Measure theory approach in sliding mode control for nonlinear systems

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Abstract

A new sliding mode control (SMC) design approach using measure theory and Lyapunov functional candidate is presented for nonlinear control problems. A Lyapunov function is supposed for designing a sliding surface (SS). In fact the problem that is considered is as follows. A state trajectory from a given initial point reaches into a given point on a sliding surface in the minimum time, and then tends to the origin (equilibrium point) along the sliding surface. A measure theory approach with embedding process is used to solve such a problem in two phases. In the first phase, after designing an appropriate SS by a suggested Lyapunove function, and using measure theoty, an embedding is constructed to solve a time optimal control problem such that the system trajectory reaches a SS in minimum time, then in the second phase, using SS, a control is designed such that the system trajectory tends to the origin along the SS. A numerical example is presented to illustrate the effectiveness of the proposed method.

Keywords: Sliding surface design, Lyapunov function, Time optimal control, Measure theory, Sliding mode control.

1. Introduction

Variable structure control systems were first studied in Russia by Emel'yanov and Barbashin in the early 1960s [1, 2]. The pioneering work did not presented outside of Russia until the mid 1970s when a book by Itkis [1] and a survey paper by Utkin [2] were published in the West.

Designing sliding modes that guarantee the desired performances has been one of the major issues in control theory. The typical approach to sliding mode design is to handle the reduced order system through the nonsingular transformation to the regular form. Also, in handling the reduced order system, many of the standard approaches to sliding mode control have been proposed, such neural network [3], fuzzy sliding control [4].

This paper is organized as follows: Section 2 addresses the designing of a sliding surface. Section 3 describes the functional space and measure theory used in this paper to solve a time-optimal control problem. In Section 4, a numerical example is given to illustrate the procedure and its validity. Finally, conclusion is presented in Section 5.

2. Sliding surface design

Assume sliding surface be as

$$s(x) = x_2 + f_2(x_1, x_2, t) = 0. \tag{2.1}$$

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The following theory shows the stability of dynamical system

$\dot{x}_1 = f_1(t, x_1, x_2\ ) + B_1 u(t)$ (2.2)

$\dot{x}_2 = f_2(t, x_1, x_2\ ).$ (2.3)

with respect to the SS (2.1).

Theorem 2.1. The dynamical system (2.2)-(2.3) is asymptotically stable if the sliding surface be as (2.1) and the system (2.2) satisfied in $\lim_{t\to\infty} f_1\big(t, x_1, -f_2(t, x_1, x_2)\big) = 0$.

Proof. To show the stability of dynamical system (2.1) on sliding surface (2.1), one can define a suitable Lyapunov function $V(x)$ from R^n to R as:

$V(x_2) = \frac{1}{2}\ {x_2}^T x_2.$ (2.4)

Then,

$\dot{V}(x_2) = {x_2}^T \dot{x_2} = {x_2}^T f_2(x_1, x_2, t),$

now, associated with the SS (2.1) we have

$\dot{V}(x_2) = -{f_2}^T(x_1, x_2, t)\ f_2(x_1, x_2, t) < 0.$

The above inequality and the condition $\lim_{t\to\infty} f_1\big(t, x_1, -f_2(t, x_1, x_2)\big) = 0$, gurantee the asymptotically stable of the system with respect to sliding surface (2.1).■

In the sequel, we select the SS as (2.1). In the next section, we briefly describe the functional space that is used in our embedding method.

3. Obtaining optimal time's reaching

Consider the dynamical system (2.2)-(2.3) with boundary value conditions:

$X(0) = X_0, X(T) = X_T,$ (3.1)

where $h = \binom{f_1}{f_2} : [0, T] \times A \times U \to \mathbb{R}^n$, $t \,\epsilon\, (0, T) \subseteq \mathbb{R}$, $X(t) = \binom{x_1}{x_2} \,\epsilon\, A \subseteq \mathbb{R}^n$ and $u(t) \,\epsilon\, U \subseteq \mathbb{R}^m$. The operating regions A and U are compact sets such that X_0, $X_T \,\epsilon\, A$. It is desired to design the control u such that the system trajectory starting from the initial state X_0 reaches to the final state X_T. An optimal time control problem is presented as following:

Minimize: $J = \int_0^T\ dt$ (3.2)

subject to:

$\dot{x}_1 = f_1(t, x_1, x_2\ ) + B_1 u(t)$ (3.3)

$\dot{x}_2 = f_2(t, x_1, x_2\ )$ (3.4)

$\int_0^T \dot{s}\ (x)\ dt = -s(0)$ (3.5)

$X(0) = X_0, X(T) = X_T,$ (3.6)

