European Journal of Mechanics A/Solids 29 (2010) 348–355

Contents lists available at ScienceDirect

European Journal of Mechanics A/Solids

journal homepage: www.elsevier.com/locate/ejmsol

A novel approach for forward position analysis of a double-triangle spherical parallel manipulator

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ARTICLE INFO

Article history:
Received 19 March 2008
Accepted 11 June 2009
Available online 3 December 2009

Keywords:
Spherical parallel manipulator
Forward position analysis
Rodrigues formula
Bezout's elimination method

ABSTRACT

In this paper, we introduce a new approach for forward position analysis of a double-triangle (DT) spherical parallel manipulator. Utilizing spherical geometry of the manipulator, two coupled trigonometric equations are obtained through using special form of Rodrigues formula. Next, the two coupled equations are solved using Bezout's elimination method which leads to a polynomial of eight degree. Finally, we provide an example having eight real solutions, the polynomial thus being minimal.

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1. Introduction

Parallel manipulators are closed-loop mechanical chains, which generally have good performance in terms of accuracy, rigidity and ability to manipulate large loads. These mechanisms consist of two main bodies connected by several legs. One body is assumed to be fixed while the other is regarded as moveable and hence they are respectively called base and moving platform.

For two decades, parallel manipulators have attracted attention of more and more researches that consider them as valuable alternative design for robotic mechanisms (Chablat and Wenger, 2003; Coiffet, 1992). Compared with serial robots, parallel mechanisms offer greater structural rigidity, lower moving mass, better accuracy, higher natural frequencies, larger dynamic charge capacity, simpler modular mechanical construction as well as the possibility to mount all actuators at or near the fixed base. However, most existing parallel robots have limited and complicated workspace volume with singularities and highly non-isotropic input–output relations (Angeles, 2002).

Many industrial applications require orientating a rigid body around a fixed point such as; orienting a tool or a workpiece in machine tools, solar panels, space antenna and telescopic mechanisms, flight simulator mechanism and camera devices. A spherical manipulator is one in which the end-effector is moved on the surface of a sphere. In other words, the end-effector can rotate around any axis passing through a fixed point, center of sphere. Therefore, a spherical manipulator can be used as a device to orient the end-effector. Spherical manipulators can be either serial (Chablat and Angeles, 2003) or parallel (Gosselin and Angeles, 1989; Gosselin et al., 1994, 1995; Alizade et al., 1994; Karouia and Hervé, 2000, 2002; Gallardo et al., 2008; Di Gregorio, 2001; Pierrot and Dombre, 1990; Wiitala and Stanisic, 2000; Innocenti and Parenti-Castelli, 1993; Wohlhart, 1994). Generally, a spherical parallel manipulator is composed of three legs connecting a moving platform (end-effector) to a fixed base. The end-effector has three degrees of freedom for rotation around a fixed point. The fixed point of rotation, denoted by O, is the center of spherical parallel manipulator.

The forward position analysis (FPA) of spherical parallel manipulator has attracted much attention among researchers. Many studies (Innocenti and Parenti-Castelli, 1993; Wohlhart, 1994; Gosselin et al., 1992a,b; Gosselin and Gagné, 1995; Di Gregorio, 2003, 2000, 2004; Husain and Waldron, 1992; Huynh and Herve, 2005; Mohammadi Daniali et al., 1993) have addressed this problem for different manipulator architectures. They showed that the FPA of these mechanisms can be solved in echelon form (finding all possible solutions of the FPA).

The solution of the FPA for 3-RRR spherical parallel manipulator can be found in the literature. Gosselin et al. (1994) derived at a polynomial of eight degree and gave an example having eight real solutions, the polynomial thus being minimal.

Mohammadi Daniali et al. (1993) proposed a spherical double-triangle parallel manipulator and solved its direct kinematics problem. They derived at a polynomial of sixteen degree. Coefficients of the polynomial were too large (more than 100 pages in the most compact form). In addition, they gave an example having four real

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doi:10.1016/j.euromechsol.2009.12.001

solutions. Also, they showed that the polynomial leads to a maximum of eight real solutions, the polynomial thus not being minimal.

In this paper, we introduce a new approach for forward position analysis of a double-triangle (DT) spherical parallel manipulator that is optimum. Utilizing spherical geometry of the manipulator, we will obtain two coupled trigonometric equations using special form of Rodrigues formula (equivalent axis–angle representation). Next, we solve the two coupled equations using Bezout's elimination method, which leads to a polynomial of eight degree. Lastly, we give an example having eight real solutions, the polynomial thus being minimal.

2. Spherical double-triangle (DT) parallel manipulator

Consider a unit sphere with center at O and a fixed spherical triangle, $P_1P_2P_3$, on its surface. Next consider a second spherical triangle, $Q_1Q_2Q_3$, called moving platform (see Fig. 1). The fixed base and the moving platform are connected via three legs. Each of the three moving legs is made of CP–R–CP (curved prismatic–revolute–curved prismatic) joints. We use the term curved prismatic to denote a motion that slides on a curved path. An example of this joint used in industry is CURVILINE, which is a curved linear bearing (rollonbearings). Also, what is called curved prismatic joint in the manipulator can be viewed as a special realization of revolute joint. Therefore, the mechanism may also be viewed as a parallel wrist of architectural type 3-RRR with special realizations of revolute joints. This manipulator was introduced by Mohammadi Daniali et al. (1993), called spherical double-triangle (DT) parallel manipulator. To develop the kinematics model of this manipulator, Mohammadi Daniali et al. (1993) used principles of spherical trigonometry. However, we propose a different model for the kinematics that uses unit vectors to define the spherical structure.

To develop the kinematics model of this manipulator, we define a total of 15 unit vectors, all of which pass through the center of the sphere. We know that a spherical triangle is a figure formed on the surface of a sphere by three great circular arcs intersecting pair wise in three vertices. Therefore, the three circular arcs of the fixed spherical triangle, P_iP_{i+1}, and the three circular arcs of the moving spherical triangle, Q_iQ_{i+1}, all lie on great circles. The first curved prismatic joint which is also the motorized joint moves along circular arc, P_iP_{i+1}, located on the surface of the sphere. In practice, it is difficult to manufacture an actuated curved prismatic joint, which moves on a circular arc. However, by closer inspection, one can see that each of the motorized joints can also be viewed as a revolute joint with its axis passing through the origin of the sphere (see Figs. 2 and 3).

Consider Figs. 1 and 3. The corners of the fixed spherical triangle are denoted by P_i. Direction of OP_i can be defined by unit vector $\mathbf{v}_i$. Actuators stroke which can travel along the arc P_iP_{i+1} are defined by ρ_i. The corners of the moving spherical triangle are specified by P_i. Direction of OQ_i can be defined by unit vector $\mathbf{u}_i$. The corner angles of the moving spherical triangle are defined by $\alpha_1, \alpha_2, \alpha_3$. The arcs of the base triangle, P_iP_{i+1}, cross over the corresponding arcs of the moving spherical triangle platform, Q_iQ_{i+1}, at point R_i. The angular position of the actuators are defined by unit vector $\mathbf{r}_i$. Direction of the unit vector is defined along OR_i. Furthermore, R_i is a joint, which allows rotation about $\mathbf{r}_i$ axis as well as a rotation about the axis that passes through center of sphere, O, and is perpendicular to OQ_iQ_{i+1} plane. The nine unit vectors $\mathbf{v}_i$, $\mathbf{u}_i$ and $\mathbf{r}_i$ (for $i = 1, 2, 3$) help to describe the structure and configuration of the manipulator. Using these unit vectors, six more unit vectors will be defined, in Section 4, in order to complete the forward position analysis of the manipulator.

