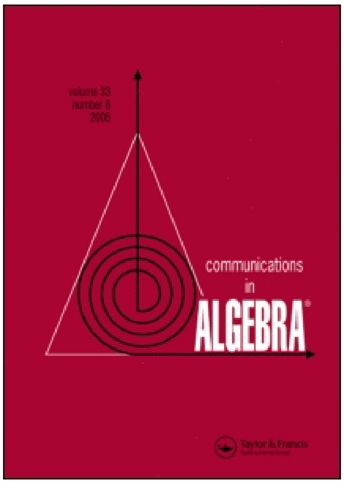

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CHARACTERIZATION OF HIGHER DERIVATIONS ON ALGEBRAS

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Let $\mathscr{A}$ be an algebra. A sequence $\{d_n\}$ of linear mappings on $\mathscr{A}$ is called a higher derivation if $d_n(ab) = \sum_{k=0}^{n} d_k(a)d_{n-k}(b)$ for each $a, b \in \mathscr{A}$ and each non-negative integer n. In this article, we show that if $\{d_n\}$ is a higher derivation on an algebra $\mathscr{A}$ such that d_0 is the identity mapping on $\mathscr{A}$, then there is a sequence $\{\delta_n\}$ of derivations on $\mathscr{A}$ such that

$$d_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_j + \cdots + r_i}\right)\delta_{r_1}\ldots\delta_{r_i}\right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

Key Words: Algebra; Derivation; Higher derivation; (σ, τ)-derivation.

2000 Mathematics Subject Classification: 16W25.

1. INTRODUCTION

Let $\mathscr{A}$ be an algebra. A linear mapping $\delta : \mathscr{A} \to \mathscr{A}$ is called a *derivation* if it satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathscr{A}$. If we define a sequence $\{d_n\}$ of linear mappings on $\mathscr{A}$ by $d_0 = I$ and $d_n = \frac{\delta^n}{n!}$, where I is the identity mapping on $\mathscr{A}$, then the Leibniz rule ensures us that d_n's satisfy the condition

$$d_n(ab) = \sum_{j=0}^{n} d_j(a)d_{n-j}(b) \qquad (*)$$

for each $a, b \in \mathscr{A}$ and each non-negative integer n. This motivates us to consider the sequences $\{d_n\}$ of linear mappings on an algebra $\mathscr{A}$ satisfying $(*)$. Such a sequence is called a *higher derivation*. Higher derivations were introduced by Hasse and Schmidt [5], and algebraists sometimes call them *Hasse–Schmidt derivations*. Though, if $\delta : \mathscr{A} \to \mathscr{A}$ is a derivation, then $d_n = \frac{\delta^n}{n!}$ is a higher derivation; this is not the only example of a higher derivation. This kind of higher derivation is called an *ordinary*

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higher derivation. For an account on higher derivations the reader is referred to the book [3].

In this article, we characterize all higher derivations on an algebra $\mathscr{A}$ in terms of the derivations on $\mathscr{A}$. Indeed, we show that each higher derivation is a combination of compositions of derivations. The importance of our work is to transfer the problems such as innerness (for a definition and discussion see [12]) and automatic continuity (see [7] and [8]) of higher derivations into the same problems concerning derivations. As a corollary we characterize all higher derivations which are ordinary. Throughout the article, all algebras are assumed over a field of characteristic zero.

2. THE RESULTS

Throughout the article, $\mathscr{A}$ denotes an algebra over a field of characteristic zero, and I is the identity mapping on $\mathscr{A}$. A linear mapping $\delta : \mathscr{A} \to \mathscr{A}$ is called a *derivation* if it satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathscr{A}$. A sequence $\{d_n\}$ of linear mappings on $\mathscr{A}$ is called a *higher derivation* if $d_n(ab) = \sum_{k=0}^{n} d_k(a)d_{n-k}(b)$ for each $a, b \in \mathscr{A}$ and each non-negative integer n.

Proposition 2.1. *Let $\{d_n\}$ be a higher derivation on an algebra $\mathscr{A}$ with $d_0 = I$. Then there is a sequence $\{\delta_n\}$ of derivations on $\mathscr{A}$ such that*

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k}$$

for each nonnegative integer n.

Proof. We use induction on n. For $n = 0$, we have

$$d_1(ab) = d_0(a)d_1(b) + d_1(a)d_0(b) = ad_1(b) + d_1(a)b.$$

Thus if $\delta_1 = d_1$, then δ_1 is a derivation on $\mathscr{A}$.

