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Strong approximation of stochastic differential equations with Runge–Kutta methods

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Abstract. In this paper, we describe stochastic Runge–Kutta (SRK) methods with strong order 1.0 for strong solutions of Stratonovich stochastic differential equations (SDEs) which was first introduced by Burrage and Burrage in 1996. In particular, three new SRK methods with strong order 1.0 are constructed. They are an explicit two–stage method, an explicit three–stage method with minimum principal error coefficients and an implicit three–stage method with minimum principal error coefficients. Numerical results for two test problems with our methods and Burrage's method and Platen method will be compared.

Keywords: stochastic differential equations, strong approximation, rooted trees theory, Runge–Kutta methods

1 Introduction

Physical phenomena of interest in science and technology are very often theoretically simulated by means of models which correspond to ordinary differential equations (ODEs). These equations are in general nonlinear and, as such, their solution is usually a different task. Moreover, many times some of the parameters and initial data are not known with complete certainly due to lack of information, uncertainty in the measurements or incomplete knowledge of the the mechanism themselves, and in practice any system undergoes perturbations from the surrounding ambient and, therefore, the behavior of the system itself is, in several circumstances, far away from the simple conditions of the ideal deterministic representation. To compensate this lack of information and to have a more realistic description of the system one introduces stochastic noise in the equation. This results in SDEs. Some areas where SDEs have been used extensively in modelling include chemistry, physics, engineering, mathematical biology and finance. Consider the scalar autonomous Itô SDE given by

$$dy(t) = g_0(y(t))\,dt + g_1(y(t))\,dW(t)\,, \qquad y(t_0) = y_0, \qquad t \in [t_0, T], \tag{1}$$

where g_0 is called the drift coefficient, g_1 is called the diffusion coefficient, and $W(t)$ is a standard Wiener process. In [10] explicit and semi–implicit two–stage SRK methods of strong order 0.5 with minimum principal error coefficients, and explicit, semi–implicit two–stage SRK methods of strong order 1 for the SDE (1) were constructed. Moreover order conditions for coefficients of a class of SRK methods with strong order 1 for the SDE (1) were obtained (see [11]), especially the explicit two–stage and three–stage SRK methods of this class with minimum principal error coefficients were constructed.

The integral formulation of (1) can be written as

$$y(t) = y_0 + \int_{t_0}^{t} g_0(y(s))\,ds + \int_{t_0}^{t} g_1(y(s))\,dW(s), \tag{2}$$

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where the second integral in (2) is Itô stochastic integral, with respect to the Wiener process $W(t)$ (see [7]). For the SDE (1) the associated Stratonovich type given by

$$dy(t) = \bar{g}_0(y(t))\, dt + g_1(y(t))\, odW(t)\,, \qquad y(t_0) = y_0, \tag{3}$$

where

$$\bar{g}_0(y) = g_0(y) - \frac{1}{2}\, g_1'(y)\, g_1(y).$$

In the other words, two different SDE (1) and (3), under different rules of calculus, have the same solution. There are many different methods to solve these kinds of differential equations (see, for example, [1], [6]). In this paper, we will present three new classes of methods for solving (3). Numerical methods for SDEs are recursive methods where trajectories, i.e the sample paths of solution are computed at discrete time step. These schemes are classified according to their type (strong or weak) and order of convergence (see [8]). In some applications the solution is required for a given path that is called path–wise or strong solutions. In [9], we have analyzed an estimate of the absolute error for SDEs by path–wise approximations. The organization of this paper is as follows: In the next section, we describe Runge–Kutta methods for ODEs. In particular, explicit and implicit Runge–Kutta methods for ODEs based on the rooted trees theory are introduced (see [3]). In section three, the rooted tree analysis of strong schemes for SDEs with a scalar Wiener process is introduced (see [2]). In section four the new classes of explicit and implicit SRK methods for SDEs are constructed. Also we show that 1–norm of principal error coefficients of our three–stage SRK methods are less than the 1–norm of principal error coefficients the Platen and the Burrage methods. In continuation, the fixed–point iteration algorithm be used to improve of our implicit method. Some numerical results which show the efficiency of our methods will be presented in the last section.