3. Representing rotation

Before presenting forward position analysis of spherical double-triangle, it is useful to consider matrix representation of a rotation presented by Rodrigues (Wampler, 2006). The Rodrigues form states that rotation matrix can be represented as

$$\mathbf{Q}(\mathbf{e}, \varphi) = \mathbf{I}_{3\times 3} + (1 - \cos\varphi)\mathbf{E}^2 + \sin\varphi\mathbf{E} \tag{1}$$

where the unit vector $\mathbf{e}$ is the axis of rotation and φ is the angle of rotation about the unit vector $\mathbf{e}$. Also, $\mathbf{E}$ is skew symmetric matrix that is formed by the unit vector $\mathbf{e}$ as

$$\mathbf{E} = \begin{bmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{bmatrix} \tag{2}$$

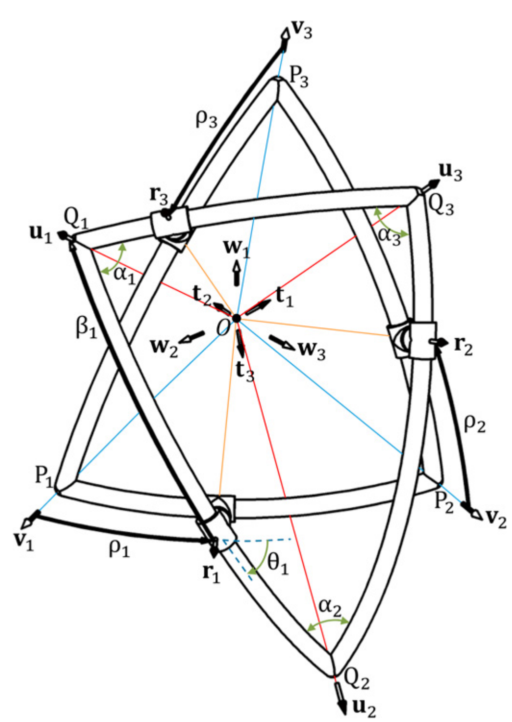

Fig. 1. General model of spherical DT.

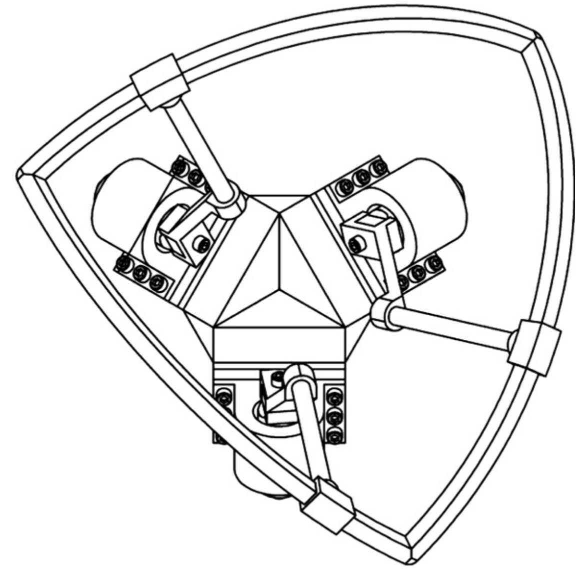

Fig. 2. Physical model of spherical DT.

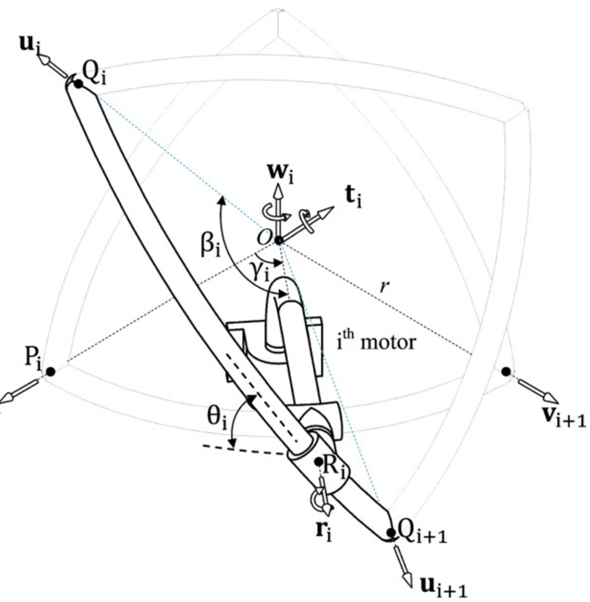

Fig. 3. Parameters description of *i*th leg.

where e_x, e_y, e_z are Cartesian components of the unit vector $\mathbf{e}$. Consider a vector $\mathbf{v}$ that is transformed by matrix $\mathbf{Q}$ ($\mathbf{e}$, φ). The resulting vector $\mathbf{v}'$, transformed vector, can be written as

$$\mathbf{v}' = \mathbf{Q}(\mathbf{e},\varphi)\mathbf{v} = \mathbf{v} + 1(1-\cos\varphi)\mathbf{E}^2\mathbf{v} + \sin\varphi\mathbf{E}\mathbf{v} \tag{3}$$

The above equation can also be written as

$$\mathbf{v}' = \mathbf{v} + (1-\cos\varphi)(\mathbf{e}\times(\mathbf{e}\times\mathbf{v})) + \sin\varphi(\mathbf{e}\times\mathbf{v}) \tag{4}$$

where sign $\times$ represents cross-product. Furthermore, we know that $\mathbf{e}\times(\mathbf{e}\times\mathbf{v}) = \mathbf{e}(\mathbf{e}^{\mathrm{T}}) - \mathbf{v}$. Therefore, we have

$$\mathbf{v}' = \mathbf{v} + (1-\cos\varphi)(\mathbf{e}(\mathbf{e}^{\mathrm{T}}\mathbf{v}) - \mathbf{v}) + \sin\varphi(\mathbf{e}\times\mathbf{v}) \tag{5}$$

Finally, we can write

$$\mathbf{v}' = \cos\varphi\mathbf{v} + (1-\cos\varphi)\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{v} + \sin\varphi(\mathbf{e}\times\mathbf{v}) \tag{6}$$

Therefore, linear operator of rotation matrix can be written as follow

$$\mathbf{Q}(\mathbf{e},\varphi) = \cos\varphi\mathbf{I}_{3\times3} + (1-\cos\varphi)\mathbf{e}\mathbf{e}^{\mathrm{T}} + \sin\varphi\mathbf{e}\times \tag{7}$$

Eq. (7) is special form of Rodrigues formula, which is equivalent to angle–axis representation (Angeles, 2002; Craig, 1989). This form will be used in the next section.

