



Global Error Minimization method for solving strongly nonlinear oscillator differential equations

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ABSTRACT

A modified variational approach called Global Error Minimization (GEM) method is developed for obtaining an approximate closed-form analytical solution for nonlinear oscillator differential equations. The proposed method converts the nonlinear differential equation to an equivalent minimization problem. A trial solution is selected with unknown parameters. Next, the GEM method is used to solve the minimization problem and to obtain the unknown parameters. This will yield the approximate analytical solution of the nonlinear ordinary differential equations (ODEs). This approach is simple, accurate and straightforward to use in identifying the solution. To illustrate the effectiveness and convenience of the suggested procedure, a cubic Duffing equation with strong nonlinearity is considered. Comparisons are made between results obtained by the proposed GEM method, the exact solution and results from five recently published methods for addressing Duffing oscillators. The maximal relative error for the frequency obtained by the GEM method compared with the exact solution is 0.0004%, which indicates the remarkable precision of the GEM method.

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1. Introduction

Most phenomena in our world are essentially nonlinear and are described by nonlinear ordinary differential equations. Solving nonlinear ODEs is thus of great importance for gaining insight into real-world or engineering problems. However, generally speaking, it is difficult to obtain accurate solutions of nonlinear problems. Consequently, solutions are approximated using numerical techniques, analytical techniques and a combination of these. In this context some useful analytical approximation methods have been developed. The most well known technique is the perturbation method [1]. By means of perturbation methods, a lot of important properties and interesting phenomena of nonlinear problems have been revealed. But the classical perturbation methods apply only to weakly nonlinear problems. Therefore, considerable attention has been directed towards the approximate analytical solutions for strongly nonlinear equations. Some researchers have modified the classical perturbation methods. For instance, He [2] modified the Lindstedt–Poincaré method for strongly nonlinear oscillations. Lakrad and Belhaq [3] extended the multiple-scales method to the case of strongly nonlinear self-excited systems. Liu et al. [4] utilized the high dimensional harmonic balance method for analysis of the Van Der Pol oscillator. Recently, a homotopy based approach has gained popularity in solving nonlinear problems. Liao [5] proposed a powerful analytic method, namely the Homotopy Analysis Method (HAM), for solving nonlinear problems. HAM has been utilized by many researchers [6–12]. In 1999, He [13] proposed another application of the homotopy technique for solving nonlinear problems, namely the Homotopy Perturbation Method (HPM). HPM has been applied with great success to obtain the solutions of a large variety of nonlinear problems [14–20]. There are various other methods for solving nonlinear

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ODEs such as non-perturbative methods [21], variational iteration methods [22,23], He's variational approach [24–26], the energy balance method [27–29] and the iteration perturbation method [30,31]. A review of the recently developed nonlinear analytical methods can be found in detail in Refs. [21,32–34].

Our concern in this paper is the derivation of an approximate analytical solution for a nonlinear oscillatory differential equation. To do this we modify the variational approach proposed by He [24] and develop a method called GEM (Global Error Minimization). In the proposed method, the nonlinear differential equation is converted to an equivalent minimization problem. We combine the general idea of global error minimization in the AVK method [35] and He's variational approach [24] for solving the nonlinear ODEs. The idea of error minimization is a natural process. Therefore, we believe that GEM provides a natural way to obtain a solution.

In the first part of the GEM method, a simple sine or cosine term with unknown parameters is selected as the trial solution. The unknown parameters are identified via the minimization of the global error. Next, more sine or cosine terms are added to increase the desired accuracy of the approximated solution. We will demonstrate that by using a few terms a solution with high accuracy is obtained.

The remainder of this paper is organized as follows. In Section 2, the main idea with the necessary definitions and theorems is presented. The applicability and effectiveness of our approach are shown in Section 3 by solving a nonlinear Duffing equation with various degrees of nonlinearity. Section 4 discusses the similarities of and differences between the GEM method, the AVK method and He's variational approach. Finally, concluding remarks relating to the overall study are made in the last section.