Indeed the condition (3.5) is the described form of the integral equation that force the system reaches the SS. The pair $w = \big(X(\cdot), u(\cdot)\big)$ satisfying the conditions (3.3)-(3.6) is termed admissible. The set of all admissible pairs, is denoted by W. Now, one seeks to find an optimal pair $w^* = \big(X^*(\cdot), u^*(\cdot)\big) \,\epsilon\, W$ which gives the minimum value of J defined in (3.2). Let $w = \big(X(\cdot), u(\cdot)\big)$ be an admissible pair and B an open ball containing $I \times A$ where $I = [\ 0, T\ ]$. Let $\Phi \in C'(\ B\ )$ where $C'(\ B\ )$ denotes the space of real-valued continuously differentiable functions on B. Define the function Φ^g as

$\Phi^g\big(\ t, X(\ t\ ), u(\ t\ )\big) = \frac{d}{dt}\ \Phi\big(\ t, X(t)\big) = \Phi_x\big(\ t, X(t)\big)\ h\big(\ t, X(\ t\ ), u(\ t\ )\big) + \Phi_t\big(\ t, X(t)\big),$ (3.7)

with $\big(\ t, X(\ t\ ), u(\ t\ )\big) \in \Omega = I \times A \times U$ for all $t \,\epsilon\, I$. The function Φ^g is in the space $C(\Omega)$, the set of all continuous functions on the compact set Ω. For each admissible pair,

$\int_0^T \Phi^g\big(\ t, X(\ t\ ), u(\ t\ )\big)\ \ dt = \Phi\big(\ T, X(T)\big) - \Phi\big(\ 0, X(0)\big) = \Delta\Phi \qquad \forall \Phi \in C'(\ B\ ).$ (3.8)

Let $D(I^0)$ be the space of infinitely differentiable real-valued functions with a compact support in I^0, where $I^0 = (0, T)$. Define

$$\psi_j\left(t, X(t), u(t)\right) = \frac{d}{dt}\left(\psi(t) X_j(t)\right)$$
$$= \psi'(t)\, X_j(t) + \psi(t)\, h_j\left(t, X(t), u(t)\right), \quad \forall\, \psi \in D(I^0), \quad j = 1, 2, \ldots, n. \tag{3.9}$$

Then, if $w = (X(\cdot), u(\cdot))$ is an admissible pair, for every $\psi \in D(I^0)$

$$\int_0^T \psi_j\left(t, X(t), u(t)\right) dt = 0, \; j = 1, 2, \ldots, n. \tag{3.10}$$

Let $C_1(\Omega)$ be a subspace of the space $C(\Omega)$ of all bounded continuous functions on Ω depending only on the variable t. Now, by selecting the function $f \in C_1(\Omega)$, we have;

$$\int_0^T f\left(t, X(t), u(t)\right) dt = a_f \qquad \left(f \in C_1(\Omega)\right), \tag{3.11}$$

The set of equalities (3.8) excludes the special cases (3.10) and (3.11) and provides the properties of the admissible pairs in the classical formulation of optimal control problems. In the following a transformation is developed to a non-classical problem to obtain enhanced properties in some aspects (see [5] for the details).

For each admissible pair w, there is a positive linear continuous functional Λ_w on $C(\Omega)$ such that

$$\Lambda_w : F \rightarrow \int_I F\left(t, x(t), u(t)\right) dt \qquad (F \in C(\Omega)).$$

By the Riesz representation theorem (see [6]) there exists a unique positive Borel measure μ on Ω such that

$$\int_I F\left(t, x(t), u(t)\right) dt = \int_\Omega F\, d\mu \equiv \mu(F) \qquad (F \in C(\Omega)). \tag{3.12}$$

Thus, the optimal control problem (3.2)-(3.6) is equivalent to the minimization of

$$J(\mu) = \int_\Omega d\mu \equiv \mu(1) \in \mathbb{R} \tag{3.13}$$

over the set of measures μ, associated with the admissible pair w, which satisfy

$$\begin{aligned}
&\mu(\Phi^g) = \Delta\Phi && \Phi \in C'(B) \\
&\mu(\psi_j) = 0 && \psi \in D(I^0), \quad \mathrm{j} = 1, 2, \ldots, \mathrm{n} \\
&\mu(\dot{s}) = -s(0) && \\
&\mu(f) = a_f && f \in C_1(\Omega).
\end{aligned} \tag{3.14}$$

The set $\Omega = I \times A \times U$ is covered with a grid, where the grid will be defined by taking all points in Ω as $z_j = (t_j, X_j, u_j)$. Instead of the infinite-dimensional linear programming problem (3.13)-(3.14), the following finite dimensional linear programming (LP) problem is considered where $z_j \in \omega$, in which ω is an approximately dense subset of Ω.

