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TOPOLOGICAL GAMES AND CONTINUITY OF QUASI-CONTINUOUS MAPPINGS

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Abstract: Let φ be a quasi-continuous function from a topological space B to $C_p(K)$, where K is a compact Hausdorff space. In this note we discuss about conditions which imply continuity of φ on a dense subset D of B to $C(K)$ with the norm topology.

1. Introduction

Let B and X be topological spaces. A mapping $\varphi : B \to X$ is said to be *quasi-continuous* at $b \in B$, if for every open subset $W \subset X$, containing the point $\varphi(b)$, there exists some open subset V of B such that $b \in \overline{V}$ and $\varphi(V) \subset W$.

Suppose that (X, τ) is a topological space and τ' is another topology on X, not necessarily related to the topology of X. We say that the topological space (X, τ) has Baire (respectively α-favorable) τ'-quasi-continuity property, if for each quasi-continuous mapping φ from a Baire (an α-favorable) space B into (X, τ), there exists a dense G_δ subset D of B such that $\varphi : D \to (X, \tau')$ is continuous.

In 1985, Talagrand [20] provided an example of a continuous mapping φ from an α-favorable (hence Baire) space B to $(C(K)$, pointwise) where K is compact Hausdorff space such that for each point $x \in B$, $\varphi : B \to (C(K), \|\cdot\|)$ is not continuous. The result of Talagrand raises the following question:

What are compact spaces K such that for every Baire (or α-favorable) space B and (quasi-)continuous mapping $\varphi : B \to (C(K)$, pointwise) must be norm continuous at each point of some dense subset G_δ subset D of B?

Several partial answer to the above question have been obtained by several authors under some geometrical restrictions (see e.g. [1, 2], [5]–[11], [14]–[17] and [19, 20]).

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A. Bouziad in [2] has shown that every continuous mapping from a Baire space B to $C(K)$ with the pointwise topology is norm continuous on a dense subset of B if there is a family $\mathscr{K}$ of compact subsets of K such that

(1) $\bigcup \mathscr{K}$ is dense in K,

(2) for each $L \in \mathscr{K}$ and a continuous mapping φ from a Baire space B to $(C(L)$, pointwise), there is a dense G_δ subset D of B such φ is norm continuous on D and

(3) for any sequence $\{L_n\}$ in $\mathscr{K}$, $\bigcup_{i \geq 1} L_i$ is a closed subset of K.

Namioka and Pol in [17] have obtain similar result for norm σ-fragmentability in $C(K)$ with the pointwise topology.

Kendrov and Moors [13] have shown that if a compact space K has a representation of the form $K = \bigcup_{n \geq 1} K_n$ such that each $n \in \mathbf{N}$, $C(K_n)$ is sigma-fragmented by the supremum norm then so is $C(K)$. In [10] and [11], Kenderov et al. have modified the winning rule of fragmenting game [12] on a topological space (X, τ) to give a characterization for α-favorable τ'-quasi-continuity property. In fact they have shown that the absence of a winning strategy for one of the player implies α-favorable quasi-continuity property. In the next section, we will prove that if a topological space does not have Baire quasi-continuity property, then the other player has no winning strategy. Moreover, we apply this characterization to show that following similar result of Bouziad [2] is true for α-favorable quasi-continuity property:

Let $\mathscr{K}$ be a family of compact Hausdorff subspace of a compact space K such that for each $L \in \mathscr{K}$, $C_p(L)$ has α-favorable norm-quasi-continuity property. If $\bigcup \mathscr{K}$ is dense in K and for each countable subset $\{L_n\}$ of $\mathscr{K}$, $\bigcup L_n$ is a closed subset of K, then $C(K)$ with the pointwise topology has α-favorable norm quasi-continuity property.

2. Results

Let us recall the following topological game, which is known as *"Banach-Mazur"* or *"Choquet" game* (see [4] or [18]).