2. The basic idea of the GEM method

In this section the Global Error Minimization (GEM) method is introduced and developed. The method is systematically described and will result in an approximate analytic solution for the strongly nonlinear oscillator ODEs. Consider a general second-order nonlinear oscillator differential equation

$$\ddot{x} + F(\dot{x}, x, t) = 0 \quad (1)$$

with initial conditions

$$x(0) = A, \quad \dot{x}(0) = B. \quad (2)$$

2.1. Preliminary

Definition 1. Consider the nonlinear system (1); we define the following functional for the oscillator equation, called the global error functional [35]. Let

$$E(\dot{x}, x, t) = \int_0^T \|\ddot{x} + F(\dot{x}, x, t)\| dt, \quad T = \frac{2\pi}{\omega}, \quad \omega \text{ is the primary natural frequency} \quad (3)$$

where E is a continuous functional.

Definition 2. We convert the nonlinear ODE in Eqs. (1) and (2) to the following minimization problem:

$$\begin{aligned} &\text{Minimize} && E(\dot{x}, x, t) \\ &\text{s.t.} && x(0) = A, \quad \dot{x}(0) = 0. \end{aligned} \quad (4)$$

Lemma 1. If h is a nonlinear continuous function on $[0, T]$ and non-negative ($h \geq 0$), then the necessary and sufficient condition for $\int_0^T h dx = 0$ is $h \equiv 0$ on $[0, T]$ [35].

Proof. Let us assume $\int_0^T h dx = 0$ but $h \neq 0$, and by assumption at a point x in $[0, T]$, $h(x) > 0$, since for continuity of $h(x)$ it is positive in some neighborhood of x , i.e. $h(x) > 0$, and for all $x \in (x_1 - \varepsilon, x_1 + \varepsilon)$ that ε is a positive number. Therefore $\int_0^T h dx \geq \int_{x_1 - \varepsilon}^{x_1 + \varepsilon} h dx > 0$, i.e., $\int_0^T h dx > 0$, a contradiction to our assumption. Thus h must be zero on $[0, T]$. On the other hand if $h \equiv 0$ on $[0, T]$ then obviously $\int_0^T h dx = 0$ [35]. \square

Theorem 1. The necessary and sufficient condition for x to be a solution of the nonlinear ODE (1) with initial condition $x(0)$ and $\dot{x}(0)$ is $E(\dot{x}, x, t) = 0$ in the minimization problem (4).

Proof. It is sufficient to define h in Lemma 1 as follows:

$$h(t) = [\ddot{x} + F(\dot{x}, x, t)]^2. \quad (5)$$

Since $[\cdot]^2$ is a continuous function and non-negative and also $F(\dot{x}, x, t)$ is a continuous function, then $h(t)$ is continuous with respect to variables x, \dot{x} . Additionally, since x, \dot{x} are continuous functions on $[0, T]$, then the total function $\ddot{x} + F(\dot{x}, x, t)$ is

continuous in the interval $[0, T]$. Therefore, using Lemma 1, $\int_0^T h(t)dt = 0$ is equivalent to $h \equiv 0$ for all $t \in [0, T]$, i.e.

$$\ddot{x} + F(\dot{x}, x, t) = 0 \quad \forall t \in [0, T]. \tag{6}$$

Therefore, Theorem 1 is proved [35]. \square

2.2. Outline of the procedure

The solution of Eq. (1) can be expressed in the form of Fourier series [36]:

$$x = a_0 \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t)) \tag{7}$$

where a_0, a_n, b_n are constants. These unknown constants could not be determined for the case of infinite Fourier series. However, we can approximate Eq. (7) by a finite series [30,37]:

$$\tilde{x} = a_0 + \sum_{n=1}^m (a_n \cos(n\omega t) + b_n \sin(n\omega t)). \tag{8}$$

Various methods have been developed for determining the unknown constants used in Eq. (8) [1,30,31,34]. In this paper, a natural and efficient method will be developed for determining these unknowns.

