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A RANGE FUNCTION APPROACH TO SHIFT-INVARIANT SPACES ON LOCALLY COMPACT ABELIAN GROUPS

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This paper develops several aspects of shift-invariant spaces on locally compact abelian groups. For a second countable locally compact abelian group G we prove a useful Hilbert space isomorphism, introduce range functions and give a characterization of shift-invariant subspaces of $L^2(G)$ in terms of range functions. Utilizing these functions, we generalize characterizations of frames and Riesz bases generated by shifts of a countable set of generators from $L^2(\mathbb{R}^n)$ to $L^2(G)$.

Keywords: Second countable locally compact abelian group; shift-invariant space; range function; frame; Riesz family.

AMS Subject Classification: 43A25, 42C15

1. Introduction and Preliminaries

Shift-invariant (SI) subspaces of $L^2(\mathbb{R}^n)$ are the spaces which are invariant under integer translations. The theory of shift-invariant spaces plays an important role in many areas, most notably in the theory of wavelets, spline systems, Gabor systems, radial function approximation and sampling theory. The general structure of these spaces in $L^2(\mathbb{R}^n)$ was revealed in the work of de Boor, DeVore and Ron with the use of fiberization techniques based on range functions.[2] The study of analogous spaces for $L^2(\mathbb{T}, H)$ with values in a separable Hilbert space H, in terms of range functions, is quite classical and goes back to Helson.[8] Recently Bownik gave a characterization of shift-invariant subspaces of $L^2(\mathbb{R}^n)$ following an idea from Helson's book.[3] So far the theory of SI spaces has been investigated on $\mathbb{R}^n$ but to work with other concrete examples of locally compact abelian (LCA) groups, it is essential for the theory to be extended to the general setting. Some general properties of SI spaces on LCA groups, have been studied by the authors.[10] The present paper is devoted to the study of structural properties of SI spaces on second countable LCA groups using a range function approach.

Let G be an LCA group equipped with a Haar measure m_G. We shall use the notations and constructions of Ref. 6 associated to LCA groups. The dual group of G is denoted by $\hat{G}$. The Fourier transform $\hat{f}$ of any function $f \in L^1(G)$ is defined by $\hat{f}(\xi) = \int_G f(x)\overline{\xi}(x)dm_G(x)$, where ξ is an element in $\hat{G}$. The Plancherel Theorem asserts that the Fourier transform $\hat{}\, : L^1(G) \cap L^2(G) \to C_0(\hat{G})$, $f \mapsto \hat{f}$ extends uniquely to a Hilbert space isomorphism from $L^2(G)$ onto $L^2(\hat{G})$, the so-called Plancherel isomorphism again denoted by $\hat{}$.[6]

Let K be a closed subgroup of G and G/K be the quotient group whose Haar measure υ is unique up to a constant factor. If this factor is suitably chosen we have

$$\int_G f(x)dx = \int_{G/K}\int_K f(xy)dm_K(y)d\upsilon(xK), \quad f \in L^1(G). \tag{1.1}$$

This identity is known as Weil's formula.[6]

A subgroup L of G is called a uniform lattice if it is discrete and co-compact (i.e. G/L is compact). For a uniform lattice L in G, a fundamental domain is a measurable set S_L in G such that every $x \in G$ can be uniquely written in the form $x = ks$, where $k \in L$ and $s \in S_L$. The existence of a fundamental domain for a uniform lattice in a LCA group is guaranteed by Ref. 11.

Let L be a subgroup of G. Then the subgroup $L^\perp = \{\xi \in \hat{G};\ \xi(L) = \{1\}\}$ is called the annihilator of L in $\hat{G}$. For a uniform lattice L in G the subgroup $L^\perp$ is a uniform lattice in $\hat{G}$.

Now we define a SI space on a LCA group.

Let G be a LCA group and L be a uniform lattice in G. A closed subspace $V \subseteq L^2(G)$ is called SI (with respect to L) if $f \in V$ implies $T_k f \in V$, for any $k \in L$, where T_k is the translation operator defined by $T_k f(x) = f(k^{-1}x)$ for all $x \in G$. For any subset $\phi \subseteq L^2(G)$, let $S(\phi) = \overline{\text{span}}\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$ be the SI space generated by ϕ.

The rest of this paper is organized as follows. In Sec. 2, we prove a Hilbert space isomorphism, which is famous on $\mathbb{R}^n$, in the setting of LCA groups, under the aspects of direct integral and group theory. In fact we have found a formulation that does not require the concepts that are peculiar to $\mathbb{R}^n$. In Sec. 3, we give a definition of a range function in a LCA group. The main result of Sec. 3 is Theorem 3.1, which sets out a characterization of SI subspaces of $L^2(G)$ in terms of range functions, using various tools in abstract harmonic analysis. In Sec. 4, we generalize a characterization of frames generated by shifts of a countable set of generators in terms of their behavior on subspaces of $l^2(L^\perp)$. In Sec. 5, we give some examples which reveal the strength of our generalization.

