Numerical Simulation of the Impingement of a Vertical Liquid Jet on a Solid Surface

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Abstract
The impingement of a vertical liquid jet on a solid surface which leads to the occurrence of a circular hydraulic jump (CHJ) is numerically simulated in this study by using the volume-of-fluid (VOF) method. The results show that increasing the volumetric flow rate will increase both the radius and height of the jump which is confirmed by the experimental observations. Also, different types of circular hydraulic jumps, captured by the numerical method, are discussed in the paper.

Keywords: Vertical Jet Impingement, Circular Hydraulic Jump, Numerical Simulation, Volume-of-Fluid Method.

1- Introduction

At the beginning of the nineteenth century, the great British physicist, Lord Rayleigh encountered a discontinuity in the geometry of linear one dimensional flow. The structure is called *river bore* if moving, and *hydraulic jump*, if stationary and is created due to e.g. variation in river bed. The classical planar hydraulic jump which occurs in open-channel flows is a very old and well-known phenomenon thoroughly considered in the literature. However, the Circular Hydraulic Jump (CHJ) and consequently Polygonal Hydraulic Jump (PHJ), although having a similar name, are completely different Phenomena. When a circular vertical liquid jet impacts on a solid horizontal surface, which is called *target plate*, the flow spreads radially away everywhere – from the stagnation point – until at a particular radius, which is called the radius of the jump, the thickness of the liquid film increases abruptly and a so-called circular or axisymmetric hydraulic jump occurs.

As mentioned earlier, the first person who considered CHJ was probably Lord Rayleigh (1914) who proposed his model by using the continuity and momentum equations and assuming the flow as being inviscid [1]. He assumed that mass and momentum are conserved across the jump, but energy is not. He finally could derive some relations for the inviscid jump. Rayleigh's method was based on the analogy of shallow water and gas theories. The complete theory of inviscid circular hydraulic jump was presented by Birkhoff and Zarantonello in 1957 [2].

However, it is clear that the flow in such a problem is viscous and the inviscid theory is not adequate for predicting the location of the circular hydraulic jump occurrence, since the fluid layer thickness before the jump is typically sufficiently thin, so that the diffusion of vorticity from the lower boundary is dynamically significant. Therefore, the viscosity must be taken into account.

The first person, who considered the effect of viscosity in CHJ, was Watson in 1964 who solved the problem analytically. He, in a strong, long and highly-referred paper, described the flow in terms of a Blasius sublayer developing in the vicinity of the stagnation point, as on a flat plate, and also in terms of a similarity solution. By using the momentum equation, he could finally obtain some relations for predicting the radius of the jump. Watson's model will be considered in detail in the next section.

The validity of Watson's theory has been investigated experimentally by many different researchers throughout the world in the last four decades such as Watson himself [3], Olson and Turkdogan [4], Ishigai *et al.* [5], Nakoryakov [6], Bouhadef [7], Craik *et al.* [8], Errico [9], Vasista [10], Liu and Lienhard [11], Ellegaard *et al.* [12], and in particular Bush and Aristoff [13, 14]. The agreement between the theory and experiment has been very diverse, from good to bad, depending on the jump conditions. Even Watson himself has presented some data that are in poor agreement with his own theory.

Some other investigators also considered the problem from different aspects. Bowles and Smith studied the circular hydraulic jump -with surface tension considerations- and the small standing waves preceding the jump [15]. Higuera also proposed a model for planar jump by studying the flow in transition region in the limit of infinite Reynolds number [16]. Bohr *et al.* in 1993 obtained a scaling relation for the radius of the jump [17]. In 1997, they also proposed a simple viscous theory for free-surface flows that can accommodate regions of separated flow and yield the structure of stationary hydraulic jumps [18].

Watanabe *et al.* presented integral methods for shallow free-surface flows with separation in the application of circular hydraulic jump and also the flow down an inclined plane [19]. Ellegaard *et al.* who in 1996 investigated the CHJ empirically [12], for the very first time, observed the polygonal hydraulic jumps in their experiments [20] and reported them in detail in 1999 [21]. In the same year, Yokoi and Xiao considered the transition in the circular hydraulic jump numerically [22]. Three years later, they also studied numerically the structure formation in circular hydraulic jumps with moderate Reynolds numbers [23]. Brechet and Neda also investigated the circular hydraulic jumps and compared their theory and experiments [24].