The nonlinear problem (1) is first converted to the minimization problem (4). We directly substitute the trial solution (8) in the minimization problem. The solutions of the minimization problem are the unknown constants of Eq. (8). Consider the case where $E(\dot{x}, x, t) = 0$; then, with respect to Theorem 1, \tilde{x} happens to be the exact solution. Generally such a case will not arise for nonlinear problems. However, if $E(\dot{x}, x, t) \cong 0$ we find an excellent analytical approximation of the original nonlinear ODE. It is worth noting that we know the desired answer of our minimization problem in advance, which is zero. Therefore, we have a valuable measure for comparing the accuracy of the approximated solutions. Note that $E(\dot{x}, x, t)$ is the global error and any reduction in this functional, by choosing a better trial solution, would greatly improve the approximation of the analytical solution.

3. Numerical experiments and discussion

In this section, we demonstrate the effectiveness of our approach by showing results for the cubic Duffing equation. Duffing equations describe many kinds of nonlinear oscillatory systems in physics, mechanics and engineering [38]. They are well known in the field of nonlinear dynamics and have been presented for various types of nonlinearity. To execute our example, we use a PC with 2.67 GHz CPU, 1 GB RAM memory and the *Mathematica* package.

The cubic Duffing equation can be expressed as follows:

$$\ddot{x}(t) + x(t) + \varepsilon x^3(t) = 0 \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{9}$$

The solution of Eq. (9) is only defined for $\varepsilon A^2 > -1$. Note that the maximum oscillation amplitude of the Duffing oscillator in Eq. (9) for $\varepsilon = -1$ is $A = 1$ which corresponds to the heteroclinic orbit with period $+\infty$ [39]. In this paper, we only consider $\varepsilon > 0$. It should also be noted that the restoring force function ($F(\dot{x}, x, t) = x(t) + \varepsilon x^3(t)$) in Eq. (9) is an odd function of x ; thus the periodic solution contains only odd multiples of ω , (i.e. $x(t) = \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n + 1)\omega t]$) [40].

3.1. First-order approximation

We begin the procedure with the simplest trial solution:

$$\tilde{x}_1(t) = b \cos(\omega t). \tag{10}$$

Next, we convert Eq. (9) to the minimization problem (11):

$$\begin{aligned} \text{Minimize} \quad & E(\dot{\tilde{x}}_1, \tilde{x}_1, t) = \int_0^T (\ddot{\tilde{x}}_1 + \tilde{x}_1 + \varepsilon \tilde{x}_1^3)^2 dt, \quad T = \frac{2\pi}{\omega}, \\ \text{s.t.} \quad & \tilde{x}_1(0) = A, \quad \dot{\tilde{x}}_1(0) = 0. \end{aligned} \tag{11}$$

The constraints of the minimization problem are readily satisfied by choosing $b = A$. Therefore, by replacing $\tilde{x}(t) = A \cos(\omega t)$ in Eq. (11) and performing the integration we get

$$\text{Minimize} \quad E(\dot{\tilde{x}}_1, \tilde{x}_1, t) = \frac{A^2\pi (5A^4\varepsilon^2 - 12A^2\varepsilon(-1 + \omega^2) + 8(-1 + \omega^2)^2)}{8\omega}. \tag{12}$$

The solution of Eq. (12) could be found through the condition $\frac{\partial E(\dot{\tilde{x}}, \tilde{x}, t)}{\partial \omega} = 0$:

$$\omega = \sqrt{\frac{1}{3} + \frac{A^2\varepsilon}{4} + \frac{1}{12}\sqrt{64 + 96A^2\varepsilon + 39A^4\varepsilon^2}}. \tag{13}$$

Eq. (13) is the approximate frequency of Eq. (9) provided by the GEM method. The exact frequency, ω , of the cubic Duffing equation was found by [1]:

$$\omega_{ex} = \frac{2\pi}{T_{ex}}, \quad T_{ex} = \frac{4}{\sqrt{1 + \varepsilon A^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - m \sin^2 x}}, \quad m = \frac{\varepsilon A^2}{2(1 + \varepsilon A^2)}. \tag{14}$$

