On nondiscreteness of a higher topological homotopy group and its cardinality

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Abstract

Here, we are going to extend Mycielski's conjecture to higher homotopy groups. Also, for an $(n-1)$-connected locally $(n-1)$-connected compact metric space X, we assert that $\pi_n^{top}(X)$ is discrete if and only if $\pi_n(X)$ is finitely generated. Moreover, $\pi_n^{top}(X)$ is not discrete if and only if it has the power of the continuum.

1 Introduction

In 1998, J. Pawlicowski [5] presented a forcing free proof of a conjecture of Mycielski [4] that the fundamental group of a connected locally connected compact metric space is either finitely generated or has the power of the continuum. In [1], Biss equipped the loop space of X with the compact open topology. Then he put a canonical topology on the fundamental group of X as a quotient of $Hom((S^1, 1), (X, x))$ which is invariant under the homotopy type of X and denoted it by $\pi_1^{top}(X, x)$. He proved among the other things that $\pi_1^{top}(X, x)$ is a topological group which is independent of the base point. Recently, P. Fabel [2] using the Mycielski's conjecture, showed that if X is a peano continuum, then either X has a finitely generated discrete topological fundamental group, or it has a non-discrete topological fundamental group, having the power of the continuum. In [3], we et al. introduced a topology on higher homotopy groups of a pointed space (X, x) as a quotient of $Hom((I^n, \dot{I}^n), (X, x))$ equipped with the

compact-open topology and denoted it by $\pi_n^{top}(X, x)$. We proved that $\pi_n^{top}(X, x)$ is a topological group. Also, we found necessary and sufficient conditions for which the topology is discrete. In this note, we are going to extend Mycielski's conjecture to higher homotopy groups. At the end, we generalize Fabel's results for topological higher homotopy groups: Suppose X is an $(n-1)$-connected locally $(n-1)$-connected compact metric space. Then $\pi_n^{top}(X)$ is discrete if and only if $\pi_n(X)$ is finitely generated. Moreover, $\pi_n^{top}(X)$ is not discrete if and only if $\pi_n(X)$ has the power of the continuum.

2   Main results

We recall a topological space X is called *n-semilocally simply connected at a point* x if there exists an open neighborhood U of x for which any n-loop in U is null-homotopic in X. Moreover, X is said to be *n-semilocally simply connected* if it is n-semilocally simply connected at each point (see [3]). A space X is called *n-connected* for $n \geq 0$ if it is path connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$. X is called *locally n-connected* if for each $x \in X$ and each neighborhood U of x, there is a neighborhood $V \subseteq U \subseteq X$ containing x so that $\pi_k(V) \longrightarrow \pi_k(U)$ is zero map for all $0 \leq k \leq n$ and for all basepoint in V (see [6]).

Lemma 2.1. *Suppose X is an $(n-1)$-connected, locally $(n-1)$-connected compact metric space and $\pi_n(X)$ is not finitely generated. Then there exists $x \in X$ such that for each positive integer m, there exists an n-loop f_m at x with diameter $< 2^{-m}$ which is not nullhomotopic. In particular, X is not n-semilocally simply connected at x.*

Proof. For simplicity, we prove the assertion for $n = 2$. Similar argument gives the result in general case. Suppose otherwise. Then for each $x \in X$, there exists $m(x) \in \mathbb{N}$ such that every 2-loop at x which has diameter less than $2^{-m(x)}$ is homotopic to the constant loop at x. Let U_x be an open ball containing x. By local 1-connectivity of X, for each $x \in X$ there is a path connected neighborhood V_x containing x, so that $V_x \subseteq U_x$ and $\pi_1(V_x) \longrightarrow \pi_1(U_x)$ is zero map. Suppose $m(x)$ is sufficiently big and by shrinking V_x, if it is necessary, we may assume that V_x has diameter $2^{-(m(x)+1)}$. Therefore each k-loop, $k = 1, 2$, contained in V_x is nullhomotopic in U_x. Suppose that W_x is a path connected neighborhood of x with diameter less than $2^{-(m(x)+3)}$. Now by compactness of X, there is a finite cover of X by subsets W_i's, containing x_i, $i \leq N_0$, with the above property. If it is necessary, we can omit some W_i's such that the number of remainders is at least to cover X. For each i, j with $W_i \cap W_j \neq \phi$, fix a path h_{ij} in $W_i \cup W_j$ going from x_i to x_j. Note that any path g from x_i to x_j which is contained in $W_i \cup W_j$ is homotopic to h_{ij} in X. Indeed, suppose $m(x_i) \leq m(x_j)$. Then $g * (h_{ij})^{-1}$ is a 1-loop at x_i with diameter $< 2^{-(m(x_i)+1)}$ contained in V_i, so $g * (h_{ij})^{-1}$ is homotopic to constant loop at x_i in U_i. This homotopy gives a homotopy from g to h_{ij} in U_i. If $m(x_i) > m(x_j)$, consider $(h_{ij})^{-1} * g$, an 1-loop at x_j (similar argument used in [5] for $n = 1$). Each four of these paths $\{h_{ij}, h_{jk}, h_{kl}, h_{li}\}$ with common vertices $\{x_i, x_j, x_k, x_l\}$ induce an inessential 1-loop $f : S^1 \longrightarrow X$. Indeed, suppose that