4. Forward position analysis

The forward position analysis, FPA, finds the orientation of the end-effector given the actuators position. Unlike serial manipulators, the FPA for parallel manipulators is more complex and results in a set of nonlinear equations. The nonlinear equations are usually transformed into a single high degree polynomial. For any given manipulator, there exist different modeling methods to derive the forward position problem. Among these methods, the method that results in the lowest order polynomial is the superior method. Furthermore, the modeling method is optimal if we can find an example where the number of real answers is equal to the order of the polynomial.

Mohammadi Daniali et al. (1993) used the principle of spherical trigonometry to solve direct kinematics problem of DT and derived at a polynomial of sixteen degree. Furthermore, coefficients of the polynomial were quite large (more than 100 pages in the most compact form). They also showed that the polynomial leads to a maximum of eight real solutions, thus the polynomial not being minimal.

In this section, we propose a new modeling approach for forward position problem of the spherical DT. For this purpose, we obtain two trigonometric equations by using special form of Rodrigues formula as well as the structure of the manipulator. Using Bezout's elimination method, the two trigonometric equations are transformed into a single algebraic equation. The solutions of this equation are answers of the forward position analysis.

In Section 2, we defined unit vectors ($\mathbf{v}_i$, $\mathbf{u}_i$ and $\mathbf{r}_i$) that help to describe the structure and configuration of the manipulator. Two additional unit vectors, $\mathbf{w}_i$ and $\mathbf{t}_i$ need to be defined in order to describe the position of the passive and actuated curved prismatic joints (special realization of revolute joints). It is important to note that all these unit vectors are along axes that pass through origin of the sphere. To define rotation between these unit vectors, special form of Rodrigues formula (Eq. (7)) will be used.

Consider Figs. 1 and 3. As stated before, the motion of the curved prismatic actuator can also be viewed as a revolute joint with an axis that passes through the origin of the sphere. This axis is defined by a unit vector $\mathbf{w}_i$ and is perpendicular to the plane OP_iP_{i+1}. Therefore

$$\mathbf{w}_i = \frac{\mathbf{v}_i\times\mathbf{v}_{i+1}}{\|\mathbf{v}_i\times\mathbf{v}_{i+1}\|} \quad \text{or} \quad \mathbf{w}_i = \frac{\mathbf{v}_i\times\mathbf{r}_i}{\|\mathbf{v}_i\times\mathbf{r}_i\|} \tag{8}$$

The motion of the passive curved prismatic joint can also be viewed as a special realization of revolute joint with an axis that passes through the origin of the sphere. This axis is defined by a unit vector $\mathbf{t}_i$. This unit vector is perpendicular to the plane OQ_iQ_{i+1}. Therefore, this unit vector can be defined as

$$\mathbf{t}_i = \frac{\mathbf{u}_i\times\mathbf{u}_{i+1}}{\|\mathbf{u}_i\times\mathbf{u}_{i+1}\|} \quad \text{or} \quad \mathbf{t}_i = \frac{\mathbf{u}_i\times\mathbf{r}_i}{\|\mathbf{u}_i\times\mathbf{r}_i\|} \tag{9}$$

In forward position problem, values of the actuators stroke ρ_i and radius of sphere, r, are known, therefore the angle, γ_i, which also represents motor rotation can be defined by

$$\gamma_i = \rho_i/r \tag{10}$$

As shown in Fig. 3, if we rotate unit vector $\mathbf{v}_i$ about unit vector $\mathbf{w}_i$ in positive direction by angle γ_i, unit vector $\mathbf{r}_i$ can be obtained. Therefore, we can write

$$\mathbf{r}_i = \mathbf{Q}(\mathbf{w}_i,\gamma_i)$$
$$\mathbf{v}_i = \cos\gamma_i\mathbf{v}_i + (1-\cos\gamma_i)\mathbf{w}_i\mathbf{w}_i^{\mathrm{T}}\mathbf{v}_i + \sin\gamma_i(\mathbf{w}_i\times\mathbf{v}_i) \tag{11}$$

According to Eq. (8), $\mathbf{v}_i$ is perpendicular to $\mathbf{w}_i$, therefore above equation can be simplified as

$$\mathbf{w}_i\perp\mathbf{v}_i \Rightarrow \mathbf{r}_i = \cos\gamma_i\mathbf{v}_i + \sin\gamma_i(\mathbf{w}_i\times\mathbf{v}_i) = [a_i \quad b_i \quad c_i]^{\mathrm{T}}$$
$$\text{for } i = 1,2,3 \tag{12}$$

where a_i, b_i and c_i are Cartesian components of the unit vector $\mathbf{r}_i$. For simplicity, and without loss of generality, we assume that unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the X–Y plane. Therefore

$$\mathbf{w}_1 = [0 \quad 0 \quad 1]^{\mathrm{T}}, \mathbf{r}_1 = [a_1 \quad b_1 \quad 0]^{\mathrm{T}} \tag{13}$$

As shown in Fig. 3, unit vector $\mathbf{t}_1$ can now be obtained by rotating unit vector $\mathbf{w}_1$ about unit vector $\mathbf{r}_1$ by negative angle θ_1. Note that θ_1 is also rotation of the passive revolute joint.

$$\mathbf{t}_1 = \mathbf{Q}(\mathbf{r}_1, -\theta_1)$$
$$\mathbf{w}_1 = \cos\theta_1 \mathbf{w}_1 + (1-\cos\theta_1)\mathbf{r}_1\mathbf{r}_1^T\mathbf{w}_1 - \sin\theta_1(\mathbf{r}_1 \times \mathbf{w}_1) \quad (14)$$

According to Eq. (8), $\mathbf{w}_1$ is perpendicular to $\mathbf{r}_1$, therefore, above equation can be simplified as

$$\mathbf{r}_1 \perp \mathbf{w}_1 \Rightarrow \mathbf{t}_1 = \cos\theta_1 \mathbf{w}_1 - \sin\theta_1(\mathbf{r}_1 \times \mathbf{w}_1) \quad (15)$$

As shown in Fig. 3, we can obtain unit vector $\mathbf{u}_1$ by rotating unit vector $\mathbf{r}_i$ about unit vector $\mathbf{t}_1$ by negative angle β_1. Note that angle β_1 is the equivalent rotation of the passive curved prismatic joint (special realization of revolute joint).

$$\mathbf{u}_1 = \mathbf{Q}(\mathbf{t}_1, -\beta_1)$$
$$\mathbf{r}_1 = \cos\beta_1 \mathbf{r}_1 + (1-\cos\beta_1)\mathbf{t}_1\mathbf{t}_1^T\mathbf{r}_1 - \sin\beta_1(\mathbf{t}_1 \times \mathbf{r}_1) \quad (16)$$

According to Eq. (9), $\mathbf{t}_i$ is perpendicular to $\mathbf{r}_i$, therefore the above equation can be simplified as

$$\mathbf{t}_1 \perp \mathbf{r}_1 \Rightarrow \mathbf{u}_1 = \cos\beta_1 \mathbf{r}_1 - \sin\beta_1(\mathbf{t}_1 \times \mathbf{r}_1) \quad (17)$$

The steps taken thus far have defined:

- Unit vector $\mathbf{w}_1$: using known $\mathbf{v}_1$ and $\mathbf{v}_2$ information about the fixed base, $\mathbf{w}_1$ is determined.
- Angle γ_1: the rotation of actuated joint. This information is given in FPA.
- Unit vector $\mathbf{r}_i$: calculated by Eq. (12).
- Unit vector $\mathbf{t}_1$: which is a function of angle θ_1, the rotation of the passive revolute joint. This is unknown at this stage of FPA.
- Unit vector $\mathbf{u}_1$: which is a function of angles θ_1 and β_1. This is unknown at this stage of FPA.