For small values of εA^2 , it is possible to do the power series expansions of the exact (Eq. (14)) and approximate (Eq. (13)) angular frequencies. Doing these expansions, the following equations are obtained:

$$\omega_{ex}(\varepsilon A) = 1 + \frac{3}{8}\varepsilon A^2 - \frac{21}{256}\varepsilon^2 A^4 + \frac{81}{2048}\varepsilon^3 A^6 + \dots \tag{15}$$

$$\omega(\varepsilon A) = 1 + \frac{3}{8}\varepsilon A^2 - \frac{16}{256}\varepsilon^2 A^4 + \frac{36}{2048}\varepsilon^3 A^6 + \dots \tag{16}$$

In Eq. (16), the first two terms are the same as the terms obtained from the expansion of the exact frequency (Eq. (15)), whereas the third term of the expansion of the exact frequency is $\frac{21}{256}$ compared with $\frac{16}{256}$ obtained in this study, that is, the relative error in this term is less than 24%. It should be noted that the 24% error in the third term is for the first-order approximation.

For large values of $\lambda = \varepsilon A^2$, it is also possible to do the power series expansions of the exact and approximate angular frequencies and obtain

$$\omega_{ex}(\lambda) = \frac{\Gamma^2(3/4)\sqrt{\lambda}}{\sqrt{\pi}} + \frac{\Gamma^4(3/4) + 4\Gamma^2(3/4)\Gamma^2(5/4)}{8\sqrt{\pi}\lambda\Gamma^2(5/4)} + \dots = 0.847213\sqrt{\lambda} + \frac{0.617172}{\sqrt{\lambda}} + \dots \tag{17}$$

$$\omega(\lambda) = \frac{1}{2}\sqrt{1 + \sqrt{\frac{13}{3}}}\lambda + \frac{1}{\sqrt{\varepsilon A}\sqrt{3 + \sqrt{39}}}\left(\frac{1}{\sqrt{3}} + \frac{4}{\sqrt{13}}\right) + \dots = 0.877734\sqrt{\lambda} + \frac{0.55475}{\sqrt{\lambda}} + \dots \tag{18}$$

The relative error of the first term in Eq. (18) is less than 3.6%. Furthermore we have

$$\lim_{\lambda \rightarrow 0} \frac{\omega(\lambda)}{\omega_{ex}(\lambda)} = 1 \tag{19}$$

$$\lim_{\lambda \rightarrow \infty} \frac{\omega(\lambda)}{\omega_{ex}(\lambda)} = 1.03602. \tag{20}$$

For large values of εA^2 , the relative error of the approximate solution is 3.6%. This error is comparable with the error of the first-order approximation of other methods [13,24]. He [13] solved Eq. (9) utilizing the HPM and obtained the frequency as

$$\omega = \sqrt{\frac{10 + 7\varepsilon A^2 + \sqrt{64 + 104\varepsilon A^2 + 49\varepsilon^2 A^4}}{18}}. \tag{21}$$

The relative error of Eq. (21) is 4.08%. He approximated Eq. (21) for small ε and obtained $\sqrt{1 + \frac{3\varepsilon A^2}{4}} + O(\varepsilon^2)$. Additionally, He [24] solved the Duffing problem (Eq. (9)) by means of a variational approach and obtained $\omega = \sqrt{1 + \frac{3\varepsilon A^2}{4}}$. These two frequencies are exactly the same as our approximate solution for small ε .