Let X be a topological space. The Banach-Mazur game $BM(X)$ is played by two players α and β, who select alternately, nonempty open subsets of X. β starts a game by selecting a nonempty open subset V_1 of X. In response to this move the player α replies by selecting some nonempty open subset W_1 of V_1. At the n-th stage of the game, $n \geq 1$, the player β chooses a nonempty open subset $V_n \subset W_{n-1}$ and α answers by choosing a nonempty open subset W_n of V_n. Proceeding in this fashion, the players generate a sequence $(V_n, W_n)_{n=1}^{\infty}$ which is called a *play*. The player α is said to have *won* the play $(V_n, W_n)_{n=0}^{\infty}$ if $\bigcap_{n \geq 1} V_n = \bigcap_{n \geq 1} W_n \neq \varnothing$; otherwise the player β is said to have won this play. A *partial play* is a finite sequence of sets consisting of the first few moves of a play. A *strategy* for a player is a rule by means of which the player makes his/her choices. Here is a more formal definition of the notion strategy. A strategy s for the player α is a sequence of mappings $s = \{s_n\}$, which is inductively defined as follows:

The domain of s_1 is the set of all open subsets of X and s_1 assigns to each nonempty open set $V_1 \subset X$, a nonempty open subset $W_1 = s_1(V_1)$ of V_1. In general, if a partial play $(V_1, \ldots, W_{n-1})$ has already been specified, where $W_i = s_i(V_1, \ldots, V_i)$, $1 \leq i \leq n-1$. Then the domain of s_n would be the set

$$\{(V_1, W_1, \ldots, W_{n-1}, V) : V \subset W_{n-1} \text{ can be the next move of } \beta\text{-player}\}$$

and it assigns to each choice $V_n \subset W_{n-1}$ some nonempty open subset

$$W_n = s_n(V_1, W_1, \ldots, W_{n-1}, V_n)$$

of V_n.

A *s-play* is a play in which α selects his/her moves according to the strategy s. The strategy s for the player α is said to be a *winning strategy* if every s-play is won by α. A space X is called *α-favorable* if there exists a winning strategy for α in $BM(X)$.

It is easy to verify that every α-favorable space X is a Baire space, that is, a space in which the intersection of countably many dense and open subsets is dense in the space. There are examples of Baire spaces which are not α-favorable (see for example [9]). It is known that X is a Baire space if and only if the player β does not have a winning strategy in the game $BM(X)$ (see [19] Theorems 1 and 2).

We refer the reader to [3] for a survey on topological games and their applications in analysis.

We need to the following result which may be known in the literature, however, we mentioned it here for the sake of completion:

LEMMA 1. *Let φ be a mapping from a Baire space B into a metric space (X, ρ). Then $\varphi : B \to (X, \rho)$ is continuous on a dense subset of B, if and only if for each $\epsilon > 0$ and nonempty open subset W of B, there exists a nonempty open subset U of W, such that ρ-diam $\varphi(U) < \epsilon$.*

PROOF. Let D be a dense subset of B, such that φ is ρ-continuous at each point of D. Then for each non empty open subset W of B, $W \cap D \neq \emptyset$. Let $x_0 \in W \cap D$, by the ρ-continuity of φ at x_0, for each $\varepsilon > 0$, there exists a nonempty open subset U of W (containing x_0), such that ρ-diam $\varphi(U) < \epsilon$.

Conversely, suppose that for each $\varepsilon > 0$ and each nonempty open subset U of B, there exists a non empty open subset U of W, such that ρ-diam $\varphi(U) < \varepsilon$. For each $n \in N$, we define

$$G_n = \bigcup\{V : V \subset B \text{ is open and } \rho\text{-diam}\, \varphi(V) < 1/n\}.$$

Clearly each A_n is open subset of B. We will show that each G_n is dense in B. Let W be a nonempty open subset of B. By our assumption, there exists a non empty open subset V of W such that ρ-diam $\varphi(V) < 1/n$. Therefore $\varnothing \neq V \subset G_n \cap W$. This shows that each G_n is dense in B. Let $D = \bigcap_{n\geq 1} G_n$. Since B is Baire, D is dense in B. Clearly, $\varphi : D \to (X,\rho)$ is continuous. □

Kenderov et al. in [11] introduced the following topological game: Let (X,τ) be a topological space and τ' be a another topology on X, which is not necessarily related to τ, the game $KM(X,\tau,\tau')$ is played between two players Σ and Ω as follows:

Σ starts the game by choosing some non-empty subset A_1 of X. Then Ω chooses some nonempty τ-relatively open subset B_1 of A_1. In general, if the selection $B_n \neq \varnothing$ of the player Ω is already specified, the player Σ makes the next move by selecting an arbitrary non-empty set A_{n+1} contained in B_n. By continuing this procedure, the two players generate a sequence of sets

$$A_1 \supset B_1 \supset \cdots \supset A_n \supset B_n \supset \cdots$$

which is called a play and is denoted by $p = (A_i, B_i)_{i=1}^{\infty}$. If

$$p_1 = (A_1), \dots, p_n = (A_1, B_1, \dots, A_n)$$

are the first "n" move of some play (of the game), then p_n is called the n^{th} *partial play* of the game. The player Ω is said to have won a play $p := (A_i, B_i)_{i\geq 1}$, if the set $\bigcap_{n\geq 1} A_n$ is either empty or contains exactly one point x and for every τ'-open neighborhood U of x, there is some positive n with $B_n \subset U$. Otherwise the player Σ is said to be the winner in this play.

Under the term strategy for Σ-player, we mean a mapping $\sigma = \{\sigma_n\}$, which is defined inductively as follows:

σ_1 assigns to the topological space $B_0 = X$ a nonempty subset $A_1 = \sigma_1(B_0)$. Therefore, the domain of σ_1 is the set $\{B_0\}$. If the partial play $p_n = (A_1, \dots, B_{n-1}, A_n)$ have already been specified. Then the domain of σ_{n+1} is the set

$$\{(B_0, A_1, \dots, A_n, B) \colon B \text{ can be a response of } \Omega \text{ player to } (A_1, \dots, A_n)\}$$

and σ_{n+1} assigns to each response B_n of the Ω-player to $(A_1, \dots, A_n)$, a nonempty subset $A_{n+1} = \sigma_{n+1}(A_1, \dots, B_{n-1}, A_n, B_n)$ of B_n. A play $p = (A_i, B_i)_{i\geq 1}$ is called a σ-play if $A_i = \sigma_i(B_0, A_1, \dots, B_{i-1})$ for each $i \geq 1$. σ is called a *winning strategy* for the player Σ if he/she wins every σ-play. Similarly, a winning strategy for the Ω player can be defined.

In [11], the authors characterized α-favorable quasi-continuity as follows:

THEOREM 2.1. *Let τ, τ' be two T_1 topologies on a set X. Suppose that for every τ'-open set U and every point $x \in U$ there exists a τ'-neighborhood V of x such that $\overline{V}^{\tau} \subset U$. Then the following conditions are equivalent:*

(i) *The game $KM(X, \tau, \tau')$ is Σ-unfavorable;*

(ii) *every quasi-continuous mapping $f : Z \to (X, \tau)$ from a complete metric space Z into (X, τ) has at least one point of τ'-continuity;*

(iii) *every quasi-continuous mapping $f : Z \to (X, \tau)$ from an α-favorable space Z into (X, τ) is τ'-continuous at the points of a subset which is of second category in every nonempty open subset of Z.*

When ρ is a metric on X and τ' is the ρ-topology on X, then we have the following result which is of special interest:

THEOREM 2.2. *Let (X, τ) be a topological space and ρ be a metric on X.*

(a) *If (X, τ) does not have Baire ρ-quasi-continuity property. Then there exists a strategy σ for the player Σ in the game $KM(X, \tau, \rho)$, such that the player Σ wins some σ-play $(A_i, B_i)_{i \geq 1}$.*

(b) *If (X, τ) does not have α-favorable ρ-quasi-continuity property. Then there exists a strategy σ for the player Σ in the game $KM(X, \tau, \rho)$, such that the player Σ wins all σ-plays.*

PROOF. Let B be a Baire space and $\varphi : B \to (X, \tau)$ be a quasi-continuous mapping such that for every dense subset D of B, $\varphi|D$ is not ρ-continuous. By Lemma 1, there exists some $\varepsilon > 0$ and a nonempty open set W of B, such that for each nonempty open subset V of W,