2. A Hilbert Space Isomorphism

Throughout this paper we always assume that G is a second countable LCA group, L is a uniform lattice in G, and $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$ with a measure $d\xi$ on it. We show that $L^2(G)$ is isometrically isomorphic to the space

$L^2(S_{L^\perp}, l^2(L^\perp))$ of square integrable functions from $S_{L^\perp}$ to $l^2(L^\perp)$. Notice that this space is just the direct integral $\int_A^\oplus H_\xi d\xi$, where $A = S_{L^\perp}$ and $H_\xi = l^2(L^\perp)$, for all $\xi \in S_{L^\perp}$.[6] $L^2(S_{L^\perp}, l^2(L^\perp))$ is a Hilbert space with inner product $\langle f, g\rangle = \int_{S_{L^\perp}} \langle f(\xi), g(\xi)\rangle_{l^2(L^\perp)} d(\xi)$.[5]

Proposition 2.1. *The mapping* $\mathcal{T} : L^2(G) \to L^2(S_{L^\perp}, l^2(L^\perp))$, *defined by* $\mathcal{T}f(\xi) = (\hat{f}(\xi\eta))_{\eta\in L^\perp}$ *is an isometric isomorphism, between* $L^2(G)$ *and* $L^2(S_{L^\perp}, l^2(L^\perp))$.

Proof. For every $f \in L^2(G)$, using Weil's formula and the Plancherel Theorem we have

$$\begin{aligned} \|\mathcal{T}f\|^2 &= \int_{S_{L^\perp}} \|\mathcal{T}f(\xi)\|^2_{l^2(L^\perp)} d\xi \\ &= \int_{S_{L^\perp}} \sum_{\eta\in L^\perp} |\hat{f}(\xi\eta)|^2 d\xi \\ &= \int_{\hat{G}} |\hat{f}(\xi)|^2 d\xi \\ &= \|f\|_2^2. \end{aligned} \tag{2.1}$$

So $\mathcal{T}$ is an isometry. Let $g \in L^2(S_{L^\perp}, l^2(L^\perp))$. Let f be given by $\hat{f}(\eta) = g(\xi)(\alpha)$, for every $\eta \in \hat{G}$ of the form $\eta = \xi\alpha$, for $\xi \in S_{L^\perp}, \alpha \in L^\perp$. Then obviously $\mathcal{T}f = g$. So $\mathcal{T}$ is onto, and the proof is complete. □

Applying Proposition 2.1 to $G = \mathbb{R}^n$ and $L = \mathbb{Z}^n$, the following corollary, which is stated in Ref. 3, is immediate.

Corollary 2.1. *The mapping* $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$ *defined for* $f \in L^2(\mathbb{R}^n)$ *by* $\mathcal{T}f : \mathbb{T}^n \to l^2(\mathbb{Z}^n)$, $\mathcal{T}f(x) = (\hat{f}(x+k))_{k\in\mathbb{Z}^n}$, *is an isometric isomorphism between* $L^2(\mathbb{R}^n)$ *and* $L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$.

Consider $L^2(\hat{L}, l^2(L^\perp))$ as the direct integral $\int_A^\oplus l^2(L^\perp) d\lambda$, for $A = \hat{L}$ with its Haar measure λ. It is interesting to note that this space is also isometrically isomorphic to $L^2(G)$. To prove it we use a direct integral argument.

Proposition 2.2. $L^2(\hat{L}, l^2(L^\perp))$ *is isometrically isomorphic to* $L^2(G)$.

Proof. By Ref. 5, we have

$$\left(\int_{\hat{L}}^\oplus \mathbb{C} d\lambda\right) \otimes l^2(L^\perp) \simeq \int_{\hat{L}}^\oplus (\mathbb{C} \otimes l^2(L^\perp)) d\lambda,$$

where $\otimes$ is the Hilbert space tensor product (see Ref. 12). The right-hand side is isometrically isomorphic to $\int_{\hat{L}}^\oplus l^2(L^\perp) d\lambda$. Therefore,

$$L^2(\hat{L}) \otimes l^2(L^\perp) \simeq L^2(\hat{L}, l^2(L^\perp)).$$

Let S_L denote a fundamental domain for L in G. We have $l^2(L^\perp) \simeq L^2(S_L)$, $L^2(\hat{L}) \simeq L^2(S_{L^\perp})$.[11] Thus,

$$L^2(S_{L^\perp}) \otimes L^2(S_L) \simeq L^2(\hat{L}, l^2(L^\perp)).$$

But $L^2(S_{L^\perp}) \otimes L^2(S_L) \simeq L^2(S_{L^\perp} \times S_L)$,[6] (note that S_L and $S_{L^\perp}$ are of finite measure[11]), and $L^2(G) \simeq L^2(S_{L^\perp} \times S_L)$.[11] So $L^2(G) \simeq L^2(\hat{L}, l^2(L^\perp))$. (By $\simeq$ we mean "is isometrically isomorphic to".) □