Now suppose that δ_k is defined and is a derivation for $k \leq n$. Putting $\delta_{n+1} = (n+1)d_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} d_{n-k}$, we show that the well-defined mapping δ_{n+1} is a derivation on $\mathscr{A}$. For $a, b \in \mathscr{A}$, we have

$$\begin{aligned}\delta_{n+1}(ab) &= (n+1)d_{n+1}(ab) - \sum_{k=0}^{n-1} \delta_{k+1} d_{n-k}(ab)\\ &= (n+1)\sum_{k=0}^{n+1} d_k(a)d_{n+1-k}(b) - \sum_{k=0}^{n-1} \delta_{k+1}\left(\sum_{\ell=0}^{n-k} d_\ell(a)d_{n-k-\ell}(b)\right).\end{aligned}$$

Now we have

$$\begin{aligned}\delta_{n+1}(ab) &= \sum_{k=0}^{n+1}(n+1)d_k(a)d_{n+1-k}(b) - \sum_{k=0}^{n-1} \delta_{k+1}\left(\sum_{\ell=0}^{n-k} d_\ell(a)d_{n-k-\ell}(b)\right)\\ &= \sum_{k=0}^{n+1}(k+n+1-k)d_k(a)d_{n+1-k}(b) - \sum_{k=0}^{n-1} \delta_{k+1}\left(\sum_{\ell=0}^{n-k} d_\ell(a)d_{n-k-\ell}(b)\right).\end{aligned}$$

Since $\delta_1, \ldots, \delta_n$ are derivations,

$$\delta_{n+1}(ab) = \sum_{k=0}^{n+1} k d_k(a) d_{n+1-k}(b) + \sum_{k=0}^{n+1} d_k(a)(n+1-k) d_{n+1-k}(b)$$
$$- \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \big[\delta_{k+1}(d_\ell(a)) d_{n-k-\ell}(b) + d_\ell(a) \delta_{k+1}(d_{n-k-\ell}(b))\big].$$

Writing

$$K = \sum_{k=0}^{n+1} k d_k(a) d_{n+1-k}(b) - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \delta_{k+1}(d_\ell(a)) d_{n-k-\ell}(b),$$
$$L = \sum_{k=0}^{n+1} d_k(a)(n+1-k) d_{n+1-k}(b) - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} d_\ell(a) \delta_{k+1}(d_{n-k-\ell}(b)),$$

we have $\delta_{n+1}(ab) = K + L$. Let us compute K and L. In the summation $\sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k}$, we have $0 \leq k + \ell \leq n$ and $k \neq n$. Thus if we put $r = k + \ell$, then we can write it as the form $\sum_{r=0}^{n} \sum_{k+\ell=r, k\neq n}$. Putting $\ell = r - k$, we indeed have

$$K = \sum_{k=0}^{n+1} k d_k(a) d_{n+1-k}(b) - \sum_{r=0}^{n} \sum_{0\leq k\leq r, k\neq n} \delta_{k+1}(d_{r-k}(a)) d_{n-r}(b)$$
$$= \sum_{k=0}^{n+1} k d_k(a) d_{n+1-k}(b) - \sum_{r=0}^{n-1} \sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a)) d_{n-r}(b) - \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a)) b.$$

Putting $r + 1$ instead of k in the first summation, we have

$$K + \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a)) b$$
$$= \sum_{r=0}^{n} (r+1) d_{r+1}(a) d_{n-r}(b) - \sum_{r=0}^{n-1} \sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a)) d_{n-r}(b)$$
$$= \sum_{r=0}^{n-1} \left[(r+1) d_{r+1}(a) - \sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a))\right] d_{n-r}(b) + (n+1) d_{n+1}(a) b.$$

By our assumption $(r+1) d_{r+1}(a) = \sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a))$ for $r = 0, \ldots, n-1$. We can, therefore, deduce that

$$K = \left[(n+1) d_{n+1}(a) - \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a))\right] b = \delta_{n+1}(a) b.$$

By a similar argument, we have

$$L = a\left[(n+1) d_{n+1}(b) - \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(b))\right] = a \delta_{n+1}(b).$$

Thus

$$\delta_{n+1}(ab) = K + L = \delta_{n+1}(a)b + a\delta_{n+1}(b).$$

Whence δ_{n+1} is a derivation on $\mathscr{A}$. $\square$

To illustrate the recursive relation mentioned in Proposition 2.1, let us compute some terms of $\{d_n\}$.