3.2. Higher order approximations

To improve our analytical approximation, we will add additional terms to the trial solution. First we will consider the third-order approximation:

$$\tilde{x}_3(t) = b \cos(\omega t) + c \cos(3\omega t) + d \cos(5\omega t). \tag{22}$$

With the new trial solution, Eq. (9) is converted to a minimization problem (Eq. (23)):

$$\begin{aligned} \text{Minimize} \quad & E(\dot{\tilde{x}}_3, \tilde{x}_3, t) = \int_0^T (\ddot{\tilde{x}}_3 + \tilde{x}_3 + \varepsilon \tilde{x}_3^3)^2 dt, \quad T = \frac{2\pi}{\omega} \\ \text{s.t.} \quad & \tilde{x}_3(0) = A, \quad \dot{\tilde{x}}_3(0) = 0. \end{aligned} \tag{23}$$

The constraint of this minimization problem is in the following form:

$$\tilde{x}_3(0) = b + c + d = A. \tag{24}$$

The constraint in Eq. (24) is linear and can be removed by replacing $b = A - c - d$ in the minimization problem (18). This will convert the constraint minimization problem into an unconstrained minimization problem which is easier to solve.

The solution of Eq. (23) could be found by using the conditions $\frac{\partial E(\dot{\tilde{x}}_3, \tilde{x}_3, t)}{\partial c} = \frac{\partial E(\dot{\tilde{x}}_3, \tilde{x}_3, t)}{\partial d} = \frac{\partial E(\dot{\tilde{x}}_3, \tilde{x}_3, t)}{\partial \omega} = 0$. These conditions

Table 1
Comparison of the third-order (ω_3) and fourth-order (ω_4) approximate frequency with the exact one (ω_{exact}).

εA^2	ω_{exact}	ω_3	Percentage error (%)	ω_4	Percentage error (%)
0.5	1.17078	1.17078	0.000042	1.17078	4.64964×10^{-7}
1	1.31778	1.31778	0.00027	1.31778	4.59436×10^{-6}
5	2.15042	2.1505	0.0039	2.15042	0.000111181
10	2.86664	2.86683	0.0065	2.86665	0.000203834
100	8.53359	8.53454	0.011	8.53362	0.000380843
1000	26.8107	26.8139	0.0118761	26.8108	0.000407491
5000	59.9157	59.9229	0.0119385	59.916	0.000409969
50 000	189.445	189.468	0.0119526	189.446	0.000410529

will result in a system of nonlinear algebraic equations. Various methods such as the Newton–Raphson method [41], the Adomian decomposition method [42], and the homotopy continuation method [43,44] have been developed for solving such equations. The Newton–Raphson method can be used to accurately solve a system of nonlinear algebraic equations with a proper set of initial approximations. An appropriate initial approximation is the solution of the low term trial solution with the new coefficients set to zero. If we add one more term to the third-order trial solution (Eq. (22)) we can easily obtain the fourth-order approximation. Upon, solving the minimization problem, discussed above, we obtain an astonishingly accurate solution for the third- and fourth-order approximations. Further, we consider various degrees of nonlinearity (εA^2) and a comprehensive comparison of our results is made with the exact solution [1]. The results are tabulated in Table 1.

Table 1 shows the excellent agreement of our method with the exact frequency for both small and large λ . As shown, the maximum relative error of the third-order approximate and exact solutions is 0.012%. For the fourth-order approximation the error is 0.0004%.

We also compared our results with those from three other methods which have the best results to date. Wu et al. [39] solved Eq. (9) using an improved harmonic balance method that incorporates the salient features of both Newton’s method and the harmonic balance method. They obtained the following result for the third-order approximation:

$$\omega_{\text{HBM3}}(\lambda) = \sqrt{\frac{r_0 + r_1\lambda + r_2\lambda^2 + r_3\lambda^3 + r_4\lambda^4 + r_5\lambda^5 + r_6\lambda^6 + r_7\lambda^7 + r_8\lambda^8 + r_9\lambda^9}{16(4 + 3\lambda)(s_0 + s_1\lambda + s_2\lambda^2 + s_3\lambda^3 + s_4\lambda^4 + s_5\lambda^5 + s_6\lambda^6 + s_7\lambda^7)}} \tag{25}$$

where

$$\begin{aligned} r_0 &= 1099511627776 & r_1 &= 7352984010752 & r_2 &= 21769041739776 \\ r_3 &= 37447618527232 & r_4 &= 41248951894016 & r_5 &= 30171363606528 \\ r_6 &= 14654232029184 & r_7 &= 4557352944960 & r_8 &= 823439591472 \\ r_9 &= 65856986475 & s_0 &= 17179869184 & s_1 &= 89120571392 \\ s_2 &= 197199396864 & s_3 &= 241278386176 & s_4 &= 176300392448 \\ s_5 &= 76934648832 & s_6 &= 18565607712 & s_7 &= 1911234801. \end{aligned}$$