As an immediate consequence of Propositions 2.1 and 2.2 we have:

Corollary 2.2. *Suppose G is a second countable LCA group, L is a uniform lattice in G and $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$. Then the three Hilbert spaces $L^2(G), L^2(\hat{L}, l^2(L^\perp))$ and $L^2(S_{L^\perp}, l^2(L^\perp))$ are isometrically isomorphic.*

3. A Characterization of Shift-Invariant Spaces

Let G be a LCA group and L be a uniform lattice in G. A range function is a mapping

$$J : S_{L^\perp} \to \{\text{closed subspaces of } l^2(L^\perp)\}.$$

J is called measurable if the associated orthogonal projections $P(\xi) : l^2(L^\perp) \to J(\xi)$ are measurable i.e. $\xi \mapsto \langle P(\xi)a, b\rangle$ is measurable for each $a, b \in l^2(L^\perp)$ (see Ref. 5).

The main result of this section is the following characterization theorem in $L^2(G)$.

Theorem 3.1. *Suppose G is a second countable LCA group, L is a uniform lattice in G, and $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$. A closed subspace $V \subseteq L^2(G)$ is SI (with respect to the uniform lattice L) if and only if $V = \{f \in L^2(G),\ \mathcal{T}f(\xi) \in J(\xi)$ for a.e. $\xi \in S_{L^\perp}\}$, where J is a measurable range function and $\mathcal{T}$ is the mapping as in Proposition 2.1. The correspondence between V and J is one to one under the convention that the range functions are identified if they are equal a.e. Moreover, if $V = S(\phi)$ for some countable set $\phi \subseteq L^2(G)$ then*

$$J(\xi) = \overline{\text{span}}\{\mathcal{T}\varphi(\xi);\ \varphi \in \phi\}. \tag{3.1}$$

We will prove this theorem in the sequel. For this, we need some preparations. We start with a definition.

Definition 3.1. For a given range function J, we define the space

$$M_J = \{\varphi \in L^2(S_{L^\perp}, l^2(L^\perp)),\ \varphi(\xi) \in J(\xi) \text{ for a.e. } \xi \in S_{L^\perp}\}. \tag{3.2}$$

The following proposition entails that M_J defined by (3.2) is a Hilbert subspace of $L^2(S_{L^\perp}, l^2(L^\perp))$.

Proposition 3.1. *Let J be a range function. Then M_J is a closed subspace of $L^2(S_{L^\perp}, l^2(L^\perp))$.*

Proof. Let $\varphi \in \overline{M_J}$. Take any $(\varphi_n) \subseteq M_J$ converging to φ in $L^2(S_{L^\perp}, l^2(L^\perp))$. By Ref. 5, there exists a subsequence (φ_{n_i}) of (φ_n) which converges to φ almost everywhere; that is $\varphi_{n_i}(\xi) \to \varphi(\xi)$ as $n_i \to \infty$, for a.e. $\xi \in S_{L^\perp}$. Since the space $J(\xi)$ is closed, $\varphi \in M_J$. Hence M_J is closed. $\square$

The following lemma is needed in the proof of Theorem 3.1.

Lemma 3.1. *Let J be a measurable range function with associated orthogonal projections P. Let Q denote the orthogonal projection of $L^2(S_{L^\perp}, l^2(L^\perp))$ onto M_J. Then for any $\varphi \in L^2(S_{L^\perp}, l^2(L^\perp))$,*

$$(Q\varphi)(\xi) = P(\xi)(\varphi(\xi))$$

for a.e. $\xi \in S_{L^\perp}$.

Proof. Define $\mathsf{P}' : L^2(S_{L^\perp}, l^2(L^\perp)) \to L^2(S_{L^\perp}, l^2(L^\perp))$, by $(\mathsf{P}'\varphi)(\xi) = P(\xi)(\varphi(\xi))$. Note that $\mathsf{P}'(L^2(S_{L^\perp}, l^2(L^\perp))) \subseteq L^2(S_{L^\perp}, l^2(L^\perp))$. Indeed, since $\|P(\xi)\| \leq 1$ for every $\varphi \in L^2(S_{L^\perp}, l^2(L^\perp))$, we have

$$\begin{aligned}
\|\mathsf{P}'\varphi\|^2_{L^2(S_{L^\perp}, l^2(L^\perp))} &= \int_{S_{L^\perp}} \|\mathsf{P}'\varphi(\xi)\|^2_{l^2(L^\perp)} d\xi \\
&= \int_{S_{L^\perp}} \|P(\xi)(\varphi(\xi))\|^2_{l^2(L^\perp)} d\xi \\
&\leq \int_{S_{L^\perp}} \|\varphi(\xi)\|^2_{l^2(L^\perp)} d\xi \\
&= \|\varphi\|^2_{L^2(S_{L^\perp}, l^2(L^\perp))} < \infty.
\end{aligned}$$

