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An almost optimal control for time-delay systems

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Abstract: In this paper, optimal control problems for a class of time-delay nonlinear systems, subject to mixed control-state constraints are studied. The delays in the systems are on the state and/or on the control input. The theory of optimal control based on measure theory, functional analysis and linear programming, is extended in order to optimise a definite objective function, and to design an appropriate optimal control for the nonlinear time-delay systems.

Keywords: Time-delay systems, optimal control, nonlinear systems, control design.

1. Introduction

Consider the following time-delay system

$$\dot{x}(t) = f(x(t)) + g(x(t-\tau_1)) + h(x(t), x(t-\tau_1))u(t-\tau_2) + s(x(t-\tau_1))u(t)\,, \tag{1}$$

where, $x \in A \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, while A and U are closed subsets, τ_1 and τ_2 are constant known positive real numbers, $f, g \in \mathbb{R}^n$ and $h, s \in \mathbb{R}^{n\times m}$ are smooth vector fields. It is assumed that

$$x(t) = \phi(t), \quad t \in [-\tau_1, t_0]$$
$$u(t) = \theta(t), \quad t \in [-\tau_2, t_0]$$

where t_0 is the initial time and, $\phi(t)$ and $\theta(t)$ are known piecewise continuous vector functions. One may consider

$$u(t) = 0, \quad t \in [-\tau_2, t_0]$$

It is assumed that the system (1) is controllable. Define $\tau = \max\{\tau_1, \tau_2\}$. Let $t_f \geq t_0 + \tau$ be the final time. The aim is to design an optimal control u such that the state trajectories starting from $x(t_0) = x_0$ reach a given point $x(t_f) = x_f$. It is also assumed that $u(t) = [u_1(t) \quad u_2(t) \quad \ldots \quad u_m(t)]^T$ and $x(t) = [x_1(t) \quad x_2(t) \quad \ldots \quad x_n(t)]^T$ are bounded functions on the interval $[t_0, t_f]$, i.e.

$$\begin{aligned} \alpha_{1_i} &= u_{m_i}(t) \leq u_i(t) \leq u_{M_i}(t) = \alpha_{2_i}, \quad i = 1,2,\cdots,m \\ \beta_{1_j} &= x_{m_j}(t) \leq x_j(t) \leq x_{M_j}(t) = \beta_{2_j}, \quad j = 1,2,\cdots,n \end{aligned} \tag{2}$$

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where, α_{1_i}, α_{2_i}, β_{1_j} and β_{2_j} are appropriate known real numbers. This assumption is necessary for the existence of an optimal Lebesgue-measurable control using the method proposed in this paper. Consider the cost functional

$$J\big(x(.),u(.)\big) = \int_{t_0}^{t_f} (f_x^T\left(x(z)\right) f_x(x(z)) + f_u^T(u(z)) f_u(u(z)))dz\,, \tag{3}$$

where the vector field functions $f_x \in \mathbb{R}^n$ and $f_u \in \mathbb{R}^m$ are known (piecewise) continuous measurable functions. It is desired to minimise the functional (3) over the set of admissible pairs $\big(x(.),u(.)\big)$.

2. The variational formulation

The aim is to minimise (3), subject to the time-delay system (1) with the boundary conditions (2). *Select* the auxiliary nonzero continuously differentiable n-vector function

$$\eta(t) = [\eta_1(t) \quad \eta_2(t) \quad \dots \quad \eta_n(t)]^T,$$

where, $\eta_j(t) \in C^1[-\tau,\, t_f]$, $j = 1,2,\cdots n$, are arbitrary nonzero differentiable functions. Multipling of left and right sides of (1) in $\eta^T(t)$ and adding $\dot{\eta}^T(t)x(t)$ to the both sides of (1) and integrating both sides over the time interval $I = [t_0, t_f]$ yields