Avedisian and Zhao studied in detail, the effect of gravity on the circular hydraulic jump and its different parameters experimentally [25]. Rao and Arakeri considered the CHJ empirically and measured the radius of the jump, film thickness and also the length of the transition zone and specially focused on jump formation and transition to turbulent flow [26]. In 2002, Ferreira *et al*. simulated the circular hydraulic jump numerically in order to compare the various upwind schemes for convective term of the Navier-Stokes equations [27].

Gradeck *et al*. studied the impingement of an axisymmetric jet on a moving surface both numerically and experimentally in order to simulate the cooling of a rolling process in the steel making industry [28]. Ray and Bhattacharjee also studied the standing and traveling waves in CHJ [29]. Very recently, the Mikielewicz proposed a simple dissipation model for the CHJ [30]. Also, Kate *et al*. studied experimentally the impinging of an oblique liquid jet on a solid surface which causes non-circular jumps. They also have measured the film thickness and the stagnation pressure for different angles of the incoming jet [31].

In this study, the impingement of a vertical liquid jet on a solid surface is simulated by the method of Volume-of-Fluid using Youngs' algorithm. The formation of a circular hydraulic jump and its different types are investigated.

2- Theory of Circular Hydraulic jump

Circular hydraulic jumps might take place, when a vertical descending liquid jet impacts a solid horizontal surface. Figure 1 shows a sample of an empirically observed CHJ.

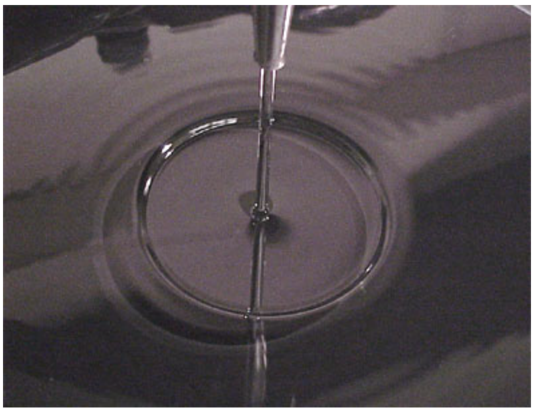

Figure 1: The circular hydraulic jump [taken from Ref. [13]

The impingement of this circular jet on the mentioned solid surface is important in a variety of processes such as the fuel tank of space shuttles, aircraft generator coils, coating flows, impingement cooling of electronic devices, laser mirrors and material processing in manufacturing. The important feature of CHJ is its potential for heat loss in downstream of the jump, especially for the processes in which the purpose is cooling a hot surface, such as the research done by Womac *et al*. [32]. The general structure of a circular hydraulic jump is shown in figure 2.

As mentioned in previous section, Watson was the first person who analyzed the viscous circular hydraulic jump and proposed two models for it. His first model was an inviscid one for downstream of the jump in which he assumed the pressure force to be equal to the rate at which momentum is increasing. In his second model which was viscous, he had used the prandtl boundary layer theory for development of the flow which is considered here in brief, since it is the first and the only valid theory for CHJ.

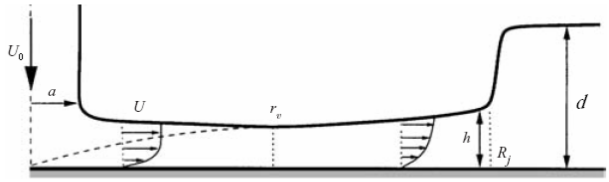

Figure 2: The generals tructure of CHJ

In upstream region where the flow is viscous, Watson divided the flow field into four different regions:

i.) The region very close to the stagnation point where the radial distance is of the same order of the jet radius $(r = O(a))$ and the boundary layer thickness is of order $\delta = O(\upsilon a / U_0)$ where a and U_0 are the radius and the velocity of the incoming jet and υ is kinematic viscosity (see figure 2);

ii) The region $r >> a$ in which the features of stagnation region are not important and the boundary layer is similar to the Blasius sublayer development over flat plate;

iii) The region from the point where the boundary layer spans the whole fluid layer to the point where the velocity becomes self-similar that can be called a transition region;

iv) The region in which the similarity solution suggested by Watson is valid.