Clearly $(\mathsf{P}')^2 = \mathsf{P}'$ and $(\mathsf{P}')^* = \mathsf{P}'$, since $P(\xi)$ has these properties for a.e. $\xi \in S_{L^\perp}$. So P' is an orthogonal projection with range $\acute{M}$. Obviously $\acute{M} \subseteq M_J$ (since $P(\xi)$ is an orthogonal projection onto $J(\xi)$). To complete the proof we show that $M_J \subseteq \acute{M}$. Suppose by contradiction that there is $0 \neq \psi \in M_J$ which is orthogonal to $\acute{M}$. Then for each $\varphi \in L^2(S_{L^\perp}, l^2(L^\perp))$ we have

$$\begin{aligned}
0 = \langle \mathsf{P}'\varphi, \psi \rangle_{L^2(S_{L^\perp}, l^2(L^\perp))} &= \int_{S_{L^\perp}} \langle \mathsf{P}'\varphi(\xi), \psi(\xi) \rangle_{l^2(L^\perp)} d\xi \\
&= \int_{S_{L^\perp}} \langle P(\xi)(\varphi(\xi)), \psi(\xi) \rangle_{l^2(L^\perp)} d\xi \\
&= \int_{S_{L^\perp}} \langle \varphi(\xi), P(\xi)\psi(\xi) \rangle_{l^2(L^\perp)} d\xi \\
&= \int_{S_{L^\perp}} \langle \varphi(\xi), \psi(\xi) \rangle_{l^2(L^\perp)} d\xi \\
&= \langle \varphi, \psi \rangle_{L^2(S_{L^\perp}, l^2(L^\perp))}.
\end{aligned}$$

So $\psi = 0$ which is a contradiction. This completes the proof. $\square$

Proof of Theorem 3.1. Suppose $V = S(\phi)$ is a SI space for some countable set $\phi \subseteq L^2(G)$, $M = \mathcal{T}V$ and $J(\xi)$ is given by (3.1). It is enough to show that $M = M_J$. Let $\varphi \in M$. Then there exists a sequence $\{\varphi_n\}$ converging to φ such that $\mathcal{T}^{-1}\varphi_n \in \text{span}\{T_k\varphi; \varphi \in \phi,\ k \in L\}$. Since $\mathcal{T}T_k\varphi(\xi) = ((\widehat{T_k\varphi})(\xi\eta))_{\eta\in L^\perp} = (\hat{\varphi}(\xi\eta)\overline{\xi}(k))_{\eta\in L^\perp} = \overline{\xi}(k)\mathcal{T}\varphi(\xi)$, thus $\varphi_n(\xi) \in J(\xi)$ and so $\varphi(\xi) \in J(\xi)$. This implies that $M \subseteq M_J$.

To show that $M_J \subseteq M$, we observe that $M^\perp = \{0\}$. Take any $\psi \in L^2(S_{L^\perp}, l^2(L^\perp))$ which is orthogonal to M. For any $\varphi \in \mathcal{T}\phi$ and $k \in L$, we have $M_k\varphi \in \mathcal{T}V$, where $M_k\varphi(\xi) = \overline{\xi}(k)\varphi(\xi)$, so $0 = \langle M_k\varphi, \psi\rangle = \int_{S_{L^\perp}} \overline{\xi}(k)\langle\varphi(\xi), \psi(\xi)\rangle_{l^2(L^\perp)} d\xi$. Hence $\langle\varphi(\xi), \psi(\xi)\rangle = 0$ for a.e. $\xi \in S_{L^\perp}$ and any $\varphi \in \mathcal{T}\phi$. Thus $\psi(\xi) \in J(\xi)^\perp$ for a.e. $\xi \in S_{L^\perp}$. This implies that there is no $0 \neq \psi \in M_J$ which is orthogonal to M. Therefore $M = M_J$. Moreover we need to show that J, given by (3.1) is measurable. Let $P(\xi)$ be the orthogonal projection of $l^2(L^\perp)$ onto $J(\xi)$ and $\psi \in L^2(S_{L^\perp}, l^2(L^\perp))$. By Ref. 5, it is enough to show that $\xi \mapsto P(\xi)\psi(\xi)$ is measurable. Let Q denote the orthogonal projection of $L^2(S_{L^\perp}, l^2(L^\perp))$ onto M. Since the field $\xi \mapsto Q\psi(\xi)$ is measurable, by Lemma 3.1, so is $\xi \mapsto P(\xi)\psi(\xi)$. Thus J is measurable.

