF. The first part is proved, the second is similar.
$RCSD \Rightarrow RR_2$: Let (X,Y) be $RCSD$ then by definition (1.4), for all $(x,y) \in I\!R^2$, $P(X>x,Y>y|X>x',Y>y')$ is decreasing in x' and in y' . So that for $y=-\infty$, $P(X>x|X>x',Y>y')$ decreasing in x' and in y', for all x . Therefore if $x>x'$, then $P(X>x|X>x',Y>y') = \frac{P(X>x,Y>y')}{P(X>x',Y>y')}$ is decreasing in y' and hence for $y'<y$,

$$\frac{P(X>x,Y>y)}{P(X>x',Y>y)} \leq \frac{P(X>x,Y>y')}{P(X>x',Y>y')} \tag{11}$$

which is equivalent to (3) with $h=\bar{F}$.
$RR_2 \Rightarrow RCSD$: Since $\bar{F}(x,y)$ is RR_2 hence for $x>x'$ and $y>y'$, (11) valid and so for $x>x'$ $P(X>x|X>x',Y>y') = \frac{P(X>x,Y>y')}{P(X>x',Y>y')}$ is decreasing in y' . Similarly for $y>y'$ by RR_2-property of $\bar{F}(x,y)$

$$P(Y>y|X>x',Y>y') \geq P(Y>y|X>x,Y>y')$$

i.e. $P(Y>y|X>x',Y>y')$ is decreasing in x' . Now, if $x>x'$, $y<y'$

$$\begin{aligned} P(X>x,Y>y|X>x',Y>y') &= \frac{P(X>x,Y>y')}{P(X>x',Y>y')} \\ &\leq \frac{P(X>x,Y>y)}{P(X>x',Y>y)} \\ &= P(X>x,Y>y|X>x',Y>y), \end{aligned}$$

thus $P(X>x,Y>y|X>x',Y>y')$ is decreasing in y'. Similarly for $x\leq x'$, $y>y'$, $P(X>x,Y>y|X>x',Y>y')$ is decreasing in x' . Also for $x<x'$, $y<y'$, $P(X>x,Y>y|X>x',Y>y')=1$, therefore (X,Y) is $RCSD$. □

COROLLARY 2.4. *Under the assumptions of Theorem 2.3 and Proposition 2.2*

a: *$RCSD(X,Y) \Rightarrow HND(X,Y)$.*
b: *$LCSI(X,Y) \Rightarrow LND(X,Y)$.*

THEOREM 2.5. *Let $F_\theta(x)$ and $G_\theta(y)$ be two families of distribution functions. For any mixing distribution K , consider the distribution*

$$H(x,y) = \int_\Omega F_\theta(x)G_\theta(y)dK(\theta),$$

where Ω is a Borel set in $I\!R^n$ and K is a probability measure on Ω .

*(i)***:** *If one of the family is $MILR$ and the other is $MDLR$, then $H(x,y)$ is LND.*
*(ii)***:** *If $F_\theta(x)$ and $G_\theta(y)$ are both $MDLR$ or $MILR$, then $H(x,y)$ is LPD.*

Proof. We prove part (i) . The proof of part (ii) is similar . Let $F_\theta(x)$ be $MDLR$ and $G_\theta(y)$ be $MILR$, so that for $x < x'$, $y < y'$ and $\eta > \theta$ $(\eta, \theta \in \Omega)$, we have

$$[F_\eta(x)F_\theta(x') - F_\eta(x')F_\theta(x)][G_\eta(y)G_\theta(y') - G_\eta(y')G_\theta(y)] \leq 0.$$

After some simple calculation we obtain $H(x,y)H(x',y') \leq H(x,y')H(x',y)$. Therefore the distribution function H is RR_2, and hence H is LND. $\square$