Once θ_1 and β_1 are calculated, the orientation of moving spherical triangle with respect to base can be determined. Therefore, for solving forward position problem of the spherical DT manipulator, we must obtain θ_1 and β_1 angles. These two angles are obtained by simultaneously solving two trigonometric equations.

The next step of the solution will utilize the structure of the moving spherical triangle in order to find one of the two trigonometric equations. According to Fig. 1 and Eq. (9), the unit vector $\mathbf{t}_3$ is perpendicular to the plane OQ_3Q_1 of the moving spherical triangle. The unit vector $\mathbf{t}_3$ can now be obtained by rotating the unit vector $\mathbf{t}_1$ about the unit vector $\mathbf{u}_1$ by angle α_1.

$$\mathbf{t}_3 = -\mathbf{Q}(\mathrm{u}_1, \alpha_1)$$
$$\mathbf{t}_1 = -\cos\alpha_1 \mathbf{t}_1 - (1-\cos\alpha_1)\mathbf{u}_1\mathbf{u}_1^T\mathbf{t}_1 - \sin\alpha_1(\mathbf{u}_1 \times \mathbf{t}_1) \quad (18)$$

According to Eqs. (15) and (17), the unit vector $\mathbf{t}_1$ is a function of θ_1 and unit vector $\mathbf{u}_1$ is a function of θ_1 and β_1. Therefore, with the known α_1 the unit vector $\mathbf{t}_3$ defined in Eq. (18) is also a function of the unknowns θ_1 and β_1.

According to Fig. 1 and Eq. (9), the unit vector $\mathbf{t}_2$ is perpendicular to the plane OQ_2Q_3 of the moving spherical triangle. To find $\mathbf{u}_2$, we must first rotate the unit vector $\mathbf{r}_1$ about the unit vector $\mathbf{t}_1$ by equivalent angle ($<Q_1OQ_2 - \beta_1$) until the unit vector $\mathbf{u}_2$ is obtained.

$$\mathbf{u}_2 = \mathbf{Q}(\mathbf{t}_1, < Q_1OQ_2 - \beta_1)\mathbf{r}_1 = \cos(< Q_1OQ_2 - \beta_1)\mathbf{r}_1 + (1 - \cos(< Q_1OQ_2 - \beta_1))\mathbf{t}_1\mathbf{t}_1^T\mathbf{r}_1 + \sin(< Q_1OQ_2 - \beta_1)(\mathbf{t}_1 \times \mathbf{r}_1) \quad (19)$$

According to Eq. (9), $\mathbf{t}_1$ is perpendicular to $\mathbf{r}_1$, therefore the above equation can be simplified as

$$\mathbf{t}_1 \perp \mathbf{r}_1 \Rightarrow \mathbf{u}_2 = \cos(< Q_1OQ_2 - \beta_1)\mathbf{r}_1 + \sin(< Q_1OQ_2 - \beta_1)(\mathbf{t}_1 \times \mathbf{r}_1) \quad (20)$$

Similarly, the unit vectors $\mathbf{t}_2$ can now be obtained by rotating $\mathbf{t}_1$ about $\mathbf{u}_2$ by $-\alpha_2$.

$$\mathbf{t}_2 = -\mathbf{Q}(\mathbf{u}_2, -\alpha_2)$$
$$\mathbf{t}_1 = -\cos\alpha_2 \mathbf{t}_1 - (1-\cos\alpha_2)\mathbf{u}_2\mathbf{u}_2^T\mathbf{t}_1 + \sin\alpha_2(\mathbf{u}_2 \times \mathbf{t}_1) \quad (21)$$

According to Eqs. (15) and (20), the unit vector $\mathbf{t}_1$ is a function of θ_1 and unit vector $\mathbf{u}_2$ is a function of θ_1 and β_1. Therefore, with the known α_2, the unit vector $\mathbf{t}_2$ defined in Eq. (21) is also a function of the unknowns θ_1 and β_1. Note that α_2 is a structural parameter of the manipulator and therefore it is a known quantity.

The unit vector $\mathbf{t}_1$ is perpendicular to unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ (see Eq. (9)). Therefore, Eqs. (18) and (21) can be simplified as

$$\mathbf{t}_1 \perp \mathbf{u}_1 \Rightarrow \mathbf{t}_3 = -\cos\alpha_1 \mathbf{t}_1 - \sin\alpha_1(\mathbf{u}_1 \times \mathbf{t}_1) \quad (22)$$

$$\mathbf{t}_1 \perp \mathbf{u}_2 \Rightarrow \mathbf{t}_2 = -\cos\alpha_2 \mathbf{t}_1 - \sin\alpha_2(\mathbf{u}_2 \times \mathbf{t}_1) \quad (23)$$

The two trigonometric equations are now formulated by noting that $\mathbf{t}_3$ is perpendicular to $\mathbf{r}_3$ and $\mathbf{t}_2$ is perpendicular to $\mathbf{r}_2$. Therefore, upon multiplication of the two sides of Eqs. (22) and (23) by $\mathbf{r}_3^T$ and $\mathbf{r}_2^T$, respectively.

$$\mathbf{r}_3^T\mathbf{t}_3 = -\cos\alpha_1 \mathbf{r}_3^T\mathbf{t}_1 - \sin\alpha_1 \mathbf{r}_3^T(\mathbf{u}_1 \times \mathbf{t}_1) = 0 \quad (24)$$

$$\mathbf{r}_2^T\mathbf{t}_2 = -\cos\alpha_2 \mathbf{r}_2^T\mathbf{t}_1 - \sin\alpha_2 \mathbf{r}_2^T(\mathbf{u}_2 \times \mathbf{t}_1) = 0 \quad (25)$$

Next, we substitute the kinematics parameters of the spherical DT parallel manipulator into Eqs. (24) and (25) and rewrite them as following

$$d_1 \sin\beta_1 + d_2 \cos\beta_1 + d_3 = 0 \quad (26)$$

$$d_4 \sin\beta_1 + d_5 \cos\beta_1 + d_6 = 0 \quad (27)$$

where

$$d_1 = (a_1a_3 + b_1b_3)\sin\alpha_1 \quad (28)$$

$$d_2 = \sin\alpha_1[(a_1b_3 - b_1a_3)\cos\theta_1 - c_3\sin\theta_1] \quad (29)$$

$$d_3 = \cos\alpha_1[(a_3b_1 - b_3a_1)\sin\theta_1 - c_3\cos\theta_1] \quad (30)$$

$$d_4 = \sin\alpha_2\Big\{\sin(< Q_1OQ_2)[c_2\sin\theta_1 + (a_2b_1 - b_2a_1)\cos\theta_1] - (a_1a_2 + b_1b_2)\sin(< Q_1OQ_2)\Big\} \quad (31)$$

$$d_5 = \sin\alpha_2\Big\{\cos(< Q_1OQ_2)[c_2\sin\theta_1 + (a_2b_1 - b_2a_1)\cos\theta_1] + (a_1a_2 + b_1b_2)\sin(< Q_1OQ_2)\Big\} \quad (32)$$

$$d_6 = \cos\alpha_2[(a_2b_1 - b_2a_1)\sin\theta_1 - c_2\cos\theta_1] \quad (33)$$