Example 2.2. Using Proposition 2.1, the first five terms of $\{d_n\}$ are

$$\begin{aligned}
d_0 &= I,\\
d_1 &= \delta_1,\\
2d_2 &= \delta_1 d_1 + \delta_2 d_0 = \delta_1\delta_1 + \delta_2,\\
d_2 &= \frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2,\\
3d_3 &= \delta_1 d_2 + \delta_2 d_1 + \delta_3 d_0 = \delta_1\left(\frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2\right) + \delta_2\delta_1 + \delta_3,\\
d_3 &= \frac{1}{6}\delta_1^3 + \frac{1}{6}\delta_1\delta_2 + \frac{1}{3}\delta_2\delta_1 + \frac{1}{3}\delta_3,\\
4d_4 &= \delta_1 d_3 + \delta_2 d_2 + \delta_3 d_1 + \delta_4 d_0\\
&= \delta_1\left(\frac{1}{6}\delta_1^3 + \frac{1}{6}\delta_1\delta_2 + \frac{1}{3}\delta_2\delta_1 + \frac{1}{3}\delta_3\right) + \delta_2\left(\frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2\right) + \delta_3\delta_1 + \delta_4,\\
d_4 &= \frac{1}{24}\delta_1^4 + \frac{1}{24}\delta_1^2\delta_2 + \frac{1}{12}\delta_1\delta_2\delta_1 + \frac{1}{12}\delta_1\delta_3 + \frac{1}{8}\delta_2\delta_1^2 + \frac{1}{8}\delta_2^2 + \frac{1}{4}\delta_3\delta_1 + \frac{1}{4}\delta_4.
\end{aligned}$$

Theorem 2.3. *Let $\{d_n\}$ be a higher derivation on an algebra $\mathscr{A}$ with $d_0 = I$. Then there is a sequence $\{\delta_n\}$ of derivations on $\mathscr{A}$ such that*

$$d_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_j + \cdots + r_i}\right)\delta_{r_1}\ldots\delta_{r_i}\right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

Proof. We show that if d_n is of the above form, then it satisfies the recursive relation of Proposition 2.1. Since the solution of the recursive relation is unique, this proves the theorem. Simplifying the notation, we put $a_{r_1,\ldots,r_i} = \prod_{j=1}^{i}\frac{1}{r_j+\cdots+r_i}$. Note that if $r_1 + \cdots + r_i = n+1$, then $(n+1)a_{r_1,\ldots,r_i} = a_{r_2,\ldots,r_i}$. Moreover, $a_{n+1} = \frac{1}{n+1}$.

Now we have

$$(n+1)d_{n+1} = \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j = n+1}(n+1)a_{r_1,\ldots,r_i}\delta_{r_1}\ldots\delta_{r_i}\right) + \delta_{n+1}$$

$$
\begin{aligned}
&= \sum_{i=2}^{n+1}\left(\sum_{r_1=1}^{n+2-i}\delta_{r_1}\sum_{\sum_{j=2}^{i} r_j=n+1-r_1} a_{r_2,\ldots,r_i}\delta_{r_2}\ldots\delta_{r_i}\right)+\delta_{n+1}\\
&= \sum_{r_1=1}^{n}\delta_{r_1}\sum_{i=2}^{n-(r_1-1)}\left(\sum_{\sum_{j=2}^{i} r_j=n-(r_1-1)} a_{r_2,\ldots,r_i}\delta_{r_2}\ldots\delta_{r_i}\right)+\delta_{n+1}\\
&= \sum_{r_1=1}^{n}\delta_{r_1}d_{n-(r_1-1)}+\delta_{n+1}\\
&= \sum_{k=0}^{n}\delta_{k+1}d_{n-k}.
\end{aligned}
$$

□

Example 2.4. We evaluate the coefficients $a_{r_1,\ldots,r_i}$ for the case $n=4$.

For $n=4$, we can write

$$4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1.$$

By the definition of $a_{r_1,\ldots,r_i}$ we have

$$
\begin{aligned}
a_4 &= \frac{1}{4},\\
a_{1,3} &= \frac{1}{1+3}\cdot\frac{1}{3} = \frac{1}{12},\\
a_{3,1} &= \frac{1}{3+1}\cdot\frac{1}{1} = \frac{1}{4},\\
a_{2,2} &= \frac{1}{2+2}\cdot\frac{1}{2} = \frac{1}{8},\\
a_{1,1,2} &= \frac{1}{1+1+2}\cdot\frac{1}{1+2}\cdot\frac{1}{2} = \frac{1}{24},\\
a_{1,2,1} &= \frac{1}{1+2+1}\cdot\frac{1}{2+1}\cdot\frac{1}{1} = \frac{1}{12},\\
a_{2,1,1} &= \frac{1}{2+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{8},\\
a_{1,1,1,1} &= \frac{1}{1+1+1+1}\cdot\frac{1}{1+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{24}.
\end{aligned}
$$

We can, therefore, deduce that

$$d_4 = \frac{1}{4}\delta_4+\frac{1}{12}\delta_1\delta_3+\frac{1}{4}\delta_3\delta_1+\frac{1}{8}\delta_2\delta_2+\frac{1}{24}\delta_1\delta_1\delta_2+\frac{1}{12}\delta_1\delta_2\delta_1+\frac{1}{8}\delta_2\delta_1\delta_1+\frac{1}{24}\delta_1\delta_1\delta_1\delta_1.$$