$$\rho\text{-diam } \varphi(V) \geq \varepsilon.$$

We will show that for every strategy t for Ω-player in $KM(X, \tau, \rho)$. There exists a strategy σ for Σ-player such that he/she wins some play. Let $W_1 = W$ be the first choice of β-player in $BM(B)$ and let $V_1 = t(W_1)$ be the answer of α according to his/her strategy t. Let $A_1 = \varphi(V_1)$ be the first choice of Σ-player. Then by our assumption, ρ-diam $A_1 \geq \varepsilon$. Suppose that the answer of Ω-player to A_1 is B_1, a nonempty relatively open subset of A_1. Then by the continuity of φ, there exists some nonempty open subset W_2 of V_1 such that $\varphi(W_2) \subset B_1$. We define W_2 as the next move of β-player. Let $V_2 = t(W_1, V_1, W_2)$ be the answer of α according to his/her strategy t. Let $\sigma(A_1, B_1) = A_2$, where $A_2 = \varphi(V_2)$. Since V_2 is a nonempty open subset of W, ρ-diam $A_2 \geq \varepsilon$. Proceeding like this, we construct inductively the strategy σ for the player Σ. Together, with each σ-play $(A_i, B_i)_{i \geq 1}$, we construct also a t-play $(W_i, V_i)_{i \geq 1}$ in Banach-Mazur game, with $A_n = \varphi(V_n)$

and $V_n = t(W_1, V_1, \dots, W_n) \subset W$ for $n = 1, 2, \dots$, such that ρ-diam $A_n \geq \varepsilon$. The strategy t for β-player is not winning, hence there exists a play $(W_i, V_i)_{i\geq 1}$ such that α wins this play. Hence $\bigcap_{i\geq 1} V_i \neq \varnothing$. It follows that $\varnothing \neq \varphi(\bigcap_{i\geq 1} V_i) \subset \bigcap_{i\geq 1} \varphi(V_i) = \bigcap_{i\geq 1} A_i$. Therefore (a) holds.

(b) follows from Theorem 2.1 for $\rho = \tau'$. □

LEMMA 2. *Let $\{L_n\}$ be a sequence of compact subsets of a compact space K such that $\bigcup_{n\geq 1} L_n$ is closed and for each $n \in \mathbf{N}$, $C_p(L_n)$ has α-favorable norm-quasi-continuity property. Then $C_p(K)$ has α-favorable $\|\cdot\|_{\bigcup_{n\geq 1} L_n}$-quasi-continuity property.*

PROOF. Let B be an α-favorable space and $\varphi : B \to C_p(K)$ be a quasi-continuous mapping. Then for each $n \in \mathbf{N}$, $\varphi|_{L_n} : B \to C_p(K)|_{L_n}$ is quasi-continuous. Hence, there is a dense G_δ subset $D_n \subset B$ such that $\varphi|_{L_n} : D_n \to (C(K), \|\cdot\|_{L_n})$ is continuous. Define $D = \bigcap_{n\geq 1} D_n$. Since B is α-favorable, D is a dense G_δ subset of B. Clearly $\varphi : D \to (C(K), \|\cdot\|_{\bigcup_{n\geq 1} L_n})$ is continuous. □

COROLLARY 2.3. *Under conditions of Lemma 2, $(C(K)$, pointwise on $\bigcup_{n\geq 1} L_n$, $\|\cdot\|_{\bigcup_{n\geq 1} L_n})$ is Σ-unfavorable.*

PROOF. Follows from Theorem 2.1 and Lemma 2. □

We are ready to state the main result of this paper:

THEOREM 2.4. *Let $\mathscr{K}$ be a family of compact Hausdorff subspace of a compact space K such that for each $L \in \mathscr{K}$, $C_p(L)$ has α-favorable norm-quasi-continuity property. If $\bigcup \mathscr{K}$ is dense in K and for each countable subset $\{L_n\}_{n\geq 1}$ of $\mathscr{K}$, $\bigcup_{n\geq 1} L_n$ is a closed subset of K, then $C_p(K)$ has α-favorable norm quasi-continuity property.*

PROOF. Let $C_p(K)$ do not have α-favorable norm quasi-continuity property. Then by Theorem 2.2(b) in the case when τ is the pointwise topology on $C(K)$ and ρ is the supremum norm on $C(K)$, Σ has a winning strategy σ such that for each σ-play $(A_i, B_i)_{i\geq 1}$,

$$\bigcap_{i\geq 1} A_i \neq \varnothing \ \& \ \text{norm-diam}\, A_i > \epsilon, \qquad \forall i \geq 1$$