Beléndez et al. [45] proposed a new procedure for applying the generalized, rational harmonic balance method for constructing approximate analytical solutions to conservative nonlinear oscillations. In this method, the approximate solution obtained approximates all of the harmonics in the exact solution. They obtained the following approximate frequency for the Duffing oscillator:

$$\omega_{\text{RHB}}(\lambda) = \sqrt{1 + \frac{3}{8}\lambda \left(1 + \frac{\sqrt{1+B_2}}{\sqrt{1-B_2}}\right)} \tag{26}$$

where B_2 is defined by Eq. (27)

$$\begin{aligned} &\frac{2B_2\sqrt{1+B_2}}{(\sqrt{1-B_2} + \sqrt{1+B_2})^3} \left(64 + 27\lambda \left(1 + \frac{\sqrt{1+B_2}}{\sqrt{1-B_2}}\right)\right) + \frac{\lambda}{B_2^3} \left(4 - 4\frac{\sqrt{1+B_2}}{\sqrt{1-B_2}} + B_2 \left(12 - 8\frac{\sqrt{1+B_2}}{1-B_2}\right)\right) \\ &+ 2B_2^2 \left(6 - \frac{\sqrt{1+B_2}}{\sqrt{1-B_2}}\right) + B_2^3 \left(4 + 5\frac{\sqrt{1+B_2}}{\sqrt{1-B_2}}\right) = 0. \end{aligned} \tag{27}$$

Beléndez et al. [46] used a second-order modified rational harmonic balance method to solve the nonlinear differential equation that governs the oscillations of a conservative autonomous system with one degree of freedom. They analyzed the Duffing oscillator to illustrate the usefulness and effectiveness of the technique and found the approximate solution for the frequency of the oscillator (Eq. (28)); Beléndez et al. [46] claimed that their results are the best reported for this type of nonlinear oscillator:

$$\omega_{\text{RHBM2}}(\lambda) = \sqrt{\frac{h_0 + h_1\lambda + h_2\lambda^2 + h_3\lambda^3 + h_4\lambda^4 + h_5\lambda^5 + h_6\lambda^6 + h_7\lambda^7 + h_8\lambda^8 + h_9\lambda^9}{8(g_0 + g_1\lambda + g_2\lambda^2 + g_3\lambda^3 + g_4\lambda^4)(k_0 + k_1\lambda + k_2\lambda^2 + k_3\lambda^3 + k_4\lambda^4)}} \tag{28}$$

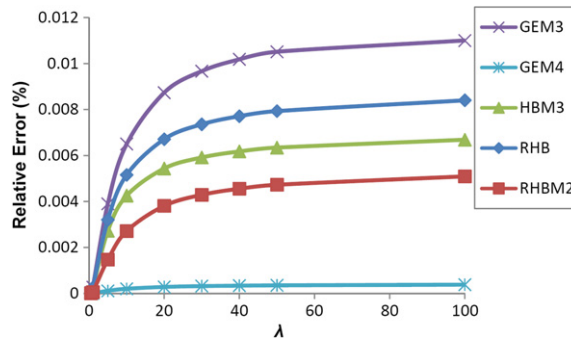


Fig. 1. Relative error for approximate frequencies.