Theorem 2.5. *Let $\mathscr{A}$ be an algebra, D be the set of all higher derivations $\{d_n\}_{n=0,1,\ldots}$ on $\mathscr{A}$ with $d_0 = I$ and Δ be the set of all sequences $\{\delta_n\}_{n=0,1,\ldots}$ of derivations on $\mathscr{A}$ with $\delta_0 = 0$. Then there is a one to one correspondence between D and Δ.*

Proof. Let $\{\delta_n\} \in \Delta$. Define $d_n : \mathscr{A} \to \mathscr{A}$ by $d_0 = I$ and

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) \delta_{r_1} \ldots \delta_{r_i} \right).$$

We show that $\{d_n\} \in D$. By Theorem 2.3, $\{d_n\}$ satisfies the recursive relation

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k}.$$

To show that $\{d_n\}$ is a higher derivation, we use induction on n. For $n = 0$, we have $d_0(ab) = ab = d_0(a)d_0(b)$. Let us assume that $d_k(ab) = \sum_{i=0}^{k} d_i(a)d_{k-i}(b)$ for $k \leq n$. Thus we have

$$\begin{aligned}
(n+1)d_{n+1}(ab) &= \sum_{k=0}^{n} \delta_{k+1} d_{n-k}(ab) \\
&= \sum_{k=0}^{n} \delta_{k+1} \sum_{i=0}^{n-k} d_i(a) d_{n-k-i}(b) \\
&= \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} \delta_{k+1} d_{n-k-i}(a) \right) d_i(b) + \sum_{i=0}^{n} d_i(a) \left(\sum_{k=0}^{n-i} \delta_{k+1} d_{n-k-i}(b) \right).
\end{aligned}$$

Using our assumption, we can write

$$\begin{aligned}
(n+1)d_{n+1}(ab) &= \sum_{i=0}^{n} (n-i+1) d_{n-i+1}(a) d_i(b) + \sum_{i=0}^{n} d_i(a)(n-i+1) d_{n-i+1}(b) \\
&= \sum_{i=1}^{n+1} i d_i(a) d_{n+1-i}(b) + \sum_{i=0}^{n} (n+1-i) d_i(a) d_{n+1-i}(b) \\
&= (n+1) \sum_{k=0}^{n+1} d_k(a) d_{n+1-k}(b).
\end{aligned}$$

Thus $\{d_n\} \in D$.

Conversely, suppose that $\{d_n\} \in D$. Define $\delta_n : \mathscr{A} \to \mathscr{A}$ by $\delta_0 = 0$ and

$$\delta_n = n d_n - \sum_{k=0}^{n-2} \delta_{k+1} d_{n-1-k}.$$

Then Proposition 2.1 ensures us that $\{\delta_n\} \in \Delta$.

Now define $\varphi : \Delta \to D$ by $\varphi(\{\delta_n\}) = \{d_n\}$, where

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) \delta_{r_1} \ldots \delta_{r_i} \right).$$

Then φ is clearly a one to one correspondence. $\square$

Recall that a higher derivation $\{d_n\}$ is called *ordinary* if there is a derivation δ such that $d_n = \frac{\delta^n}{n!}$ for all n.

Corollary 2.6. *A higher derivation* $\{d_n\} = \varphi(\{\delta_n\})$ *on an algebra* $\mathscr{A}$ *is ordinary if and only if* $\delta_n = 0$ *for* $n \geq 2$. *In this case* $d_n = \frac{d_1^n}{n!}$.

Remark 2.7. Let $\mathscr{A}$ be an algebra and $\sigma, \tau : \mathscr{A} \to \mathscr{A}$ be two homomorphisms. As a generalization of the notion of a derivation, a linear mapping $\delta : \mathscr{A} \to \mathscr{A}$ is called a (σ, τ)-*derivation* if it satisfies the generalized Leibniz rule $\delta(ab) = \delta(a)\sigma(b) + \tau(a)\delta(b)$ for all $a, b \in \mathscr{A}$. (For other approaches to generalized derivations and their applications see [1, 2, 4, 10, 11] and references therein. In particular, an automatic continuity problem for (σ, τ)-derivations is considered in [9], and an achievement of continuity of (σ, τ)-derivations without linearity is given in [6].) A simple modification in our approach shows that if we do not have the assumption $d_0 = I$, then $\delta_1 : \mathscr{A} \to \mathscr{A}$ defined by $\delta_1(a) = d_1(a)$ is a d_0-derivation, and we can, therefore, deduce that the sequence $\{\delta_n\}$ corresponding to $\{d_n\}$ consists of d_0-derivations.

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