for some $\varepsilon > 0$. Let A_1 be the first choice of Σ. By our assumption, $\| \ \|$-diam $A_1 > \varepsilon$. Therefore, there are f_0 and f_1 in A_1 such that $\|f_0 - f_1\| > \ \varepsilon/2$. Since $\bigcup \mathscr{K}$ is dense in K, there is some $x_1 \in L_1 \in \mathscr{K}$ such that $|f_0(x_1) - f_1(x_1)| > \varepsilon/2$. Let $\eta_1 = |f(x_1) - f_1(x_1)| - \varepsilon/2$ and define

$$V_1 = \{f \in A_1 : |f(x_1) - f_1(x_1)| < \eta_1\}.$$

Then for each $f \in V_1$,

$$|f(x_1) - f_0(x_1)| \geq |f_0(x_1) - f_1(x_1)| - |f(x_1) - f_1(x_1)|$$
$$> |f_0(x_1) - f_1(x_1)| - \eta_1 = \varepsilon/2.$$

Put $A_1(1) = \{f|_{L_1} : f \in V_1\} \subset C(L_1)$. Let $B_1(1)$ be the answer of Ω-player to $A_1(1)$. In general, if $A_1(1), B_1(1), \ldots, A_{n-1}(1)$ and $B_{n-1}(1)$ are specified, define

$$A_n(1) = \sigma_{n,1}(A_1(1), B_1(1), \ldots, A_{n-1}(1), B_{n-1}(1)) = B_{n-1}(1).$$

Since $C_p(L_1)$ has norm-quasi-continuity property, there exists a play, which is denoted by $p(1) = (A_i(1), B_i(1))$, such that $p(1)$ is won by Ω and $p(1)$ is a continuation of $(A_1(1), B_1(1))$. Take some $i_1 \geq 1$ such that

$$\|\cdot\|_{L_1}\text{-diam}(B_{i_1}(1)) < \frac{1}{2}.$$

Define $B_1 = \{f \in V_1; f|_{L_1} \in B_{i_1}(1)\}$. Then B_1 is a nonempty relatively pointwise open subset of A_1. Let B_1 be the answer of Ω to A_1. Suppose that, in the step n, $A_1, \ldots, A_n$ together with points $f_1, \ldots, f_{n-1}$ in $C(K)$, $x_1, \ldots, x_{n-1}$ in K, pointwise open subsets $V_1, \ldots, V_{n-1}$ and compact subsets $L_1 \subset \cdots \subset L_{n-1}$ of K have already been specified. By our assumption, norm-diam$(A_n) > \varepsilon$, hence we can take some $f_n \in A_n$ such that $\|f_n - f_{n-1}\| > \varepsilon/2$. By Lemma 2, we can assume that $\mathscr{K}$ is closed under finite union. Hence we can select some point $x_n \in L_n \in \mathscr{K}$ such that $L_{n-1} \subset L_n$ and $|f_n(x_n) - f_{n-1}(x_n)| > \varepsilon/2$. Define

$$V_n = \{f \in A_n : |f(x_n) - f_n(x_n)| < \eta_n\},$$

where $\eta_n = |f_n(x_n) - f_{n-1}(x_n)| - \varepsilon/2$. Hence for each $f \in V_n$,

$$|f(x_n) - f_{n-1}(x_n)| \geq |f_n(x_n) - f_{n-1}(x_n)| - |f(x_n) - f_n(x_n)|$$
$$> |f_{n-1}(x_n) - f_n(x_n)| - \eta_n = \varepsilon/2.$$

Define $A_1(n) = \{f|_{L_n} : f \in V_n\} \subset C(L_n)$. If $B_1(n)$ is the answer of Ω-player to $A_1(n)$, by applying similar argument used above, we can find a play $p(n) = (A_i(n), B_i(n))_{i \geq 1}$, which is a continuation of the partial play $(A_1(n), B_1(n))$, such that $A_{i+1}(n) = B_i(n)$ for each $i \geq 1$ and for some $i_n > 1$,

$$\|\cdot\|_{L_n}\text{-diam}(B_{i_n}(n)) < \frac{1}{n+1}.$$

Define

$$B_n = \{f : f \in V_n, f|_{L_n} \in B_{i_n}(n)\}$$

as the answer of Ω player to $(A_1, \dots, A_n)$. Hence by induction on n, the play $p = (A_i, B_i)_{i \geq 1}$ is determined.