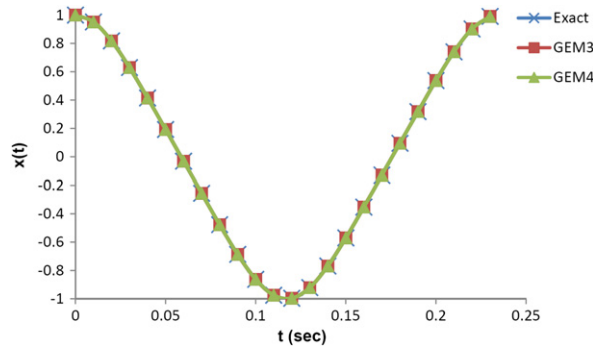


Fig. 2. Comparison of approximate periodic solutions with the exact solution.

where

$$\begin{array}{lll}
 h_0 = 137438953472 & h_1 = 906238099456 & h_2 = 2642344411136 \\
 h_3 = 4472235360256 & h_4 = 4843127439360 & h_5 = 3480739848192 \\
 h_6 = 1660534023168 & h_7 = 507162127392 & h_8 = 90002965320 \\
 h_9 = 7072335423 & g_0 = 65536 & g_1 = 192512 \\
 g_2 = 209664 & g_3 = 100548 & g_4 = 17955 \\
 k_0 = 262144 & k_1 = 761856 & k_2 = 820480 \\
 k_3 = 388896 & k_4 = 68589. &
 \end{array}$$

The comparison of the relative errors of our third- and fourth-order approximate frequencies with ω_{HBM3} , ω_{RHB} and ω_{RHBM2} is shown in Fig. 1.

As shown in Fig. 1, the third-order approximate solution provided by the GEM method is comparable with the other methods. However, the relative error of the fourth-order approximation provided by the GEM method is 0.00038%. This error is more accurate than the error provided by all other methods, about 13 times less than the relative error of ω_{RHBM2} , and offers excellent approximations of the exact frequency.

Thus far, we have been investigating the frequency of the approximate solution. Next, we will examine our approximate solution, $x(t)$. The exact solution of the cubic Duffing oscillator is given as follows [47]:

$$x_e(t) = cn[(1 + \lambda)^{\frac{1}{2}}t; k], \quad k = \frac{\lambda}{2(1 + \lambda)} \tag{29}$$

where cn is the Jacobi elliptic function.

In order to compare our results with the exact solution, we select $A = 1$ and $\varepsilon = 1000$ ($\lambda = \varepsilon A^2 = 1000$). Our third-order and fourth-order approximate solutions obtained using the GEM method are given by Eqs. (30) and (31), respectively. The frequency, ω , as shown in Table 1 is equal to 26.8139 and 26.8108 for the third- and fourth-order approximations, respectively.

$$\tilde{x}_3(t) = 1.91035 \cos(\omega t) + 0.0859606 \cos(3\omega t) + 0.00368755 \cos(5\omega t) \tag{30}$$

$$\tilde{x}_4(t) = 0.95508 \cos(\omega t) + 0.04299 \cos(3\omega t) + 0.00185 \cos(5\omega t) + 0.00008 \cos(7\omega t). \tag{31}$$

The comparison of the third- and fourth-order approximations with the exact solution is shown in Fig. 2.

It can be seen that our solution is almost identical to the exact solution. In order to better show the accuracy of the GEM solutions, the difference between the exact and approximate solutions, the solution error $= x_{\text{exact}}(t) - x_{\text{GEM}}(t)$, is plotted in Fig. 3.

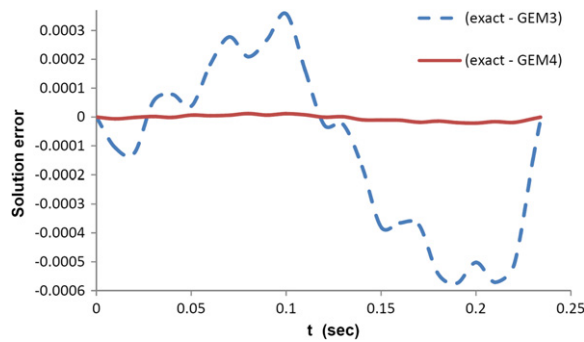


Fig. 3. Difference between exact and approximate solutions for $\varepsilon A^2 = 1000$.