$$\int_{t_0}^{t_f} \eta^T(t) f\big(x(t)\big)dt + \int_{t_0}^{t_f} \eta^T(t) g(x(t-\tau_1))dt + \int_{t_0}^{t_f} \eta^T(t) h(x(t),x(t-\tau_1))u(t-\tau_2)\,dt$$
$$+ \int_{t_0}^{t_f} \eta^T(t) s(x(t-\tau_1))u(t)\,dt + \int_{t_0}^{t_f} \dot{\eta}^T(t)x(t)\,dt = \Delta\eta\,, \tag{4}$$

where, $\Delta\eta = \eta^T(t_f)x(t_f) - \eta^T(t_0)x(t_0)$.
The integral equation (4) can be expressed as

$$\int_{t_0}^{t_f} f_\eta dt + \int_{t_0}^{t_f} g_{d_\eta}(t)\,dt + \int_{t_0}^{t_f} h_{d_\eta}(t)dt + \int_{t_0}^{t_f} x_{\dot{\eta}}(t)\mathrm{dt} = \Delta\eta\ ,$$

where, $f_\eta = \eta^T(t) f\big(x(t)\big)$, $g_{d_\eta}(t) = \eta^T(t) g\big(x(\mathrm{t}-\tau_1)\big)$,

$$h_{d_\eta}(t) = \eta^T(t)\Big(h\big(x(t),x(t-\tau_1)\big)u(t-\tau_2) + s\big(x(t-\tau_1)\big)u(t)\Big),\ x_{\dot{\eta}}(t) = \dot{\eta}^T(t)x(t)$$

Let W be the set of admissible pairs. The time-delay optimal control problem now is to find an admissible pair $p \in W$ such that minimize the cost functional (3). Consider the mapping

$$\Lambda_p:\ F \in C(\Omega) \to \int_{t_0}^{t_f} F[t,x(t),u(t)]dt \in \mathbb{R}\,,$$

Now the problem is converted into the following form:

Minimise $\Lambda_p(f_0)$
subject to

$$\Lambda_p\Big(f_\eta + g_{d_\eta}(t) + h_{d_\eta}(t) + x_{\dot{\eta}}(t)\Big) = \Delta\eta\,, \tag{5}$$

where, $f_0 = f_x^T(x(t)) f_x(x(t)) + f_u^T(u(t)) f_u(u(t))$.

To find such Λ_p it is required to enlarge the domain $C(\Omega)$, and consider all positive linear functionals Λ on $C(\Omega)$ which satisfy the constraint (5), and minimise the continuous function $\Lambda \rightarrow \Lambda(f_0)$ over the enlarged domain $C(\Omega)$. These requirements guarantee the existence of an optimal solution.

Theorem 1. Let Λ be a positive functional on $C(\Omega)$. Then there exists a positive Borel measure μ on Ω such that $\Lambda(F) = \int_{\Omega} F(t,x,u)d\mu = \mu(F),$ for any $F \in C(\Omega)$.

The extended problem can be now presented as follows:

Find a positive measure μ^* within the space $M^+(\Omega)$ which minimises the functional

$$\mu \in M^+(\Omega) \rightarrow \mu(f_0) \in \mathbb{R} ,$$

subject to the following constraint:

$$\mu\left(f_\eta + g_{d_\eta}(t) + h_{d_\eta}(t) + x_{\dot\eta}(t)\right) = \Delta\eta, \qquad \text{for any } \eta \in C^1(I).$$

Then the solution is obtained by minimising a linear form over a subset of linear equalities which resulted from applying an embedding process (for more details see [2]).

3. Example

The method is now applied to the continuous nonlinear stirred tank reactor system (CSTR). This CSTR model has originally been studied by Soliman and Ray (1971).