5. Bezout's elimination

Bezout's elimination method may be used to reduce a set of polynomials of multiple variables into a polynomial of only one variable. To apply this method to solve the nonlinear Eqs. (26) and (27), the trigonometric equations must be transformed into a set of polynomials. This transformation can be achieved by using the following trigonometric identities:

$$\sin\beta_1 = \frac{2x_1}{1+x_1^2}, \quad \cos\beta_1 = \frac{1-x_1^2}{1+x_1^2}, \quad \sin\theta_1 = \frac{2x_2}{1+x_2^2}, \quad \cos\theta_1 = \frac{1-x_2^2}{1+x_2^2} \quad (34)$$

where $x_1 = \tan(\beta_1/2)$ and $x_2 = \tan(\theta_1/2)$. Next, the above transformations are placed into Eqs. (26) and (27). This step is performed using MAPLE software. Results are manually organized into Eqs. (35) and (36), which can then be used by Bezout's elimination method.

$$\left(F_1x_2^2+F_2x_2-F_1\right)x_1^2+\left(F_4x_2^2+F_4\right)x_1+\left(F_5x_2^2+F_3x_2-F_5\right) = 0 \tag{35}$$

$$\left(F_6x_2^2+F_7x_2+F_8\right)x_1^2+\left(F_9x_2^2+F_{10}x_2+F_{11}\right) x_1+\left(F_{12}x_2^2+F_{13}x_2+F_{14}\right) = 0 \tag{36}$$

where

$F_1 = J_2 - J_4$

$F_2 = 2(J_3 - J_1)$

$F_3 = 2(J_1 + J_3)$

$F_4 = 2d_1$

$F_5 = -(J_2 + J_4)$

$F_6 = -J_{10} - J_{12} + J_9$

$F_7 = 2(J_{11} - J_8)$

$F_8 = -J_9 - J_{10} + J_{12}$

$F_9 = 2(J_7 - J_6)$

$F_{10} = 4J_5$

$F_{11} = 2(J_6 + J_7)$

$F_{12} = J_{10} - J_{12} - J_9$

$F_{13} = 2(J_8 + J_{11})$

$F_{14} = J_9 + J_{10} + J_{12}$

and

$J_1 = -c_3 \sin \alpha_1$

$J_2 = (b_3a_1 - a_3b_1)\cos \alpha_1$

$J_3 = (a_3b_1 - b_3a_1)\cos \alpha_1$

Table 1
The eight solutions of the case study.

Solution	x_2	θ_1	β_1
1	−0.124636	−14.20896124°	81.54623961°
2	−0.53794	−56.55517069°	36.54623961°
3	1.7360294	120.1137822°	7.200152117°
4	31.481525	176.3612614°	84.81471556°
5	−4.065632	−152.3631872°	180 + 86.55594156°
6	−1.041334	−92.31999291°	180 + 30.76720183°
7	0.172088	19.52855669°	180 + 88.7839801°
8	0.999405	89.96589886°	180 + 3.947725423°

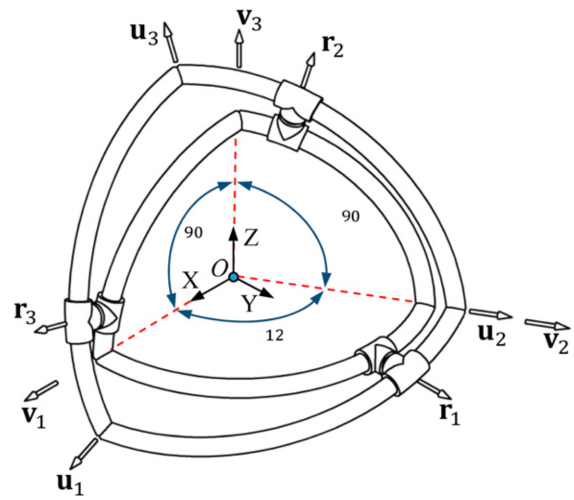

Fig. 4. Solution 1.

$J_4 = -c_3 \cos \alpha_1$

$J_5 = c_2 \sin \alpha_2 \sin(< Q_1OQ_2)$

$J_6 = (a_2b_1 - b_2a_1)\sin \alpha_2 \sin(< Q_1OQ_2)$

$J_7 = -(a_1a_2 + b_1b_2)\sin \alpha_2 \cos(< Q_1OQ_2)$

$J_8 = c_2 \sin \alpha_2 \cos(< Q_1OQ_2)$

$J_9 = (a_2b_1 - b_2a_1)\sin \alpha_2 \cos(< Q_1OQ_2)$

$J_{10} = (a_1a_2 + b_1b_2)\sin\alpha_2 \sin(< Q_1OQ_2)$

$J_{11} = (a_2b_1 - b_2a_1)\cos \alpha_2$

$J_{12} = c_2 \cos \alpha_2$

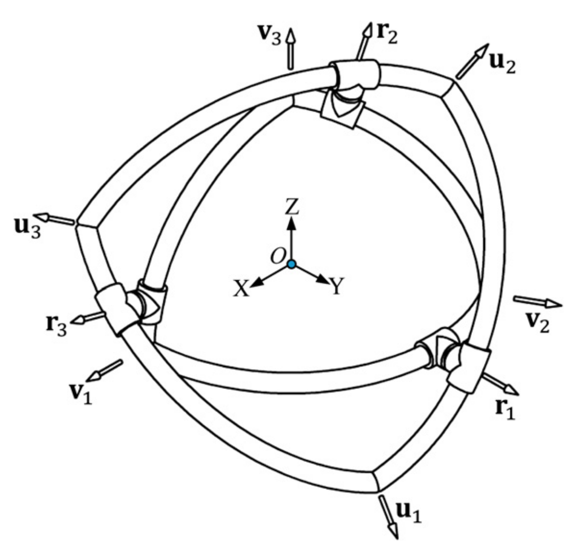

Fig. 5. Solution 2.

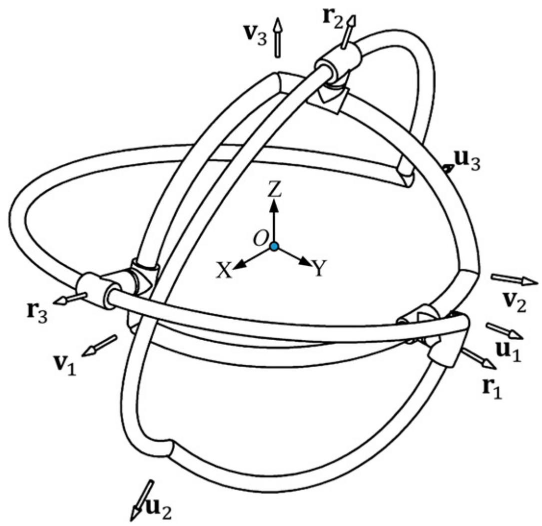

Fig. 6. Solution 3.