According to Watson's theory, the viscous solution is valid only in the second and fourth regions and for $\mathrm{Re} = Q/(\upsilon a) >> 1$ where $Q = \pi a^2 U_0$ is the volumetric flow rate. His approximate solution is clearly not correct in the first region, since the radii of the jump and the jet are of the same order. By neglecting the third region, Watson used the Karman-Pohlhausen method [33] to match the solution of the second region (from Blasius velocity profile) and the solution of the last region for which he assumed the following velocity profile:

$$u = U(r) f\left(\frac{z}{\delta}\right) \tag{1}$$

where $U(r)$ is the velocity at the free surface and f is the similarity function.

By using the above velocity profile in momentum integral equation, Watson could derive this explicit relation for the thickness of the boundary layer:

$$r^2\delta^2 - \frac{c^3\sqrt{3}}{\pi - c\sqrt{3}} \frac{\upsilon r^3}{U_0} = C \tag{2}$$

where $c = 1.402$ and C is the integration constant. By an order-of-magnitude analysis, Watson showed that in the region $r = O(a)$, $C = O(\upsilon a^3 / U_0)$ and in the region $a << r < r_0$, $C = O(a^3 / r^3)$ where $r_0 = 0.315 a \mathrm{Re}^{1/3}$ is the radial location in which the boundary layer absorbs the whole flow and is also shown by r_v in the literature. The proportionality factor for this critical radius, which is the place that the transition from the second region to the forth one is occurred, was obtained by matching the two different solutions just mentioned. A similar value was also obtained by Bowles and Smith by means of an exact numerical solution [15].

The viscosity causes the diffusion of vorticity on time scale $t \sim \delta^2/\upsilon$ across the fluid layer which is spreading radially. At this time, the flow travels the radial distance $r_0 \sim U_0 t$ which predicts that the boundary layer must include the entire fluid layer depth at the radius $r_0 \sim a \mathrm{Re}^{1/3}$. The features of the fluid flow are thoroughly altering at this critical radius. Before this point, the flow is developing as a Blasius sublayer over flat plate and the surface velocity is of the same order of the incoming jet speed, while after the critical radius, the flow is fully-developed and the surface velocity is negligible in comparison with the incoming velocity and so may be ignored, although Watson obtained a relation for it as:

$$U(r) = \frac{27c^2}{8\pi^4}\frac{Q^2}{\upsilon(r^3+l^3)} \qquad (3)$$

where $c = 1.402$ and l is an arbitrary constant which is estimated as $l = 0.567 a \mathrm{Re}^{1/3}$ by considering the initial development of the boundary layer.

Watson ignored the integration constant C and obtained two relations for predicting the fluid layer depth ξ as:

$$\xi(r) = \frac{a^2}{2r} + \left[1 - \left(\frac{2\pi}{3\sqrt{3}}c^2\right)\right]\delta \qquad r < r_0 \qquad (4)$$

$$\xi(r) = \frac{2\pi^2}{3\sqrt{3}}\frac{\upsilon(r^3+l^3)}{Qr} \qquad r \ge r_0 \qquad (5)$$

For reaching his main goal which was predicting the location of the jump occurrence, by assuming the downstream height to be known, Watson used the momentum balance and eventually could derive some relations for the jump radius.

Watson's theory for CHJ has been the subject of many different investigators. He, in his model, had used some assumptions that are very important to note. For instance, he had assumed that the flow after the jump is unidirectional which we now know that is not correct, since the boundary layer separates from the surface downstream of the jump. He also had ignored the effect of surface tension in his analysis.