$m(x_i) \leq min\{m(x_j), m(x_k), m(x_l)\}$ (the argument for other cases is similar). We show that the diam$(f) < 2^{-(m(x_i)+1)}$, where diam(f) is the diameter of $f(S^1)$. For, we have

$$
\begin{aligned}
diam(f) &\leq diam(W_i) + diam(W_j) + diam(W_k) + diam(W_l) \\
&< 2^{-(m(x_i)+3)} + 2^{-(m(x_j)+3)} + 2^{-(m(x_k)+3)} + 2^{-(m(x_l)+3)} \\
&\leq 4(2^{-(m(x_i)+3)}) = 2^{-(m(x_i)+1)}
\end{aligned}
$$

So, f is contained in V_i and this implies that f is nullhomotopic in U_i. By 1-connectivity of X, f has a continuous extension over E^2. We denote the 2-cube $f(E^2)$ by Y_{ijkl} if the family $\{h_{ij}, h_{jk}, h_{kl}, h_{li}\}$ of paths bound this cube. Note that any 2-cube Y'_{ijkl} obtained by $g : E^2 \longrightarrow X$ which is contained in $W_i \cup W_j \cup W_k \cup W_l$ is homotopic to Y_{ijkl}. Indeed, suppose $m(x_i) \leqslant min\{m(x_j), m(x_k), m(x_l)\}$. Then the 2-loop induced by Y_{ijkl} and Y'_{ijkl} is a 2-loop at x_i with diameter less than $2^{-m(x_i+1)}$, and so it is homotopic to the constant loop at x_i in X. An elementary manipulation of this homotopy gives a homotopy from f to g. Now, connect each point $x_i \in W_i$ to x_0 by path g_i. Let s_{ij} be a side of a cube with vertices x_i and x_j. Then the paths h_{ij}, g_i and g_j induce 1-loop $\lambda_{ij} : S^1 \longrightarrow X$. Again, by 1-connectivity of X, λ_{ij} has a continuous extension over E^2. We denote the cube $\lambda_{ij}(E^2)$ by Z_{ij}. Each Y-cube together with associated Z-cubes induce an 2-loop. The 1-connectivity of X implies that these 2-loops are uniquely determined up to homotopy. So we have a finite number of homotopy classes of these 2-loops and we denote them by $\alpha_1, ..., \alpha_N$. Moreover, corresponding to each Y-cube there is a homotopy class α_i which has the Y-cube as a side. Now, to get a contradiction, we show that $\pi_2(X)$ is generated by α_i's, $i = 1, ..., N$. For this, suppose that an 2-loop $\eta : (I^2, \dot{I}^2) \longrightarrow (X, x_0)$ is given. Let $\delta > 0$ be the Lebesgue's number of the covering $\{W_i : i = 1, ..., N_0\}$. We divide the cube I^2 to small subcubes $I_1, ..., I_l$ such that each $\eta(I_j)$ has diameter $< \delta$, $j = 1, ..., l$; and so it is contained in some W_{i_j}. We denote $\eta(I_j)$ by X_j and its corners by v_{j_k} , $k = 1, ..., 4$. Now, connect each vertex v_{j_k} to x_0 by a path t_{j_k}. By 1-connectivity of X, the triangles with vertices v_{j_k}, $v_{j_{k+1}}$ and x_0 and with sides t_{j_k}, $t_{j_{k+1}}$ and a side of X_j induce a sequence of 1-loops which are homotopic to constant loop at x_0. We can fill inside them by homotopy. These filled triangles with the cube X_j induce an 2-loop at x_0 which we denote its homotopy class by β_j. By 1-connectivity of X, β_j is uniquely determined up to homotopy. So we can write $[\eta] = \beta_1 * ... * \beta_l$. We show that each β_j is homotopic to one of α_i's, $i = 1, ..., N$. Indeed, corresponds to Corresponds to each v_{j_k}, there is an index $i_k \in 1, ..., N_0$ such that $v_{j_k} \in W_{i_j} \cap W_{i_k}$. We connect each v_{i_k} to x_{i_k} by a path θ_{i_k} and then we fill inside the 1-loops induced by paths $h_{i_k i_{k+1}} \theta_{i_k}, \theta_{i_{k+1}}$ and a side of X_j by homotopy. In this manner, we obtain an inessential 2-loop γ_j which has the cube Y_{i_1,i_2,i_3,i_4} as a side. (Note that γ_j has diameter less than $2^{-(m(x_{i_j})+1)}$ for some i_j and so it is nullhomotopic.) Let α_{i_j} be the homotopy class corresponding to the cube Y_{i_1,i_2,i_3,i_4}. Since γ_j is nullhomotopic, then $\alpha_{i_j} = \beta_j$. So, we have