Conversely, if J is a measurable range function and V is given by (3.1) then since $V = \mathcal{T}^{-1}M_J$, obviously it is a closed shift-invariant space.

Suppose $M_{J_1} = M_{J_2}$ for some measurable range functions J_1 and J_2 with associated projections P_1 and P_2, respectively. Then $J_1(\xi) = J_2(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Indeed, if we apply Lemma 3.1 to the constant function $\varphi(\xi) = e_\eta$, where $(e_\eta)_{\eta\in L^\perp}$ is the standard basis of $l^2(L^\perp)$, then we have $P_1(\xi)e_\eta = P_2(\xi)e_\eta$ for all $\eta \in L^\perp$ and a.e. $\xi \in S_{L^\perp}$. Therefore $P_1(\xi) = P_2(\xi)$ for a.e. $\xi \in S_{L^\perp}$. So the correspondence between V and J is one to one. □

Our goal in the next section is to generalize a characterization of frames generated by shifts of a countable set of generators in $L^2(G)$.

4. A Characterization of Frames Generated by Shifts of Functions

Suppose H is a Hilbert space. $X \subseteq H$ is called a *frame* (for $\overline{\text{span}}(X)$), if there exist two numbers A and B with $0 < A \leq B < \infty$ such that

$$A\|h\|^2 \leq \sum_{\eta\in X} |\langle h, \eta\rangle|^2 \leq B\|h\|^2 \quad \text{for } h \in \text{span}(X). \tag{4.1}$$

The numbers A and B are called the frame bounds. X is called a fundamental frame if $\text{span}(X)$ is dense in H.

Suppose G is a second countable LCA group, L is a uniform lattice in G, $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$ and $\phi \subseteq L^2(G)$ is a countable set. The following theorem allows us to reduce the problem of checking whether $\{T_k\varphi; \varphi \in \phi, k \in L\}$

is a frame in a subspace of $L^2(G)$, to analyzing the elements in subspaces of $l^2(L^\perp)$, parametrized by the base space $S_{L^\perp}$.

Theorem 4.1. *Suppose G is a second countable LCA group, L is a uniform lattice in G, $S_{L^\perp}$ is a fundamental domain for $L^\perp$ in $\hat{G}$, $\phi \subseteq L^2(G)$ is a countable set and $\mathcal{T}$ is the mapping defined in Proposition 2.1. Then $\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$ is a frame for $S(\phi)$ with bounds A and B if and only if $\{\mathcal{T}\varphi(\xi);\ \varphi \in \phi\}$ is a frame for $J(\xi)$ with bounds A and B, for a.e. $\xi \in S_{L^\perp}$. Moreover $\{T_k\varphi; \varphi \in \phi, k \in L\}$ is a fundamental frame if and only if $\{\mathcal{T}\varphi(\xi); \varphi \in \phi\}$ is a fundamental frame, for a.e. $\xi \in S_{L^\perp}$.*

Before presenting the proof, we need to establish the following lemma.

Lemma 4.1. *Retain the assumptions of Theorem 4.1. Then for any $f \in L^2(G)$, we have*

$$\sum_{\varphi\in\phi}\sum_{k\in L}|\langle T_k\varphi, f\rangle|^2 = \sum_{\varphi\in\phi}\int_{S_{L^\perp}}|\langle \mathcal{T}\varphi(\xi), \mathcal{T}f(\xi)\rangle|^2 d\xi. \tag{4.2}$$

Proof. For $f \in L^2(G)$, $k \in L$ and $\varphi \in \phi$ we have

$$\begin{aligned}
\langle T_k\varphi, f\rangle &= \langle \mathcal{T}T_k\varphi, \mathcal{T}f\rangle \\
&= \int_{S_{L^\perp}} \langle \mathcal{T}T_k\varphi(\xi), \mathcal{T}f(\xi)\rangle_{l^2(L^\perp)} d\xi \\
&= \int_{S_{L^\perp}} \langle (\widehat{T_k\varphi}(\xi\eta))_{\eta\in L^\perp}, (\widehat{f}(\xi\eta))_{\eta\in L^\perp}\rangle_{l^2(L^\perp)} d\xi \\
&= \int_{S_{L^\perp}} \overline{\xi}(k)\langle (\hat{\varphi}(\xi\eta))_{\eta\in L^\perp}, (\hat{f}(\xi\eta))_{\eta\in L^\perp}\rangle_{l^2(L^\perp)} d\xi \\
&= \int_{S_{L^\perp}} \overline{\xi}(k)\langle \mathcal{T}\varphi(\xi), \mathcal{T}f(\xi)\rangle d\xi.
\end{aligned}$$