Many researchers have mentioned later that the surface tension must be taken into consideration to improve the accuracy of Watson's model. Craik *et al.* declared in their paper that if the radius of the jump is larger than ten times of the downstream height, then the Watson's theory is accurate enough, but for smaller jumps, the accuracy of the theory is curtailed [8]. The experimental results of Errico are generally far from Watson's theory, since the flow rates in his experiments are low and his jumps are small and deep [9]. Vasista also concluded in his study that for large radius of the jet and also high outer fluid depths, the accuracy of the theory is not good [10]. Liu and Lienhard completed this result and stated that if the radius of the jump decreases or radius of the jet increases, then the upstream Froude number will be larger. They concluded finally that for jumps with large downstream height and high upstream Froude number, the Watson's model is not accurate enough. Therefore, briefly it can be said that the accuracy of Watson's theory is not appropriate for jumps of small radius and height, known as weak jumps [11].

Based on the experiments, the surface tension influence is underscored in small jumps. The empirical observations have shown that reducing the surface tension causes the radius of the circular jump increase and also makes the jump more gradual, i.e. the jump becomes less abrupt.

Bush and Aristoff in 2003 have considered the influence of surface tension on CHJ analytically and could propose a very simple valuable relation for the curvature force –which for weak jumps is comparable with pressure forces in momentum equation- and eventually were capable of modifying Watson's theory, i.e. his relations for predicting the jump radius. These modified relations are [13]:

$$\frac{r_1 d^2 g a^2}{Q^2}\left(1 + \frac{2}{Bo}\right) + \frac{a^2}{2\pi^2 r_1 d} = 0.10132 - 0.1297\left(\frac{r_1}{a}\right)^{3/2}\mathrm{Re}^{-1/2} \qquad r < r_0 \qquad (6)$$

$$\frac{r_1 d^2 g a^2}{Q^2}\left(1 + \frac{2}{Bo}\right) + \frac{a^2}{2\pi^2 r_1 d} = 0.01676\left[\left(\frac{r_1}{a}\right)^3 \mathrm{Re}^{-1} + 0.1826\right]^{-1} \qquad r \ge r_0 \qquad (7)$$

where d is the downstream height (or outer depth which is also shown by h_∞), g is the gravitational acceleration, r_1 (also shown by r_j or R_j) is the radius of the jump, $Bo = \rho g R_j \Delta H/\sigma$ is the Bond number and ΔH is the jump height.

The Bush and Aristoff relations for radius of the jump differ from those of Watson only in the term including Bond number that contains the surface tension effect which is highlighted in the weak jump regimes. By this modification to Watson's theory, Bush and Aristoff could improve the accuracy of his model in small jump regimes in which his own theory had some imperfections. According to the above relations, if the jump is big enough, then the Bond number will become large and its term in the equations becomes negligible and so the old Watson's relations will be obtained.

In 1993, Bohr *et al.* proposed a scaling relation for the circular hydraulic jump radius as:

$$R_j \sim q^{5/8}\upsilon^{-3/8}g^{-1/8} \qquad (8)$$

where $q = Q/(2\pi)$ and R_j is the radius of the jump. According to this relation, decreasing gravity and viscosity and also increasing the flow rate will result in bigger jumps. They verified the validity of this scaling relation by their own experimental observations.

3- Types of Circular Hydraulic Jumps

Many different investigators have classified the CHJ in their studies. Based on these categories, there are generally two different kinds of CHJs whose second one has two different types by itself. The type I jump is the standard circular hydraulic jump in which the surface flow is radially outward everywhere that is also marked by unidirectional surface flow in which the separation of the boundary layer occurs after the jump and on the surface. This kind of jump is also called *single jump* and the eddy formed on the surface is also known as *separation bubble* or *recirculating eddy*. The region is also called *separated region*. The experimental results of Craik *et al*. [8] and also Errico [9] have shown that this separated region may be very large and its length substantially changes with the flow conditions.

If the downstream height is increased, the jump changes its structure to type IIa jump. This recently named jump is marked by an under-surface separation bubble on the wall and also by a region of reversed surface flow adjoining the jump at its front. In other words, in this type of jump, flow has an excessive eddy which is called *surface roller* or *surfing wave* and it is like a broken wave in the ocean. In this case, the main stream with high speed flows between the two vortices.