3. EXAMPLES AND MEASURES OF DEPENDENCE

In this section we first introduce the Clayton-Oakes association measure(θ-measure) and ψ- measure and drive the relationship of these measures with hazard negative dependence,then we give some examples. Clayton[3] and Oakes[16] defined the following associated measure:

$$\theta(x,y) = \frac{\bar{F}(x,y)D_{12}\bar{F}(x,y)}{D_1\bar{F}(x,y)D_2\bar{F}(x,y)}, \tag{12}$$

where $D_{12}\bar{F}(x,y) = \frac{\partial^2}{\partial x \partial y}\bar{F}(x,y)$, $D_1\bar{F}(x,y) = \frac{\partial}{\partial x}\bar{F}(x,y)$ and $D_2\bar{F}(x,y) = \frac{\partial}{\partial y}\bar{F}(x,y)$. The function $\theta(x,y)$ measures the degree of association between X and Y, and has direct relation to local dependence function, $\gamma_{\bar{F}}(x,y)$. $\theta(x,y) = 1$ if and only if $\gamma_{\bar{F}}(x,y) = 0$ *i.e* X and Y are independent, $\theta(x,y) > 1$ if and only if $\gamma_{\bar{F}}(x,y) > 0$ *i.e* X and Y have positively dependent and $\theta(x,y) < 1$ if and only if $\gamma_{\bar{F}}(x,y) < 0$ or equivalently X and Y are negatively dependent.

Let us define some quantities to formulate $\theta(x,y)$.

$$r_1(x,y) := -\frac{\partial}{\partial x}[\log \bar{F}(x,y)] = -\frac{D_1\bar{F}(x,y)}{\bar{F}(x,y)}, r_2(x,y) := -\frac{\partial}{\partial y}[\log \bar{F}(x,y)] = -\frac{D_2\bar{F}(x,y)}{\bar{F}(x,y)}$$

By using (6)we can write:

$$\frac{\partial^2}{\partial x \partial y}\log \bar{F}(x,y) = r_1(x,y)r_2(x,y)(\theta(x,y) - 1). \tag{13}$$

So, from (12) we drive;

$$r(x,y) = r_1(x,y)r_2(x,y)\theta(x,y), \tag{14}$$

where $r(x,y) = \frac{f(x,y)}{\bar{F}(x,y)}$ is Basu's failure rate. More detail about formulate of θ is found in Gupta [6], [7].

Another measure for appearance of dependence is ψ- measure which is defined as follows:

$$\psi(x,y) = \frac{P(X > x|Y > y)}{P(X > x)} = \frac{\bar{F}(x,y)}{\bar{F}_1(x)\bar{F}_2(y)} \tag{15}$$

Under the some regular conditions, the following statements are valid for ψ- measure in (14);

- $\psi(x,y) = 1 \Leftrightarrow X$ and Y are independent.
- $\frac{\partial^2}{\partial x \partial y}\psi(x,y) = \gamma_{\bar{F}}(x,y)$.
- If $\psi(x,y) > 1$ then (X,Y) is PQD.

- If $\psi(x,y) < 1$ then (X,Y) is NQD.
- If $\theta(x,y) < (>)1$ then $\psi(x,y) < (>)1$ (the converse is not true).

The following proposition gives relationship between dependence measures.

PROPOSITION 3.1. *Let (X,Y) be an absolutely continuous random vector having reliability function $\bar{F}(x,y)$. One can verify that the following are equivalent*

(i): $\theta(x,y) < 1$,
(ii): $\gamma_{\bar{F}}(x,y) < 0$,
(iii): $\frac{\partial^2}{\partial x \partial y}\psi(x,y) < 0$,
(iv): $r(x,y) < r_1(x,y)r_2(x,y)$,
(v): (X,Y) *is* HND.