The finite dimensional LP problem, which approximates the action of the infinite dimensional LP problem (3.13)-(3.14) for a sufficient large integer N is as follows, for more details see [5].

$$\begin{aligned}
&\min \sum_{j=1}^N \beta_j \\
&s.t. \\
&\sum_{j=1}^N \beta_j\, \Phi_i^g(z_j) = \Delta\Phi_i && i = 1, 2, \ldots, M_1 && \Phi_i \in C'(B) \\
&\sum_{j=1}^N \beta_j\, \psi_r(z_j) = 0 && r = 1, 2, \ldots, M_2 && \psi \in D(I^0) \\
&\sum_{j=1}^N \beta_j\, \dot{s}(z_j) = -s(0) && && \\
&\sum_{j=1}^N \beta_j\, f_s(z_j) = a_{f_s} && s = 1, 2, \ldots, L && f_s \in C_1(\Omega) \\
&\beta_j \geq 0 && j = 1, 2, \ldots, N &&
\end{aligned} \tag{3.15}$$

As a final stage, from the dynamical system (3.3)-(3.4) and the boundry condition (3.6), one can obtain the state trajectory $X(\cdot)$. The reduced order system (2.3), that is, $\dot{x}_2 = f_2(t, x_1, x_2)$, can be solved by R-K formula for reaching origin from initial point C.

4. Numerical example

Consider the following control system:

$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = -x_2 - x_1 x_3 + 2\, x_1$$

$\dot{x}_3 = x_1x_2 + x_1.$
It is desired to design a control such that the trajectory starting from the initial point $B = (-3, 3.5, 0)$ reaches the point $C = (-2, 2, -2)$ on the SS in minimum time, then derive the system from C to the origin (equilibrium point) along the SS. We consider the Lyapunov function:
$V(x) = \frac{1}{2}(x_2^2 + x_3^2),$
hence
$\dot{V}(x) = x_2\,\dot{x}_2 + x_3\,\dot{x}_3 = x_2\,(-x_2 - x_1x_3 + 2\,x_1) + x_3(x_1x_2 + x_1) = -x_2^2 + x_1(2\,x_2 + x_3),$
now, the sliding surface suggested to be as $s(x) = x_1 + 2\,x_2 + x_3 = 0.$
Assume $A = [-3.05, -1.95] \times [1.95, 3.55] \times [-2.05, 0.05]$ and $U = [-5, 2]$. By solving the LP problem (3.15), the optimal time is found as t_s =0.87, i.e, after 0.87 seconds, the system reaches the sliding surface. In the next step, the SS and sub-control u_{eq} are designed such that the system is stable and the trajectories remain on the SS after t_s =0.87.

5. Conclusions

In this paper, a new numerical approach based on embedding process and measure theory techniques for solving SMC problems has been proposed. This approach is straightforward without requiring any predication or condition on the initials. In this paper an optimal SMC has been designed for time-varying nonlinear systems. However, since the method is independent from the linearity or non-linearity of the dynamical system, it can be applied to any linear and nonlinear system. A numerical example was used to support the theoretical results.

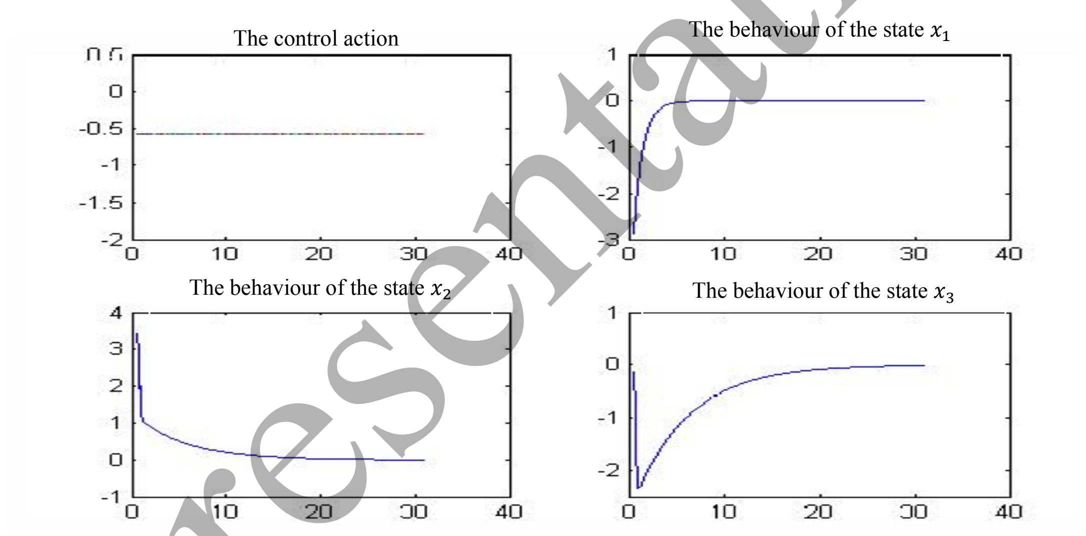

Figure 1: The action of the optimal SMC system and the behaviour of the state trajectories using this controller.

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