If the downstream height is increased further, the jump will convert to type IIb jump which is called *double* or *tiered jump* in which the thickness of the fluid layer increases twice. The interesting feature is that if the downstream height is decreased again, type IIb will turn back to type IIa. Figure 3 shows these different types of CHJ.

By increasing the outer depth even more, the flow becomes turbulent and the symmetry of the flow is broken and there will be air entrainment on the surface and generally the whole structure will be to some extent similar to that of planar classic hydraulic jump in open channels.

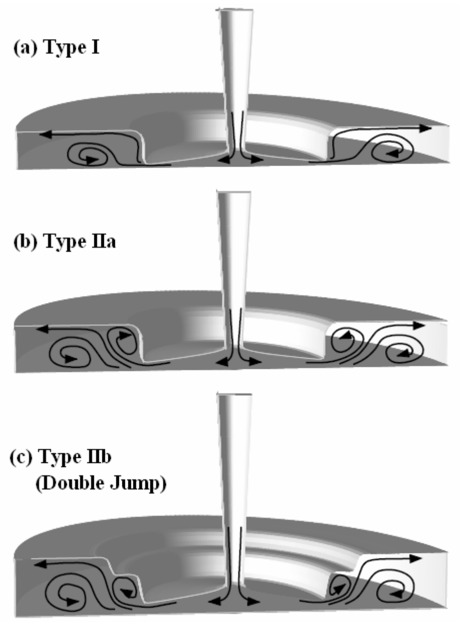

Figure 3: Schematics of different types of circular hydraulic jump

The effect of gravity on CHJ was considerably studied by Avedisian and Zhao in 2000. They have shown experimentally that reducing the gravity will make the jump radius larger and its curvature smaller. According to their observations, in low gravity conditions, the radius of the jump is higher than normal gravity conditions and also the length of the transition zone becomes larger, i.e. the jump occurs more gradually. Also at beginning of reducing gravity, there is seen a *hump* in downstream of the jump which is followed by a pattern of regular circular waves. They also mentioned that the effect of surface tension and viscosity dominate at low gravity conditions.

4- Numerical Method

In this study, the circular hydraulic jump is simulated numerically by solving the Navier-Stokes equations, along with an equation for tracking the free-surface. In this section, we present a brief account of the numerical method. The governing equations are the continuity and momentum equations:

$$\vec{\nabla} \cdot \vec{\mathbf{V}} = 0 \tag{9}$$

$$\frac{\partial \vec{\mathbf{V}}}{\partial t} + \vec{\nabla} \cdot (\vec{\mathbf{V}}\vec{\mathbf{V}}) = -\frac{1}{\rho}\vec{\nabla} p + \frac{1}{\rho}\vec{\nabla} \cdot \vec{\tau} + \vec{\mathbf{g}} + \frac{1}{\rho}\vec{\mathbf{F}}_b \tag{10}$$

where $\vec{\mathbf{V}}$ is the velocity vector, p is the pressure, τ is the stress tensor and $\vec{\mathbf{F}}_b$ represents the body forces acting on the fluid.

The free surface is tracked by using the volume-of-fluid (VOF) method by means of a scalar field f (known as volume of fluid fraction) whose value is unity in the liquid phase and zero in the vapor. When a cell is partially filled with liquid, i.e. the interface, f will have a value between zero and one:

$$f = \begin{cases} 1 & \text{in liquid} \\ > 0, < 1 & \text{at the liquid-gas interface} \\ 0 & \text{in gas} \end{cases} \tag{11}$$

The discontinuity in f is propagating through the computational domain according to:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{V} \cdot \vec{\nabla} f = 0 \tag{12}$$