Using Bezout's elimination method, we can eliminate the variable χ_1 from Eqs. (35) and (36). The resulting equation is given as follows

$$\left|\begin{matrix} \begin{vmatrix} F_1x_2^2+F_2x_2-F_1 & F_5x_2^2+F_3x_2-F_5 \\ F_6x_2^2+F_7x_2+F_8 & F_{12}x_2^2+F_{13}x_2+F_{14} \end{vmatrix} & \begin{vmatrix} F_4x_2^2+F_4 & F_5x_2^2+F_3x_2-F_5 \\ F_9x_2^2+F_{10}x_2+F_{11} & F_{12}x_2^2+F_{13}x_2+F_{14} \end{vmatrix} \\ \begin{vmatrix} F_1x_2^2+F_2x_2-F_1 & F_4x_2^2+F_4 \\ F_6x_2^2+F_7x_2+F_8 & F_9x_2^2+F_{10}x_2+F_{11} \end{vmatrix} & \begin{vmatrix} F_1x_2^2+F_2x_2-F_1 & F_5x_2^2+F_3x_2-F_5 \\ F_6x_2^2+F_7x_2+F_8 & F_{12}x_2^2+F_{13}x_2+F_{14} \end{vmatrix} \end{matrix}\right| = 0 \tag{37}$$

Therefore, we have the following 8th order single variable polynomial.

$$N_8x_2^8+N_7x_2^7+N_6x_2^6+N_5x_2^5+N_4x_2^4+N_3x_2^3+N_2x_2^2+N_1x_2+N_0=0 \tag{38}$$

The values for N_0 through N_8 are defined in appendix.

As can be seen, Eq. (38) is an eight degree polynomial where Mohammadi Daniali et al. (1993) arrived at a sixteen degree polynomial. This shows improvement in the modeling method. Furthermore, the coefficients of the polynomial shown in Eq. (38) are significantly smaller than the coefficients derived by Daniali. This greatly decreases computational time, which is necessary for dynamics and simulation. It is also important to point out that Eq. (38) admits eight solutions, which may be real and/or complex. The modeling method is therefore optimal since we can find an example having eight real solutions.

6. Example

In this section, we present an example for forward position problem of DT spherical parallel manipulator. In the forward position problem r, ρ_i, $\boldsymbol{v}_i$, and α_i are supplied. These variables represent radius of sphere, stroke of actuators, information on fixed base geometry and information on moving triangle geometry, respectively. The forward position analysis will determine orientation of the moving spherical triangle by solving for θ_1 and β_1. Then

6.1. Architecture parameters – fixed base

The OP_1P_2 plane of the fixed base is assumed to lie in the X–Y plane. This assumption is made without loss of generality. The OP_1P_3 can be on any plane however, for simplicity in this example, it is assumed to be in X–Z plane. Additionally, the length of these arcs are assumed to be $P_1P_2 = 2\pi/3$, $P_2P_3 = \pi/2$ and $P_3P_1 = \pi/2$. Therefore

$$\mathbf{v}_1 = [1 \quad 0 \quad 0], \quad \mathbf{v}_2 = \left[-1/2 \quad \sqrt{3}/2 \quad 0\right],$$
$$\mathbf{v}_3 = [0 \quad 0 \quad 1]$$

Using Eq. (8), we can calculate

$$\mathbf{w}_1 = [0 \quad 0 \quad 1], \quad \mathbf{w}_2 = \left[\sqrt{3}/2 \quad 1/2 \quad 0\right],$$
$$\mathbf{w}_3 = [0 \quad 1 \quad 0]$$

6.2. Architecture parameters – moving spherical triangle

This spherical triangle can be identified by two angles and the arc between these angles. Therefore, assume the length of the arc Q_1Q_2 and angle of the vertices of the moving spherical triangle are as follows (see Figs. 1 and 3)

$$\widehat{Q_1Q_2} = \alpha_1 = \alpha_2 = 7\pi/12$$

6.3. Position of actuators

Assume that the current location of the actuators is as follows

$$\rho_1 = \pi/2, \quad \rho_2 = \rho_3 = 5\pi/12$$

Assuming radius of sphere to be unity, equivalent angle of rotation that identifies current position of the actuators can be obtained from Eq. (10) as

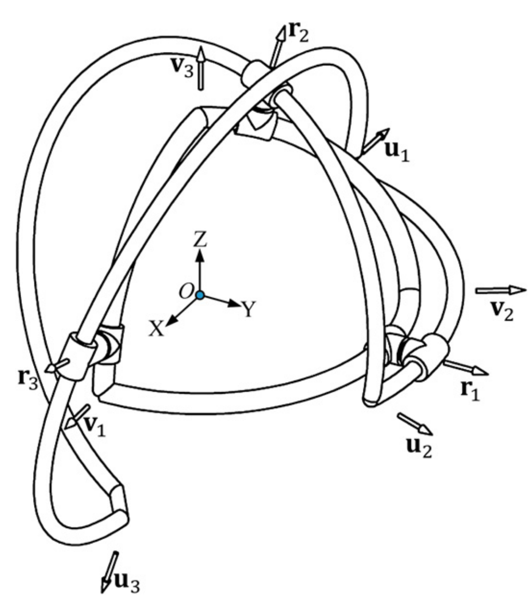

Fig. 7. Solution 4.

$$\gamma_1 = \pi/2, \quad \gamma_2 = \gamma_3 = 5\pi/12$$

The unit position vector of the actuators can now be obtained from Eq. (12) as

$$\mathbf{r}_1 = [0 \quad 1 \quad 0]$$

$$\mathbf{r}_2 = \left[\frac{-\sqrt{6}}{8}\left(1-\frac{\sqrt{3}}{3}\right) \quad \frac{3\sqrt{2}}{8}\left(1-\frac{\sqrt{3}}{3}\right) \quad \frac{\sqrt{6}}{4}\left(1+\frac{\sqrt{3}}{3}\right)\right]$$

$$\mathbf{r}_3 = \left[\frac{\sqrt{6}}{4}\left(1+\frac{\sqrt{3}}{3}\right) \quad 0 \quad \frac{-\sqrt{6}}{4}\left(1+\frac{\sqrt{3}}{3}\right)\right]$$

6.4. Computation of the orientation of the moving spherical triangle

The eight order polynomial can now be obtained by substituting variables from previous steps into Eq. (38). Therefore, we have

$$\begin{aligned}&0.08783882535\,x_2^8 - 2.513904836\,x_2^7 - 8.519069837\,x_2^6\\&+18.44430764\,x_2^5 + 19.27326032\,x_2^4 - 17.25036771\,x_2^3\\&-10.59900106\,x_2^2 + 0.859225897\,x_2 + 0.2343568586 = 0\end{aligned}$$

χ_2 can now be solved by placing the above equation into MAPLE software. Angle θ_1 can then be calculated using Eq. (34). The values for θ_1 are next placed into Eqs. (26) and (27) which results in finding β_1. Results are listed in Table 1. This completes the forward position problem. The eight degree polynomial results in eight real solutions. Therefore, the polynomial in Eq. (38) is minimal which indicates the solution method is also optimum. Additionally, four of the eight solutions are shown graphically in Figs. 4–7.