Let $g \in \bigcap A_n$. Our construction shows that g is not in the pointwise closure $\{f_1, f_2, \dots\}$ on $\bigcup_{n \geq 1} L_n$. However, for each $x \in \bigcup_{n \geq 1} L_n$, $x \in L_{n_0}$ for some n_0. But then for each $m \geq n_0$, $x \in L_m$, hence

$$\lim_{n \to \infty} |f_n(x) - g(x)| \leq \| \cdot \|_{L_m}\text{-diam}(B_m|_{L_m}) < \frac{1}{m+1}.$$

This means that g belongs to the pointwise closure of $\{f_1, f_2, \dots\}$ on $\bigcup_{n \geq 1} L_n$. Contradiction. □

REMARK 2.5. The proof of Lemma 2 shows that the result of the Lemma is true for Baire norm-quasi-continuity property. In [8] (Theorem 2), it is shown that Baire norm-quasi-continuity property is the same as the Namioka property. Hence, the Bouziad's result in [2] is true for Baire norm-quasi-continuity property:

Let K be a compact Hausdorff space, then $C(K)$ has Baire norm-quasi-continuity property if there is a family of compact subsets $\mathscr{K}$ of K such that $\mathscr{K}$ is closed under countable union and for every element $L \in \mathscr{K}$, $C(L)$ has Baire quasi-continuity property.

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References

[1] A. Bouziad, The class of co-Namioka compact spaces is stable under product, Proc. Amer. Math. Soc. Vol. 124, Number 3 (1996), 983–986.

[2] A. Bouziad, A quasi-closure preserving sum theorem about Namioka property, Topology Appl. **81** (2) (1997), 163–170.

[3] J. Cao and W. B. Moors, A survey on topological games and their applications in analysis, RACSAM, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A. Mat. **100** (2006), 39–49.

[4] G. Choquet, Lectures on Analysis, Vol. 1, W. A. Benjamin Inc., New-York (1969).

[5] J. P. R. Christensen, Joint continuity of separately continuous functions, Proc. Amer. Math. Soc. **82** (1981), 455–461.

[6] G. Debs, Pointwise and uniform convergence on Corson compact spaces, Topology Appl., **97** (1986), 299–303.

[7] R. Deville, Convergence ponctuelle et uniforme sur un espace compact, Bull. Acad. Polon. Sci. **37** (1989), 7–12.

[8] J. R. Giles and W. B. Moors, A selection theorem for quasi-lower semicontinuous set-valued mappings, J. Nonlinear and Convex Anal. **2** (2001), 345–350.

[9] R. Haydon, Baire trees, bad norms and the Namioka property, Mathematika, **42** (1995), 30–42.

[10] P. S. Kenderov, I. S. Kortezov and W. B. Moors, Continuity points of quasi-continuous mappings, Topology Appl. **109** (2001), 321–346.

[11] P. S. Kenderov, I. S. Kortezov and W. B. Moors, Norm continuity of weakly continuous mappings into Banach spaces, Topology Appl. **153** (2006), 2745–2759.

[12] P. S. Kenderov and W. B. Moors, Game characterization of fragmentability of topological spaces, Mathem. and Education in Mathem., Proceedings of the 25th Spring Conf. Union of Bulg. Mathematicians, Kazanlak, (1996), 8–18.

[13] P. S. Kenderov and W. B. Moors, Fragmentability and sigma-fragmentability of Banach spaces, J. London Math. Soc. **60** (1999), 203–223.

[14] A. K. Mirmostafaee, Fragmentability and joint continuity, The Mathematics Student, Vol. 70, No 1–4 (2001), 227–230.

[15] I. Namioka, Separate continuity and joint continuity, Pacific J. Math. **51** (1974), 515–531.

[16] I. Namioka and R. Pol, Mappings of Baire spaces into function spaces and Kadec renorming, Israel J. Math. **78** (1992), 1–20.

[17] I. Namioka and R. Pol, Sigma-fragmentability of mapping into $C_p(K)$, Topology Appl. **89** (1998), 249–263.

[18] J. C. Oxtoby, Measure and Category, Grad. Texts Math. 2. Springer-Verlag (1971).

[19] J. Saint-Reymond, Jeux topologiques et espaces de Namioka, Proc. Amer. Math. Soc. **87** (1983), 499–504.

[20] M. Talagrand, Esaces de Baire et espaces de Namioka, Math. Ann., **270** (1985), 159–164.

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