For the advection of volume fraction f based on Equation (12), different methods have been developed such as SLIC, Hirt-Nichols and Youngs' PLIC [34]. The reported literature on the simulation of free-surface flows reveals that Hirt-Nichols method has been used by many researchers. In this study, however, we used Youngs' method [34], which is a more accurate technique. Assuming the initial distribution of f to be given, velocity and pressure are calculated in each time step by the following procedure. The f advection begins by defining an intermediate value of f:

$$\tilde{f} = f^n - \delta t \vec{\nabla} \cdot (\vec{V} f^n) \tag{13}$$

Then it is completed with a "divergence correction":

$$f^{n+1} = \tilde{f} + \delta t (\vec{\nabla} \cdot \vec{V}) f^n \tag{14}$$

A single set of equations is solved for both phases, therefore, density and viscosity of the mixture are calculated according to:

$$\rho = f\rho_l + (1-f)\rho_g \qquad (15)$$

$$\mu = f\mu_l + (1-f)\mu_g \qquad (16)$$

where subscripts l and g denote the liquid and gas, respectively. New velocity field is calculated according to the two-step time projection method as follows. First, an intermediate velocity is obtained:

$$\frac{\tilde{\vec{V}} - \vec{V}^n}{\delta t} = -\vec{\nabla}\cdot(\vec{V}\vec{V})^n + \frac{1}{\rho^n}\vec{\nabla}\cdot\vec{\tau}^n + \vec{g}^n + \frac{1}{\rho^n}\vec{F}_b^n \qquad (17)$$

The continuum surface force (CSF) method is used to model surface tension as a body force ($\vec{\mathbf{F}}_b$) that acts only on interfacial cells. Pressure Poisson equation is then solved to obtain the pressure field:

$$\vec{\nabla}\cdot\left[\frac{1}{\rho^n}\vec{\nabla}p^{n+1}\right] = \frac{\vec{\nabla}\cdot\tilde{\vec{V}}}{\delta t} \qquad (18)$$

Next, new time velocities are calculated by considering the pressure field implicitly:

$$\frac{\vec{V}^{n+1} - \tilde{\vec{V}}}{\delta t} = -\frac{1}{\rho^n}\vec{\nabla}p^{n+1} \qquad (19)$$

The cell size used in this study was set based on a mesh refinement study in which the grid size was progressively increased until no significant changes were observed in the simulation results. The mesh resolution was characterized by the number of cells per radius of the jet. From the mesh refinement study, the optimum mesh size was found to be 20 cells per radius of the jet. This mesh size was used for all simulations throughout this paper.

5- Results and Discussion

By using the Volume-of-Fluid method, we could simulate the circular hydraulic jump and the results are verified by the experimental observations. The location of jump formation and also how it occurs is seen in the numerical method.

Figure 4 shows the simulated hydraulic jump which is captured for tap water ($\rho = 1000 kg/m^3$, $\upsilon = 1.122\times10^{-6} m^2/s$, $\sigma = 0.073 N/m$) for which the radius of the incoming jet is $a = 2.5mm$ and its speed is $U_0 = 1m/s$.

Figure 5 shows the formation of the jump and the velocity distributions during the jump.

Also the numerical method has been able to simulate the surface roller. The vortex of the flow is clear in Figure 6 which is also seen in the experiments.

The circular hydraulic jump is also seen in 3D case which has been obtained for a Ethylene Glycol ($\rho = 1130 kg/m^3$, $\mu = 0.016 Pa.s$, $\sigma = 0.048 N/m$) jet of radius $a = 5.6mm$ and the flow rate of $Q = 46ml/s$ on a solid surface. Figure 7 shows the simulated 3D jump and also the empirically observed jump.