7. Conclusion

We have presented a new approach for solving the forward position problem of a DT spherical parallel manipulator. First, we developed the kinematics model of the manipulator using unit vectors. Because of the spherical nature of the manipulator, all these unit vectors are on axes which pass through the origin of the sphere. Special form of Rodrigues formula was then used to show relationship between these unit vectors, which resulted in two coupled trigonometric equations. Next, using Bezout's elimination method, the two coupled equations were reduced to a polynomial of eight degree. An example having eight real solutions was provided. Therefore, the polynomial is minimal which indicates the solution method is optimum. Lastly, four of the eight solutions were shown graphically.

Appendix. Coefficients of Eq. (38).

$$\begin{aligned}N_8 = {}& F_1F_9^2F_5 - F_6F_4F_9F_5 + F_6F_4^2F_{12} + F_{12}^2F_1^2 - F_1F_{12}F_4F_9\\&- 2F_1F_{12}F_5F_6 + F_6^2F_5^2\end{aligned} \tag{A.1}$$

$$\begin{aligned}N_7 = {}& F_{13}F_4^2F_6 - 2F_{12}F_1F_5F_7 - 2F_{13}F_1F_5F_6 - F_{12}F_4F_9F_2\\&- F_5F_{10}F_4F_6 - 2F_{12}F_2F_5F_6 - F_5F_9F_4F_7 - 2F_{12}F_1F_3F_6\\&+ 2F_5F_9F_{10}F_1 + 2F_5^2F_6F_7 + F_3F_9^2F_1 + 2F_{12}F_2^1F_{13}\\&- F_{12}F_4F_{10}F_1 + F_{12}F_4^2F_7 - F_{13}F_4F_9F_1 + 2F_3F_6^2F_5 + F_5F_9^2F_2\\&+ 2F_{12}^2F_2F_1 - F_3F_9F_4F_6\end{aligned} \tag{A.2}$$

$$\begin{aligned}N_6 = {}& F_{12}^2F_2^2 - 2F_{12}^2F_1^2 + F_{13}^2F_1^2 + F_3^2F_6^2 - 2F_5^2F_6^2 + F_5^2F_7^2\\&+ 2F_{12}F_4^2F_6 + F_{12}F_4^2F_8 + 2F_{12}F_1^2F_{14} + 2F_5^2F_6F_8 + F_{13}F_4^2F_7\\&+ F_{14}F_4^2F_6 + F_3F_9^2F_2 - 2F_5F_9^2F_1 + F_5F_{10}^2F_1 - F_{12}F_4F_{10}F_2\\&- F_{12}F_4F_{11}F_1 + 4F_{12}F_2F_{13}F_1 - 2F_{12}F_2F_3F_6 - 2F_{12}F_2F_5F_7\\&- 2F_{12}F_1F_3F_7 + 4F_{12}F_1F_5F_6 - 2F_{12}F_1F_5F_8 - 2F_{13}F_2F_5F_6\\&- 2F_{13}F_1F_3F_6 - 2F_{13}F_1F_5F_7 + 4F_3F_6F_5F_7 - F_{13}F_4F_9F_2\\&- F_{13}F_4F_{10}F_1 - F_{14}F_4F_9F_1 - 2F_{14}F_1F_5F_6 + 2F_3F_9F_{10}F_1\\&- F_3F_9F_4F_7 - F_3F_{10}F_4F_6 + 2F_5F_9F_{10}F_2 + 2F_5F_9F_{11}F_1\\&- F_5F_9F_4F_8 - F_5F_{10}F_4F_7 - F_5F_{11}F_4F_6\end{aligned} \tag{A.3}$$

$$\begin{aligned}N_5 = {}& 2F_{12}F_4^2F_7 - 2F_{12}^2F_2F_1 + 2F_{12}F_2^2F_{13} - 4F_{12}F_1^2F_{13} + 2F_{13}^2F_2F_1\\&+ 2F_{13}F_1^2F_{14} + 2F_3^2F_6F_7 - 2F_3F_6^2F_5 + 2F_3F_7^2F_5 - 4F_5^2F_6F_7\\&+ 2F_5^2F_7F_8 + 2F_{13}F_4^2F_6 + F_{13}F_4^2F_8 + F_{14}F_4^2F_7 - F_3F_9^2F_1\\&+ F_3F_{10}^2F_1 - F_5F_9^2F_2 + F_5F_{10}^2F_2 - F_{12}F_4F_9F_2 - F_{12}F_4F_{11}F_2\\&+ 4F_{12}F_2F_{14}F_1 - 2F_{12}F_2F_3F_7 - 2F_{12}F_2F_5F_8 + 2F_{12}F_1F_3F_6\\&- 2F_{12}F_1F_3F_8 + 4F_{12}F_1F_5F_7 + 2F_{12}F_2F_5F_6 - 2F_{13}F_2F_5F_7\\&- 2F_{13}F_1F_3F_7 + 4F_{13}F_1F_5F_6 - 2F_{13}F_1F_5F_8 - 2F_{13}F_2F_3F_6\\&+ 4F_3F_6F_5F_8 - F_{13}F_4F_{10}F_2 - F_{13}F_4F_{11}F_1 - F_{14}F_4F_9F_2\\&- F_{14}F_4F_{10}F_1 - 2F_{14}F_2F_5F_6 - 2F_{14}F_1F_3F_6 - 2F_{14}F_1F_5F_7\\&+ 2F_3F_9F_{10}F_2 + 2F_3F_9F_{11}F_1 - F_3F_9F_4F_6 - F_3F_9F_4F_8\\&- F_3F_{10}F_4F_7 - F_3F_{11}F_4F_6 - 4F_5F_9F_{10}F_1 + 2F_5F_9F_{11}F_2\\&+ 2F_5F_{10}F_{11}F_1 - F_5F_{10}F_4F_8 - F_5F_{11}F_4F_7\end{aligned} \tag{A.4}$$