$$\text{Minimize} \quad J(u,x) = \int_0^{0.2} \left(\|x(t)\|_2^2 + 0.01u_2^2(t)\right)dt$$

subject to

$$\dot{x}_1(t) = -x_1(t) - R(t)$$

$$\dot{x}_2(t) = -x_2(t) + 0.9\, u_2(t-\tau_2) + 0.1\, u_2(t)$$

$$\dot{x}_3(t) = -2x_3(t) + 0.25\, R(t) - 1.05\, u_1(t)\, x_3(t-\tau_1)$$

$$R(t) = R\left(t, x_1(t), x_2(t), x_3(t)\right) = \left(1 + x_1(t)\right)\left(1 + x_2(t)\right)\exp\left(\frac{25\, x_3(t)}{1 + x_3(t)}\right)$$

$$x_3(t) = -0.02 \quad t \in [-\tau_1, 0)$$
$$u_2(t) = 1 \qquad t \in [-\tau_2, 0)$$
$$x(0) = [0.49, -0.0002, 0.02]^T$$

$$x(0.2) = [0, 0, 0]^T$$

$$|u_1(t)| \leq 500 \qquad t \in [0, 0.2]$$

Minimum value of the cost functional is given by J=0.01103 and the CPU time is 38 seconds. The proposed method in this paper yields extremely significantly superier results in comparison with the existing method. For instance, in the method proposed by Göllmann *et* *al.* (2008), 16000 grid points have been considered and the minimum value of the cost functional is J=0.01197 with a very large CPU time of 63932 seconds.

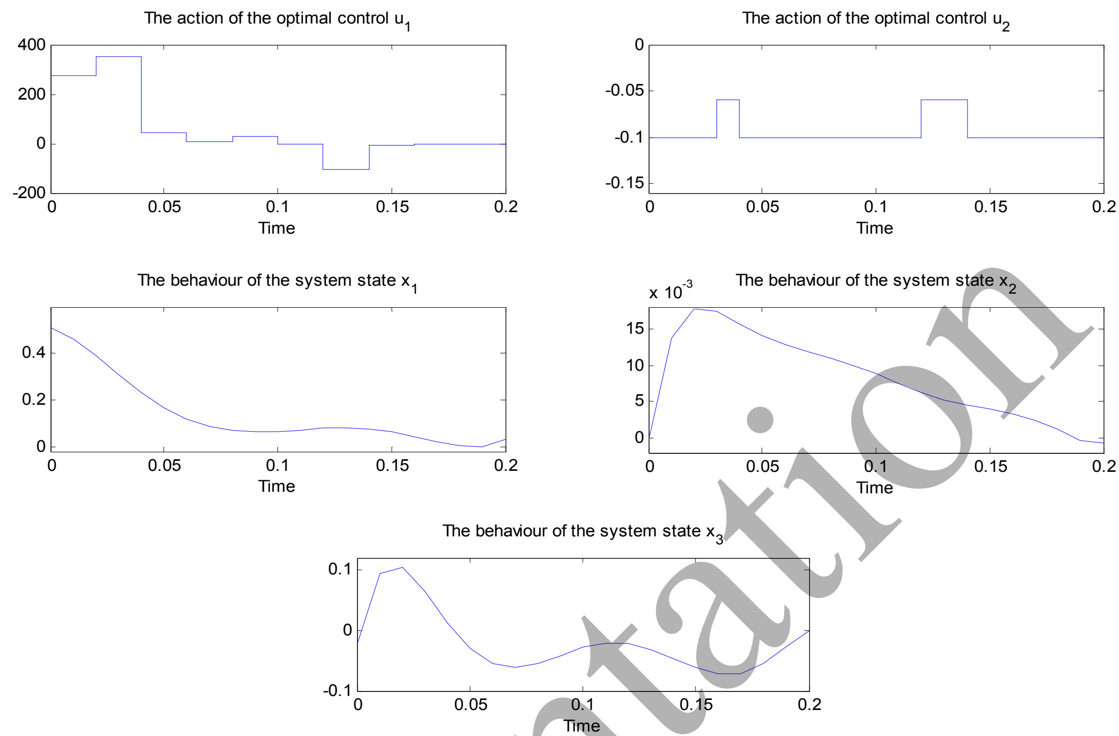

Figure 1: The stirred tank reactor system responses using the proposed optimal controller

4. Conclusions

The almost optimal control design method as presented in this paper is systematic and straightforward in comparison with the traditional and established methods. The control obtained is almost optimal and no extra initial solution is required. The significant consequence of the proposed method is that the almost optimal control, and therefore the almost optimal trajectory, is estimated in a closed form without using any iterative technique.

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