Group Theory Conference, Mashhad 2010
Ferdowsi University of Mashhad, March 10-12, 2010, p. 124-127

On Schur multipliers of pairs and triples of groups with topological approach

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Abstract
In this note, using a relation between Schur multipliers of pairs and triples of groups, the fundamental group and homology groups of a homotopy pushout of Eilenberg-MacLane spaces, we present among other things some behaviors of Schur multipliers of pairs and triples with respect to free, amalgamated free, and direct products and also direct limits with topological approach.

Keyword and phrases: Schur multiplier of a pair of groups, Schur multiplier of a triple of groups, Homology group, Homotopy group, Eilenberg-MacLane space, homotopy pushout

AMS subject Classification 2010: 20J06; 55P20; 55U25; 57M07

1 Introduction and Preliminaries

The Schur multiplier of a group G is defined to be

$$M(G) = (R \cap F')/[R, F],$$

where F/R is any free presentation of G. It is a well-known fact that $M(G)$ depends, up to isomorphism, only on G. Furthermore, it is easy to see that $M(-)$ is a functor from the category of groups to the category of abelian groups (see [5] for further details).

By a pair of groups (G, N) we mean a group G with a normal subgroup N. A homomorphism of pairs $(G_1, N_1) \to (G_2, N_2)$ is a group homomorphism $G_1 \to G_2$ that sends N_1 into N_2. The Schur multiplier of a pair of groups (G, N) which was first defined by G. Ellis [4] will be a functorial abelian group $M(G, N)$ whose principal feature is a natural exact sequence

$$\begin{gathered} \cdots \to H_3(G) \to H_3(G/N) \to M(G, N) \to M(G) \to \\ M(G/N) \to N/[N, G] \to G^{ab} \to (G/N)^{ab} \to 0 \qquad (1.1) \end{gathered}$$

in which $H_3(G)$ is the third homology group of G with integer coefficient.

There are several possible definitions of the Schur multiplier of a group and the Schur multiplier of a pair of groups. We are going to deal with topological one that we present in this note.

First, we note that for any group G one can construct functorially a connected CW-complex $K(G)$, called Eilenberg-MacLane space, whose fundamental group is isomorphic to G which has all higher homotopy groups trivial [8]. By considering $H_n(X)$ as the nth singular homology group of a topological space X, with coefficients in the group $\mathbf{Z}$, we recall the relation $H_n(G) \cong H_n(K(G))$, for all $n \geq 0$, [1, Prop. 4.1].

By Hopf formula for any CW-complex K with $\pi_1(K) = G$ and F/R as a free presentation for G we have the following isomorphism

$$\frac{H_2(K)}{h_2(\pi_2(K))} \cong \frac{R \cap F'}{[R, F]},$$

where h_2 is the corresponding Hurewicz map [3]. Hence a topological definition of the Schur multiplier of a group G can be considered as the second homology group of the Eilenberg-MacLane space $K(G)$, $H_2(K(G))$. This topological interpretation of $M(G)$ can be extended to one for $M(G,N)$ as follows.

For any two group extensions $1 \to M \to P \to Q \to 1$ and $1 \to N \to P \to R \to 1$ we consider the following homotopy pushout

$$\begin{array}{ccc} K(P) & \longrightarrow & K(\frac{P}{M}) \\ \downarrow & & \downarrow \\ K(\frac{P}{N}) & \longrightarrow & X. \end{array}$$

By Mayer-Vietoris sequence for pushout, we have the following exact sequence

$$\cdots \to H_3(P) \to H_3(Q) \oplus H_3(R) \to H_3(X) \to H_2(P) \to$$

$$H_2(Q) \oplus H_2(R) \to H_2(X) \to H_1(P) \to H_1(Q) \oplus H_1(R) \to$$

$$H_1(X) \to H_0(P) \to H_0(Q) \oplus H_0(R) \to H_0(X) \to 0. \tag{1.2}$$

Using [2, Corollary 3.4] we have

$$\pi_1(X) \cong \frac{P}{MN} \quad \text{and} \quad \pi_2(X) \cong \frac{M \cap N}{[M,N]}.$$

If we make the assumption $P = MN$, then X is $1-$connected and so by Hurewicz Theorem we have

$$H_1(X) = 0 \quad \text{and} \quad H_2(X) \cong \pi_2(X) \cong \frac{M \cap N}{[M,N]}.$$