$$\begin{aligned}N_4 = {}& F_{14}^2F_1^2 + F_5^2F_8^2 + F_{12}^2F_1^2 + F_{13}^2F_2^2 - 2F_{13}^2F_1^2 + F_3^2F_7^2 + F_5^2F_6^2\\&- 2F_5^2F_7^2 - 2F_{14}F_1F_5F_8 + F_{14}F_4^2F_8 + F_5F_{11}^2F_1 - F_{14}F_4F_{11}F_1\\&- F_5F_{11}F_4F_8 + F_{12}F_4^2F_6 + 2F_{12}F_4^2F_8 + 2F_{12}F_2^2F_{14}\\&- 4F_{12}F_1^2F_{14} + 2F_3^2F_6F_8 - 4F_5^2F_6F_8 + 2F_{13}F_4^2F_7 + 2F_{14}F_4^2F_6\\&+ F_3F_{10}^2F_2 + F_5F_9^2F_1 - 2F_5F_{10}^2F_1 + F_{12}F_4F_9F_1 - F_{12}F_4F_{10}F_2\\&- 4F_{12}F_2F_{13}F_1 - 2F_{12}F_2F_3F_8 + 2F_{12}F_2F_5F_7 + 2F_{12}F_1F_3F_7\\&- 2F_{12}F_1F_5F_6 + 4F_{12}F_1F_5F_8 + 2F_{13}F_2F_5F_6 - 2F_{13}F_2F_5F_8\\&+ 2F_{13}F_1F_3F_6 - 2F_{13}F_1F_3F_8 + 4F_{13}F_1F_5F_7 - 2F_{14}F_2F_3F_6\\&+ 4F_{13}F_2F_{14}F_1 - 2F_{13}F_2F_3F_7 - 4F_3F_6F_5F_7 + 4F_3F_7F_5F_8\\&- F_{13}F_4F_9F_2 - F_{13}F_4F_{11}F_2 - F_{14}F_4F_{10}F_2 - 2F_{14}F_2F_5F_7\\&- 2F_{14}F_1F_3F_7 + 4F_{14}F_1F_5F_6 - 2F_3F_9F_{10}F_1 + 2F_3F_9F_{11}F_2\\&- F_3F_9F_4F_7 + 2F_3F_{10}F_{11}F_1 - F_3F_{10}F_4F_6 - F_3F_{10}F_4F_8\\&- F_3F_{11}F_4F_7 - 2F_5F_9F_{10}F_2 - 4F_5F_9F_{11}F_1 + F_5F_9F_4F_6\\&+ 2F_5F_{10}F_{11}F_2\end{aligned} \tag{A.5}$$

$$\begin{aligned}N_3 = {}& F_{12}F_4^2F_7 + 2F_{12}F_1^2F_{13} - 2F_{13}^2F_2F_1 + 2F_{13}F_2^2F_{14} - 4F_{13}F_1^2F_{14}\\&+ 2F_{14}^2F_2F_1 + 2F_3^2F_7F_8 - 2F_3F_7^2F_5 + 2F_3F_8^2F_5 + 2F_5^2F_6F_7\\&- 4F_5^2F_7F_8 + F_{13}F_4^2F_6 + 2F_{13}F_4^2F_8 + 2F_{14}F_4^2F_7 - F_3F_{10}^2F_1\\&+ F_3F_{11}^2F_1 - F_5F_{10}^2F_2 + F_5F_{11}^2F_2 + F_{12}F_4F_{10}F_1 - F_{12}F_4F_{11}F_2\\&- 4F_{12}F_2F_{14}F_1 + 2F_{12}F_2F_5F_8 + 2F_{12}F_1F_3F_8 - 2F_{12}F_1F_5F_7\\&- 2F_{13}F_2F_3F_8 + 2F_{13}F_2F_5F_7 + 2F_{13}F_1F_3F_7 - 2F_{13}F_1F_5F_6\\&+ 4F_{13}F_1F_5F_8 - 4F_3F_6F_5F_8 + F_{13}F_4F_9F_1 - F_{13}F_4F_{10}F_2\\&- F_{14}F_4F_9F_2 - F_{14}F_4F_{11}F_2 - 2F_{14}F_2F_3F_7 + 2F_{14}F_2F_5F_6\\&- 2F_{14}F_2F_5F_8 + 2F_{14}F_1F_3F_6 - 2F_{14}F_1F_3F_8 + 4F_{14}F_1F_5F_7\\&- 2F_3F_9F_{11}F_1 - F_3F_9F_4F_8 + 2F_3F_{10}F_{11}F_2 - F_3F_{10}F_4F_7\\&- F_3F_{11}F_4F_6 - F_3F_{11}F_4F_8 + 2F_5F_9F_{10}F_1 - 2F_5F_9F_{11}F_2\\&+ F_5F_9F_4F_7 - 4F_5F_{10}F_{11}F_1 + F_5F_{10}F_4F_6\end{aligned} \tag{A.6}$$

$$
\begin{aligned}
N_2 = & -2F_{14}^2F_1^2 - 2F_5^2F_8^2 + F_{13}^2F_1^2 + F_{14}^2F_2^2 + F_3^2F_8^2 + F_5^2F_7^2 \\
& + 4F_{14}F_1F_5F_8 + 2F_{14}F_4^2F_8 - 2F_5F_{11}^2F_1 + F_{12}F_4^2F_8 \\
& + 2F_{12}F_1^2F_{14} + 2F_5^2F_6F_8 + F_{13}F_4F_7 + F_{14}F_4^2F_6 + F_3F_{11}^2F_2 \\
& + F_5F_{10}^2F_1 + F_{12}F_4F_{11}F_1 - 2F_{12}F_1F_5F_8 + 2F_{13}F_2F_5F_8 \\
& + 2F_{13}F_1F_3F_8 - 2F_{13}F_1F_5F_7 - 4F_{13}F_2F_{14}F_1 - 4F_3F_7F_5F_8 \\
& + F_{13}F_4F_{10}F_1 - F_{13}F_4F_{11}F_2 + F_{14}F_4F_9F_1 - F_{14}F_4F_{10}F_2 \\
& - 2F_{14}F_2F_3F_8 + 2F_{14}F_2F_5F_7 + 2F_{14}F_1F_3F_7 - 2F_{14}F_1F_5F_6 \\
& - 2F_3F_{10}F_{11}F_1 - F_3F_{10}F_4F_8 - F_3F_{11}F_4F_7 + 2F_5F_9F_{11}F_1 \\
& + F_5F_9F_4F_8 - 2F_5F_{10}F_{11}F_2 + F_5F_{10}F_4F_7 + F_5F_{11}F_4F_6 \qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
N_1 = & \ 2F_5F_{10}F_{11}F_1 + F_{13}F_4^2F_8 - F_5F_{11}^2F_2 + F_{14}F_4F_{10}F_1 - 2F_3F_8^2F_5 \\
& + F_{13}F_4F_{11}F_1 - F_{14}F_4F_{11}F_2 + 2F_5^2F_7F_8 - F_3F_{11}F_4F_8 \\
& + 2F_{14}F_2F_5F_8 - 2F_{14}F_1F_5F_7 + 2F_{13}F_1^2F_{14} + 2F_{14}F_1F_3F_8 \\
& + F_{14}F_4^2F_7 - 2F_{13}F_1F_5F_8 - 2F_{14}^2F_2F_1 + F_5F_{10}F_4F_8 \\
& - F_3F_{11}^2F_1 + F_5F_{11}F_4F_7 \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
N_0 = & \ F_{14}F_4F_{11}F_1 + F_5F_{11}F_4F_8 + F_5^2F_8^2 + F_{14}F_4^2F_8 + F_5F_{11}^2F_1 \\
& + F_{14}^2F_1^2 - 2F_{14}F_1F_5F_8 \qquad \text{(A.9)}
\end{aligned}
$$

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