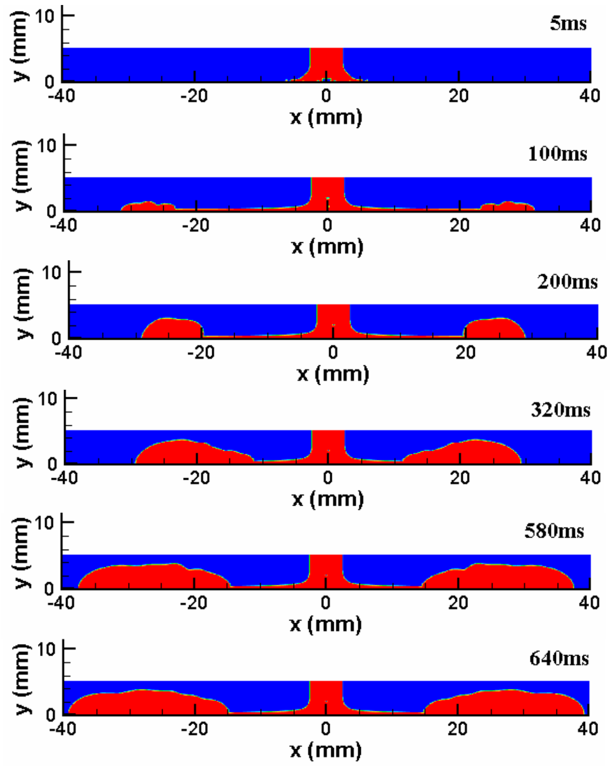

Figure 4: The evolution of a circular hydraulic jump formation using VOF method

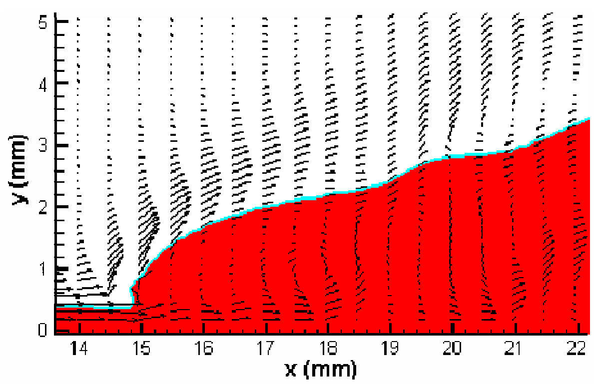

Figure 5: Velocity profiles during the jump

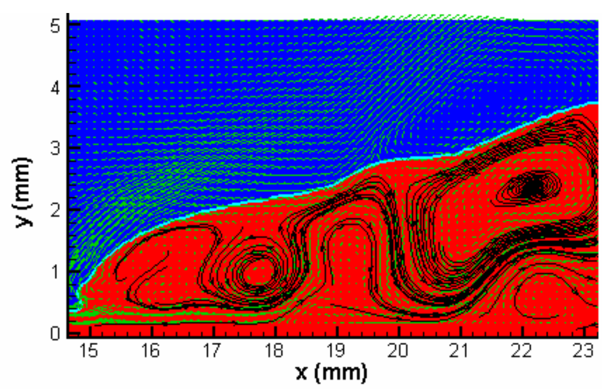

Figure 6: Calculated vortex and velocity vectors after the jump

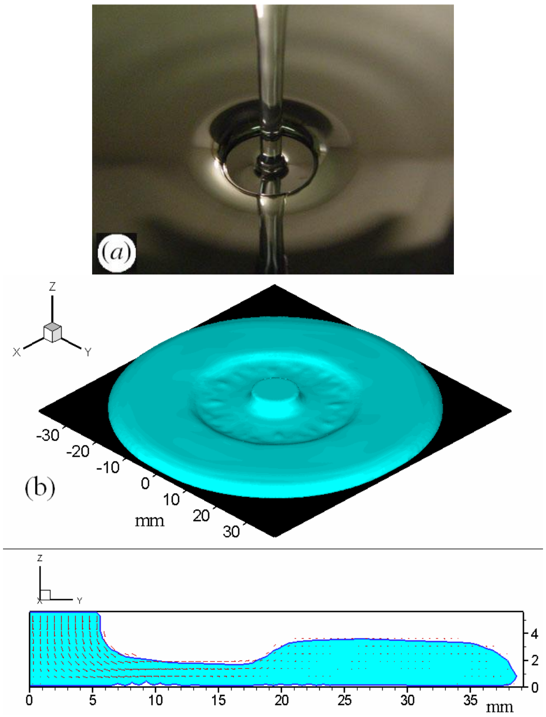

Figure 7: Thei mpingement of a vertical liquid jet on a solid surface and the formation of a circular hydraulic jump: a) Experimental result; b) Numerical simulation

In addition to simulating the formation of the jump, the VOF method could also capture the different types of CHJ introduced in the previous section. Figure 8, for instance, shows the type IIb (double) jump which is obtained for the impingement of the same liquid jet at the speed of $0.629 m/s$ on the horizontal surface that was also observed in the experiments(see figure 3(c)) .