Table 2
Comparison of the global error and maximum error of the GEM approximate solutions.

	Global error	Maximum solution error
First order	7116.6	0.218673
Second order	108.779	0.0189696
Third order	0.857511	0.00058939
Fourth order	0.00452174	0.00002258

As shown in Fig. 3 the third- and fourth-order GEM solutions offer excellent approximations to the exact solution for the Duffing nonlinear oscillator.

4. Results and discussion

As stated before, to develop the GEM method, we combined the general idea of global error minimization in the AVK method [35] and He’s variational approach [24] for solving nonlinear ODEs. Therefore, in this section the similarities of and differences between the GEM method, the AVK method and He’s variational approach are discussed.

The AVK method introduces a new approach for solving nonlinear optimal control problems (NOCP) by using the Nonlinear Programming Problem (NLPP). First, the original problem is transformed to a new problem in the form of the calculus of variations. The new problem is next discretized and solved using NLPP packages. The solution of the NLPP is used to obtain the optimal control and states. The main difference between the GEM and AVK methods is that the AVK method is a numerical approach while GEM is an analytical approach for solving nonlinear problems. Additionally, the AVK method is mostly used for addressing nonlinear optimal control problems. In developing the GEM method, we use the general idea of global error minimization presented in the AVK method.

The GEM method is similar to the variational approach for solving nonlinear oscillator differential equations. The solution procedure of the GEM method is similar to He’s variational approach [24]. He establishes a functional for the general nonlinear oscillator using a semi-inverse method. He assumes a trial solution and substitutes it in the functional. Next, the Ritz method is used and a relationship between the amplitude and frequency of the oscillator is established.

The main difference between the GEM method and He’s approach is in constructing the functional. He uses a semi-inverse method to establish a functional, while in the GEM method, constructing the functional is rather simple and straightforward. The GEM method uses an approach similar to least squares for constructing the functional. Furthermore, the functional in the GEM method has a valuable meaning. It indicates the global error in the differential equation. The defined functional enables a user to potentially use it as an index for checking the accuracy of the approximate solution. As an example, consider the case $\lambda = 1000$ for the Duffing oscillator discussed earlier. The global error of the solution for the Duffing equation (Eq. (3)) and the maximum error between the exact solution (Eq. (29)) and the approximate solutions are calculated as $(\text{Max}[x_{\text{exact}}(t) - x_{\text{GEM}}(t)])$ and reported in Table 2.

As shown in Table 2, a reduction in global error results in a lowering of the maximum solution error.

The significance of the global error index becomes more apparent when an exact solution for the nonlinear differential equation is not available. In such cases, approximate methods are used along with additional methods to assess the accuracy of the approximate solution. When using the global error index, the evaluation of the accuracy of the answer is rather inherent. The global error index gives us a window on the quality of the answer. Therefore, it is potentially possible that one may use the global error of the solution as a criterion for assessing the solution error.

5. Conclusions

A modified variational approach called Global Error Minimization (GEM) is proposed. The method provides accurate approximate analytical solutions for strongly nonlinear oscillator differential equations. We use the general idea of the

AVK numerical method [35] and convert the nonlinear ODE problem into an equivalent optimization problem to obtain an analytical solution of the ODE problem. The applicability and efficiency of our method were demonstrated by solving the strongly nonlinear cubic Duffing equation. The frequency of our approximate solution is compared with that of the exact solution as well as results obtained by [13,24,39,45,46]. The relative error for the frequency obtained by GEM is an impressive value of 0.0004% compared with exact frequency. Additionally, the approximate solution obtained by the GEM method is compared with the exact solution. The two solutions are almost identical. To the best of the authors' knowledge, the results presented in this paper are the best reported for the Duffing nonlinear oscillator at the present time.

The most significant features of this method are its approximate closed-form solution, its excellent accuracy, its simplicity, its natural nature in identifying the unknowns and it being straightforward to use in computer programs. Finally, we hope the GEM method will prove a gem in solving nonlinear problems!

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