Hence the exact sequence (1.2) becomes as follows

$$\cdots \to H_3(P) \to H_3(Q) \oplus H_3(R) \to H_3(X) \to H_2(P) \to$$

$$H_2(Q) \oplus H_2(R) \to \frac{M \cap N}{[M,N]} \to P^{ab} \to Q^{ab} \oplus R^{ab} \to 0. \tag{1.3}$$

Now, in special case, if we consider the two group extensions $1 \to N \to G \to G/N \to 1$ and $1 \to G \to G \to 1 \to 1$ corresponding to a pair of groups (G,N) and the following homotopy pushout

$$\begin{array}{ccc} K(G) & \longrightarrow & K(\frac{G}{N}) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X, \end{array} \tag{1.4}$$

then we have the following natural exact sequence as (1.1)

$$H_3(G) \to H_3(G/N) \to H_3(X) \to H_2(G) = M(G) \to$$

$$H_2(G/N) = M(G/N) \to H_2(X) = N/[N,G] \to G^{ab} \to (G/N)^{ab} \to 0.$$

Hence a topological interpretation of the Schur multiplier of a pair of groups (G,N) can be considered as the third homology group of a space X which is the homotopy pushout corresponding to the pair of groups (G,N) as (1.4).

Remark 1.1. As we mentioned before, the notion of the Schur multiplier of a pair of groups was introduced by G. Ellis in [4]. He presented several possible definitions of the notion through them the topological one is as follows.

For a pair of groups (G,N), the natural epimorphism $G \to G/N$ induces functorially the continuous map $f : K(G) \to K(G/N)$. Suppose that $M(f)$ is the mapping cylinder of f containing $K(G)$ as a subspace and is also homotopically equivalent to the space $K(G/N)$. Take $K(G,N)$ to be the mapping cone of the cofibration $K(G) \hookrightarrow M(f)$. By Mayer-Vietoris, the cofibration sequence $K(G) \hookrightarrow M(f) \to K(G,N)$ induces a natural long exact homology sequence

$$\cdots \to H_{n+1}(G/N) \to H_{n+1}(K(G,N)) \to H_n(G) \to H_n(G/N) \to \cdots$$

for $n \geq 0$. G. Ellis [4] showed that the Schur multiplier of the pair (G,N) can be considered as the third homology group of the cofiber space $K(G,N)$. We note that the mapping cone $K(G,N)$ of the cofiber $K(G) \to M(f)$ is homotopically equivalent to the space X which is the homotopy pushout corresponding to the pair of groups (G,N). Therefore our topological interpretation of the Schur multiplier of a pair of groups (G,N) is equivalent to the topological definition of G. Ellis.

In this talk, using the topological interpretations, we present among other things some behaviors of the Schur multiplier of pairs of the free product, the amalgamated free product, and the direct product. Also, we show that the Schur multiplier of a pair commutes with direct limits in some cases.

By a triple of groups (G,M,N) we mean a group G with two normal subgroups M and N. A homomorphism of triples $(G_1,M_1,N_1) \to (G_2,M_2,N_2)$ is a group homomorphism $G_1 \to G_2$ that sends M_1 into M_2 and N_1 into N_2. G. Ellis [4] defined the Schur multiplier of a triple (G,M,N) as a functorial abelian group $M(G,M,N)$ whose principle feature is a natural exact sequence

$$\cdots \to H_3(G,N) \to H_3(G/M, MN/M) \to M(G,M,N) \to M(G,N)$$

$$M(G/M, MN/M) \to M \cap N/[M \cap N, G][M,N]$$

$$\to N/[N,G] \to NM/M[N,G] \to 0. \tag{1.5}$$

He also gave a topological interpretation for $M(G,M,N)$. In the rest, first we give a topological definition for the Schur multiplier of a triple which is equivalent to the one of Ellis. Then we show that the Schur multiplier of a triple commutes with direct limits with some conditions. Second, we define a new version of the Schur multiplier of a triple (G,M,N) which is more natural generalization of the Schur multiplier of a pair (G,N) than the one of Ellis. We show that our new notion is coincide with the one of Ellis if $G = MN$. Finally, we present behaviors of this new version of the Schur multiplier of a triple with respect to free, amalgamated free and direct products and also a better behavior with respect to direct limits than the one of Ellis.

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