PROOF. By (6), (11), (12), (13) and (14) the proposition proved immediately. $\square$

EXAMPLE 3.2. *(Farli-Gumble-Morganstern* [4] *distribution (FGM))* Consider the family bivariate distributions

$$F(x,y) = F_1(x)F_2(y)[1 + \alpha(1 - F_1(x))(1 - F_2(y))]$$

where $|\alpha| \leq 1$ and $F_1(x)$ and $F_2(y)$ are continuous distributions.

$$\gamma_{F}(x,y) \leq 0 \quad \Leftrightarrow \quad \alpha\, f_1(x)f_2(y) \leq 0 \quad \Leftrightarrow \quad -1 \leq \alpha \leq 0.$$

Therefor the above bivariate family of distributions is LND if and only if $-1 \leq \alpha \leq 0$.

EXAMPLE 3.3. In the previous example the reliability function for FGM distribution is

$$\bar{F}(x,y) = \bar{F}_1(x)\bar{F}_2(y)[1 + \alpha\, F_1(x)F_2(y)], \quad |\alpha| \leq 1$$

.

$$\gamma_{\bar{F}}(x,y) = \frac{\alpha\, f_1(x)f_2(y)}{[1 + \alpha\, F_1(x)F_2(y)]^2} \leq 0 \quad \Leftrightarrow \quad -1 \leq \alpha \leq 0,$$

therefore (X,Y) is HND if and only if $-1 \leq \alpha \leq 0$.

EXAMPLE 3.4. *(Gumbel's bivariate exponential distribution) The reliability function of Gumbel's bivariate distribution is*

$$\bar{F}(x,y) = \exp\{-\alpha_1 x - \alpha_2 y - \beta xy\}, \qquad \alpha_1, \alpha_2 > 0 \quad \text{and} \quad 0 \leq \beta \leq \alpha_1\alpha_2.$$

For $x < x'$ and $y < y'$;

$$\begin{aligned}\bar{F}(x,y)\bar{F}(x',y') & -\bar{F}(x,y')\bar{F}(x',y) \\ &= \exp\{-\alpha_1(x+x') - \alpha_2(y+y')\} \\ &\qquad \times \Big[\exp\{-\beta(xy + x'y')\} - \exp\{-\beta(xy' + x'y)\}\Big] \leq 0.\end{aligned}$$

Since $xy + x'y' \geq xy' + x'y$, hence $\bar{F}$ is RR_2, and this implies that (X,Y) is HND.

EXAMPLE 3.5. *(Ali-Mikhail-Haq distribution) Consider Ali-Mikhail-Haq[1] family of distribution*

$$F(x,y)=\frac{F_1(x)F_2(y)}{1-\beta\,\bar{F}_1(x)\bar{F}_2(y)},\quad |\beta|\le 1$$

where F_1 and F_2 are continuous distribution functions and $\bar{F}_i=1-F_i \quad$ i $=1,2$. The above family of distribution is LND if and only if $-1\le\beta\le 0$, since

$$\gamma_{_F}(x,y)=\frac{\beta\,f_1(x)f_2(y)}{[1-\beta\,\bar{F}_1(x)\bar{F}_2(y)]^2}\le 0\quad\Leftrightarrow\quad -1\le\beta\le 0.$$

REMARK 3.6. In the example 3.4 we can use the proposition 3.1 and obtain

$$\begin{aligned}
r_1(x,y) &= -\frac{\partial}{\partial x}[\log\bar{F}(x,y)]=\alpha_1+\beta y\\
r_2(x,y) &= -\frac{\partial}{\partial y}[\log\bar{F}(x,y)]=\alpha_2+\beta x\\
r(x,y) &= \frac{f(x,y)}{\bar{F}(x,y)}=(\alpha_1+\beta y)(\alpha_2+\beta x)-\beta\\
\theta(x,y) &= \frac{r(x,y)}{r_1(x,y)r_2(x,y)}=\frac{(\alpha_1+\beta y)(\alpha_2+\beta x)-\beta}{(\alpha_1+\beta y)(\alpha_2+\beta x)}
\end{aligned}$$

since $\alpha_i>0,\quad i=1,2$ and $\beta\ge 0$, therefore Proposition (3.1) implies that (X,Y) is HND.

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