Utilizing the Plancherel Theorem, the Pontrjagin Duality Theorem,[6] and the fact that $L^2(S_{L^\perp}) = L^2(\hat{L})$, we have

$$\begin{aligned}
\sum_{\varphi\in\phi}\sum_{k\in L}|\langle T_k\varphi, f\rangle|^2 &= \sum_{\varphi\in\phi}\sum_{k\in L}\left|\int_{S_{L^\perp}} \overline{\xi}(k)\langle \mathcal{T}\varphi(\xi), \mathcal{T}f(\xi)\rangle d\xi\right|^2 \\
&= \sum_{\varphi\in\phi}\sum_{k\in L}\int_{S_{L^\perp}} F_\varphi(\xi)\overline{k(\xi)}d\xi \overline{\int_{S_{L^\perp}} F_\varphi(\xi)\overline{k(\xi)}d\xi} \\
&= \sum_{\varphi\in\phi}\sum_{k\in L}\hat{F}_\varphi(k)\overline{\hat{F}_\varphi(k)} \\
&= \sum_{\varphi\in\phi}\langle \hat{F}_\varphi, \hat{F}_\varphi\rangle_{l^2(L)}
\end{aligned}$$

$$= \sum_{\varphi\in\phi} \langle F_\varphi, F_\varphi\rangle_{L^2(\hat{L})}$$

$$= \sum_{\varphi\in\phi} \int_{S_{L^\perp}} |\langle \mathcal{T}\varphi(\xi), \mathcal{T}f(\xi)\rangle_{l^2(L^\perp)}|^2 d\xi$$

where $F_\varphi(\xi) = \langle \mathcal{T}\varphi(\xi), \mathcal{T}f(\xi)\rangle$. □

Proof of Theorem 4.1. Let J be the range function associated with $S(\phi)$, given by (3.1). Suppose that $\{\mathcal{T}\varphi(\xi); \varphi \in \phi\}$ is a frame for $J(\xi)$ with bounds A and B, for a.e. $\xi \in S_{L^\perp}$ i.e.

$$A\|a\|^2 \le \sum_{\varphi\in\phi} |\langle \mathcal{T}\varphi(\xi), a\rangle_{l^2(L^\perp)}|^2 \le B\|a\|^2 \quad \text{for } a \in J(\xi). \tag{4.3}$$

If $f \in S(\phi)$, then $a = \mathcal{T}f(\xi) \in J(\xi)$ for a.e. $\xi \in S_{L^\perp}$. Now by (2.1), (4.2) and (4.3), $\{T_k\varphi; \varphi \in \phi, k \in L\}$ is a frame with bounds A and B. Indeed, since

$$A\|\mathcal{T}f(\xi)\|^2 \le \sum_{\varphi\in\phi} |\langle \mathcal{T}\varphi(\xi), \mathcal{T}f(\xi)\rangle_{l^2(L^\perp)}|^2 \le B\|\mathcal{T}f(\xi)\|^2,$$

we have

$$A\int_{S_{L^\perp}} \|\mathcal{T}f(\xi)\|^2 d\xi \le \sum_{\varphi\in\phi} \int_{S_{L^\perp}} |\langle \mathcal{T}\varphi(\xi), \mathcal{T}f(\xi)\rangle_{l^2(L^\perp)}|^2 d\xi$$

$$\le B\int_{S_{L^\perp}} \|\mathcal{T}f(\xi)\|^2 d\xi.$$

Therefore, $A\|f\|_2^2 \le \sum_{\varphi\in U}\sum_{k\in L} |\langle T_k\varphi, f\rangle_{l^2(L^\perp)}|^2 \le B\|f\|_2^2$.

For the converse suppose that $\{T_k\varphi; \varphi \in \phi, k \in L\}$ is a frame with bounds A and B. Let $\{d_i\}_{i=1}^\infty$ be a dense subset of $l^2(L^\perp)$. It is enough to show that

$$A\|P(\xi)d_i\|^2 \le \sum_{\varphi\in\phi} |\langle \mathcal{T}\varphi(\xi), P(\xi)d_i\rangle_{l^2(L^\perp)}|^2$$

$$\le B\|P(\xi)d_i\|^2 \quad \text{for a.e. } \xi \in S_{L^\perp}, \quad \text{for all } i \in \mathbb{N}, \tag{4.4}$$

where $P(\xi)$ is the projection onto $J(\xi)$. By the contrary assume that (4.4) fails. Then there exist a measurable set $D \subseteq S_{L^\perp}$, with $|D| > 0$, $i_0 \in \mathbb{N}$ and $\varepsilon > 0$, such that at least one of the following holds:

$$\sum_{\varphi\in\phi} |\langle \mathcal{T}\varphi(\xi), P(\xi)d_{i_0}\rangle_{l^2(L^\perp)}|^2 > (B+\varepsilon)\|P(\xi)d_{i_0}\|^2 \quad (\xi \in D) \tag{4.5}$$

$$\sum_{\varphi\in\phi} |\langle \mathcal{T}\varphi(\xi), P(\xi)d_{i_0}\rangle_{l^2(L^\perp)}|^2 < (A-\varepsilon)\|P(\xi)d_{i_0}\|^2 \quad (\xi \in D). \tag{4.6}$$