$$[\eta] = \beta_1 * ... * \beta_l = \alpha_{i_1} * ... * \alpha_{i_l}$$

Therefore, $\pi_2(X)$ is generated by α_i's and this is a contradiction. ∎

Suppose that the nth homotopy group of X is not finitely generated. Similar to [5], we define an equivalence relation $\{0,1\}^N$ via homotopic n-loops such as follows:

First, take a point $x \in X$ and a sequence $\{f_m\}_{m \in N}$ of n-loops as claimed in Lemma 2.1. For each $\alpha \in \{0,1\}^N$, let f_m^α be the constant n-loop at x, if $\alpha(m) = 0$, otherwise let $f_m^\alpha = f_m$. Define an n-loop f_α at x as $f_0^\alpha * f_1^\alpha * \dots$. Write $\alpha \approx \beta$ if $f_\alpha \sim f_\beta$. Clearly $\approx$ is an equivalence relation and argument used in [5] , shows that it has continuum many equivalence classes; that is, there is a set of size of continuum of mutually non-homotopic n-loops. Therefore, we have the following theorem which is the extension of *Mycielski's conjecture* to higher homotopy groups.

Theorem 2.2. *Suppose X is a compact metric space, which is $(n-1)$-connected, locally $(n-1)$-connected. Then $\pi_n(X)$ is either finitely generated or has the power of the continuum.*

In [3], we et al. clarified a relationship between the cardinality of $\pi_n(X, x)$ and discreteness of $\pi_n^{top}(X, x)$. We asserted that if X is a connected separable metric space such that $\pi_n^{top}(X, x)$ is discrete, then $\pi_n(X, x)$ is countable, and as a result we showed that if X is a connected locally n-connected separable metric space, then $\pi_n(X, x)$ is countable. First, we recall the following theorem from [3].

Theorem 2.3. *Suppose X is a locally $(n-1)$-connected metrizable space and $x \in X$. Then the following are equivalent: (1) $\pi_n^{top}(X, x)$ is discrete. (2) X is n-semilocally simply connected at x.*

Now, we show that the cardinality of $\pi_n(X, x)$ and discreteness of $\pi_n^{top}(X, x)$ are relevant.

Theorem 2.4. *Suppose X is an $(n-1)$-connected locally $(n-1)$-connected compact metric space. Then $\pi_n^{top}(X)$ is discrete if and only if $\pi_n(X)$ is finitely generated. Moreover, $\pi_n^{top}(X)$ is not discrete if and only if $\pi_n(X)$ has the power of the continuum.*

Proof. The assertions follow immediately by Lemma 2.1 and Theorems 2.2 and 2.3.   ∎

Example 2.5. *Let $X = \cup_{n \in \mathbb{N}} S_n$, where $S_n = \{(x,y,z) | (x - \frac{1}{n})^2 + y^2 + z^2 = \frac{1}{n^2}\}$, be a subspace of $\mathbb{R}^3$. It is easy to see that X is 1-connected and locally 1-connected. However, the sequence $\{[S_n]\}$ is convergent to identity element of $\pi_2^{top}(X, 0)$, implying that $\pi_2^{top}(X, 0)$ is not discrete and has the power of continuum.*

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