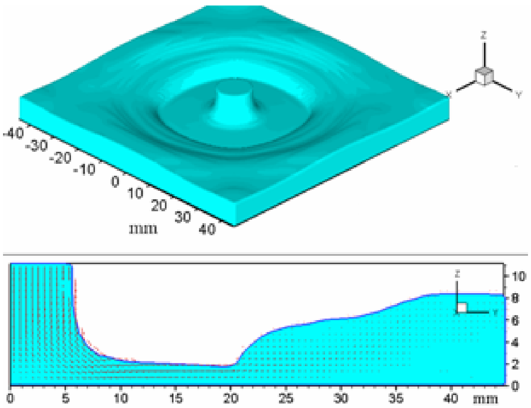

Figure 8: Simulation of a double jump by VOF method

The values of jump radius obtained for different volumetric flow rates are plotted in figure 9 and the corresponding values of jump height are shown in figure 10. The experimental observations of many researchers verify these trends [8, 10, and 13].

6- Conclusion

In this study, the impingement of vertical liquid jet on a solid horizontal surface and the occurrence of circular hydraulic jump were simulated by the method of volume-of-fluid. The results show that this numerical method is capable of simulation of the circular hydraulic jump and also the reversed flow which was considerably seen in the experiments by many different researchers. Also the recently named type IIb jump is captured by the VOF method. The numerical results also show that both the radius and height of the circular hydraulic jump increase by enhancing the volumetric flow rate which is verified by the empirical observations.

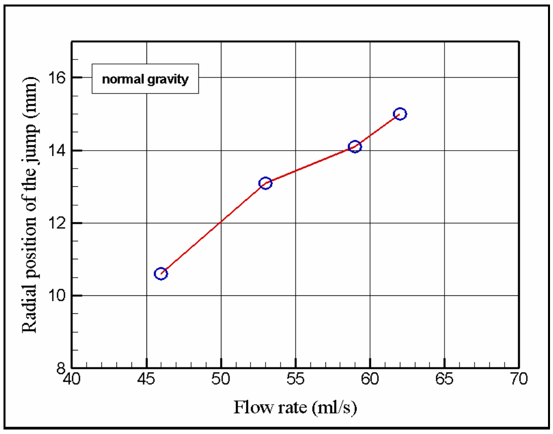

Figure 9: Variation of jump radius vs. flow rate

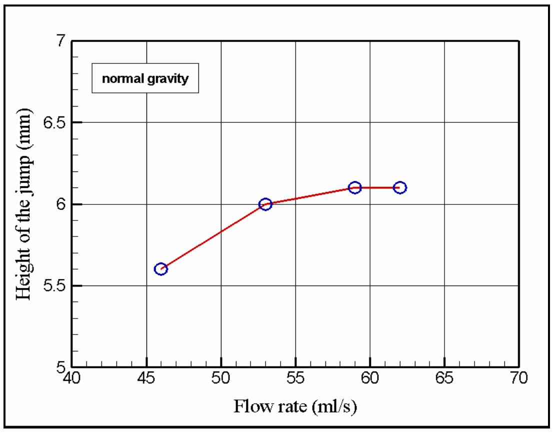

Figure 10: Variation of jump height vs. flow rate

7- Nomenclature

a	Jet Radius
Bo	Bond Number
d, h_∞	Downstream Height
f	Volume of Fluid Fraction
$\vec{\mathbf{F}}_b$	Body Force
g	Gravitational Acceleration
p	Pressure
Q	Volumetric Flow Rate
R_j, r_j, r_1	Jump Radius
r_0, r_v	Critical Radius
Re	Reynolds Number
U_0	Incoming Jet Velocity
$\vec{\mathbf{V}}$	Velocity Vector
t	Time

Greek Letters

δ	Boundary Layer Thickness
ξ	Fluid Layer Depth
υ	Kinematic Viscosity
ρ	Density
σ	Surface Tension
τ	Stress Tensor

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