First suppose that (4.5) happens. Let $f \in S(\phi)$ be given by $\mathcal{T}f(\xi) = \chi_D(\xi)P(\xi)d_{i_0}$. Then by (4.2),

$$\begin{aligned}
\sum_{k\in L}\sum_{\varphi\in\phi} |\langle T_k\varphi, f\rangle_{l^2(L^\perp)}|^2 &= \int_{S_{L^\perp}} \sum_{\varphi\in\phi} |\langle \mathcal{T}\varphi(\xi), \chi_D(\xi)P(\xi)d_{i_0}\rangle|^2 d\xi \\
&= \int_{S_{L^\perp}} \chi_D(\xi) \sum_{\varphi\in\phi} |\langle \mathcal{T}\varphi(\xi), P(\xi)d_{i_0}\rangle|^2 d\xi \\
&\geq \int_{S_{L^\perp}} (B+\varepsilon)\chi_D(\xi)\|P(\xi)d_{i_0}\|^2 d\xi \\
&= (B+\varepsilon)\int_{S_{L^\perp}} \|\chi_D(\xi)P(\xi)d_{i_0}\|^2 d\xi \\
&= (B+\varepsilon)\int_{S_{L^\perp}} \|\mathcal{T}f(\xi)\|^2 d\xi \\
&= (B+\varepsilon)\|f\|_2^2
\end{aligned}$$

which is a contradiction. A similar argument shows that (4.6) cannot hold. So (4.3) is true. The statement about fundamental frames is an immediate consequence of Theorem 3.1. □

For a Hilbert space H, $X \subseteq H$ is called a Riesz family if there exist two constants A and B such that

$$A\sum_{\eta\in X} |h_\eta|^2 \leq \left\| \sum_{\eta\in X} h_\eta \eta \right\|^2 \leq B\sum_{\eta\in X} |h_\eta|^2, \tag{4.7}$$

for all finitely supported $(h_\eta)_{\eta\in X} \subseteq \mathbb{C}$. The ideas in the proof of Theorem 4.1 can be used to prove an analogous necessary and sufficient condition for $\{T_k\varphi; \varphi \in \phi, k \in L\}$ to be a Riesz family. That is we have

Theorem 4.2. *Retain the assumptions of Theorem 4.1. Then* $\{T_k\varphi;\ \varphi \in \phi,\ k \in L\}$ *is a Riesz family with constants* A *and* B *if and only if* $\{\mathcal{T}\varphi(\xi);\ \varphi \in \phi\}$ *is a Riesz family with constants* A *and* B, *for a.e.* $\xi \in S_{L^\perp}$.

5. Application and Examples

We provide here some examples that illustrate the power of the results developed in this paper. Applying Theorem 3.1 we get to an observation about the image of continuous wavelet transform. Let G be a second countable LCA group, L be a uniform lattice in G and $S_{L^\perp}$ be a fundamental domain for $L^\perp$ in $\hat{G}$. Suppose that π is the left regular representation of G on $L^2(G)$ (see Ref. 6). For an admissible vector $\psi \in L^2(G)$, consider the continuous wavelet transform $V_\psi : L^2(G) \to L^2(G)$ defined by $V_\psi\varphi(x) = \langle\varphi, \pi(x)\psi\rangle$. By Ref. 7, $V_\psi(L^2(G))$ is a closed shift-invariant subspace of $L^2(G)$. So by Ref. 10, $V_\psi(L^2(G)) = \bigoplus_{n\in\mathbb{N}} V_{h_n}$, where $\{h_n\}_{n\in\mathbb{N}}$ is a

Parseval frame generator (see Ref. 10) of the space V_{h_n}. If we apply Theorem 3.1 to this space then $V_\psi(L^2(G)) = \{f \in L^2(G);\ \mathcal{T}f(\xi) \in J(\xi)$ a.e. $\xi \in S_{L^\perp}\}$, for the range function J given by $J(\xi) = \overline{\mathrm{span}}\{h_n(\xi); n \in \mathbb{N}\}$. (For example, let $G = \mathbb{Z}$ and $L = 2\mathbb{Z}$. Then $L^\perp = \widehat{(\mathbb{Z}/2\mathbb{Z})} = \widehat{\mathbb{Z}_2} = \mathbb{Z}_2$ and $\mathbb{T}$ is a fundamental domain for $\mathbb{Z}_2$ in $\hat{G} = \mathbb{T}$.)

Example 5.1. For applications the most important class of LCA groups is the class of compactly generated LCA Lie groups. By the Structure Theorem for compactly generated LCA Lie groups, these groups are of the form $\mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^r \times F$, where $p, q, r \in \mathbb{N}_0$ and F is a finite abelian group (see Ref. 8). Let $G = \mathbb{R}^a \times \mathbb{Z}^b \times \mathbb{T}^c \times \mathbb{Z}_d$ for $a, b, c, d \in \mathbb{N}$, where $\mathbb{Z}_d$ is the finite abelian group $\{0, 1, 2, \dots, d-1\}$ of residues modulo d. Fix $\alpha \in \mathbb{N}$. Then $\hat{G} = \mathbb{R}^a \times \mathbb{Z}^c \times \mathbb{T}^b \times \mathbb{Z}_d$ and $L = \mathbb{Z}^a \times \alpha\mathbb{Z}^b \times \mathbb{Z}_d$ is a uniform lattice in G. Thus $L^\perp = \mathbb{Z}^a \times \mathbb{Z}^c \times \mathbb{Z}^b_\alpha$. Obviously $S_{L^\perp} := \mathbb{T}^a \times \alpha\mathbb{T}^b \times \mathbb{Z}_d$ is a fundamental domain for $L^\perp$ in $\hat{G}$. Consider the orthonormal basis $B := B_1 \otimes B_2 \otimes B_3 \otimes B_4$ for $L^2(G)$, where $B_1 = \{M_\gamma T_k \chi_{[0,1)};\ k, \gamma \in \mathbb{Z}^a\}$, in which $M_\gamma T_k \chi_{[0,1)}(x) = e^{2\pi i \gamma x}\chi_{[0,1)}(x-k)$ for $x \in \mathbb{R}^a$, $B_2 = \{\chi_{\{m\}}; m \in \mathbb{Z}^b\}$, $B_3 = \{e^{2\pi i l};\ l \in \mathbb{Z}^c\}$, $B_4 = \mathbb{Z}_d$. Then $V := \bigoplus_{\varphi \in B, \gamma \in L^\perp} V_{\varphi,\gamma}$, in which $V_{\varphi,\gamma} = \overline{\mathrm{span}}\{M_\gamma T_k \varphi;\ k \in L\}$, $\varphi \in B, \gamma \in L^\perp$, is a shift-invariant subspace of $L^2(G)$. By Theorem 3.1, $V = \{f \in L^2(G),\ (\hat{f}(\xi\eta))_{\eta \in L^\perp} \in J(\xi)$ for a.e. $\xi \in S_{L^\perp}\}$, where $J(\xi) = \{\mathcal{T}(M_\gamma\varphi)(\xi);\ \varphi \in B,\ \gamma \in L^\perp\} = \overline{\mathrm{span}}\{\hat{\varphi}(\gamma^{-1}\xi\eta)_{\eta \in L^\perp};\ \varphi \in B,\ \gamma \in L^\perp\}$.

Example 5.2. Assume that A is an n by n expanding matrix which preserves $\mathbb{Z}^n$. Let $\psi \in L^2(\mathbb{R}^n)$ be a wavelet. Define $\psi_{m,k}(x) = |\det A|^{m/2}\psi(A^m x - k)$, for $m \in \mathbb{Z},\ k \in \mathbb{Z}^n$. For each $m \in \mathbb{Z}$, define $W_m := \overline{\mathrm{span}}\{\psi_{m,k};\ k \in \mathbb{Z}^n\}$. Since each W_m is a shift-invariant subspace of $L^2(\mathbb{R}^n)$ for $m \geq 0$, so is $(\bigoplus_{m \geq 0} W_m)^\perp = \bigoplus_{m<0} W_m$. So by Theorem 3.1, $(\bigoplus_{m \geq 0} W_m)^\perp = \{f \in L^2(\mathbb{R}^n);\ \mathcal{T}f(x) \in J(x)$, for a.e. $x \in \mathbb{T}^n\}$ for the range function J given by $J(x) = \overline{\mathrm{span}}\{g_m(x),\ m \geq 1\}$, where the isometric isomorphism $\mathcal{T} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{T}^n, l^2(\mathbb{Z}^n))$ is given by $\mathcal{T}f(\xi) = (\hat{f}(\xi + l))_{l \in \mathbb{Z}^n}$, and

$$g_m(x) = (\hat{\psi}((A^T)^m(x+l)))_{l \in \mathbb{Z}^n}, \quad \text{a.e.}$$

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