2 Deterministic Runge–Kutta methods

Consider the scalar autonomous ODE

$$dy(t) = g_0(y(t))dt, \qquad y(t_0) = y_0, \qquad t \in [t_0, T]. \tag{4}$$

A s–stage Runge–Kutta method for calculating a numerical approximation of the solution of the autonomous ODE (4) is given by the recursive formula:

$$\begin{cases} Y_i = y_n + h\sum_{j=1}^{s} a_{ij}\, g_0(Y_j), \qquad i = 1, 2, \ldots, s \\ y_{n+1} = y_n + h\sum_{j=1}^{s} \alpha_j\, g_0(Y_j) \end{cases} \tag{5}$$

which can be represented by the Butcher tableau:

$$\begin{array}{c|c} C & A \\ \hline & \alpha^T \end{array}$$

where $A = (a_{ij})_{s\times s}$, $\alpha^T = [\alpha_1, \alpha_2, \ldots, \alpha_s]$, $C = Ae$, $e = (1, \ldots, 1)^T \in \mathbb{R}^s$. If $a_{ij} = 0$ for all i, j with $i \leq j$, the method is explicit, otherwise we call the method implicit (semi–implicit when $a_{ij} = 0$ for all i, j with $i < j$ and least one $a_{ii} \neq 0$). In 1963, Butcher introduced the theory of rooted trees in order to compare the Taylor series expansion of the exact solution to the Taylor series expansion of the approximation solution over one step assuming exact initial values. Let τ denote the tree with one vertex and let $[t_1, \ldots, t_m]$ be the tree formed by joining the trees of t_i, $i = 1, \ldots, m$, to a common root. For each tree t corresponds to an elementary differential, $F(t)(y)$, is defined recursively by (see [3])

$$F(t)(y) = \begin{cases} g_0(y), & t = \tau, \\ {g_0}^{(m)}(F(t_1)(y), \ldots, F(t_m)(y)), & t = [t_1, \ldots, t_m]. \end{cases}$$

Let T denotes the set of all rooted trees and $\rho(t)$ the number of vertices of t, then for any $t \in T$, $\rho(t) \leq p$, the elementary weight $\Phi_D(t)$ depends only on the method and is defined by $\Phi_D(t) = \alpha^T \Psi(t)$ with Ψ defined recursively by

$$\Psi(t) = \begin{cases} e, & t = \tau, \\ (A\Psi(t_1)) * (A\Psi(t_2)) * \ldots * (A\Psi(t_m)), & t = [t_1, \ldots, t_m]. \end{cases}$$

where $e = (1, \ldots, 1)^T \in \mathbb{R}^s$ and $*$ denotes the component–by–component product (see [5]). The function $\gamma(t)$ is used to represent the density of t and is defined recursively by

$$\gamma(t) = \begin{cases} 1, & t = \tau, \\ \rho(t)\gamma(t_1)\gamma(t_2)\ldots\gamma(t_m), & t = [t_1, \ldots, t_m]. \end{cases}$$

Furthermore, if we define the symmetry of t by $\sigma(\tau) = 1$ and $\sigma(t) = m_1!m_2!\ldots m_k!$, where m_1 of $t_1, t_2, \ldots, t_m$ are identical of one kind, m_2 of $t_1, t_2, \ldots, t_m$ are identical of second kind,..., and m_k are identical of the k–th kind, then we have the Taylor series for the exact solution is given by (see [5])

$$y(t_0 + h) = y_0 + \sum_{t \in T} h^{\rho(t)} \frac{1}{\sigma(t)\gamma(t)} F(t)(y_0), \tag{6}$$

and the corresponding Taylor series numerical solution by (5), is given by (see [3])

$$y_1 = y_0 + \sum_{t \in T} h^{\rho(t)} \frac{\Phi_D(t)}{\sigma(t)} F(t)(y_0), \tag{7}$$

where y_1 is the computed approximation to $y(t_0 + h)$. The accuracy of a Runge–Kutta method can be interpreted in terms of the difference between the exact solution and numerical solution. This is, a Runge–Kutta method is said to be of order p at the integrated point $t_1 = t_0 + h$ if $y(t_0 + h) - y_1 = O(h^{p+1})$. By comparing the two series (6) and (7) term by term we can obtain the order of a Runge–Kutta method.

Theorem 1. *A Runge–Kutta method is of order p if and only if $\Phi_D(t) = \dfrac{1}{\gamma(t)}$, for all t satisfying $\rho(t) \leq p$.*

Proof. See [3].

By Theorem 1, a two–stage explicit Runge–Kutta method with order 2 has a one–parameter family in the following form:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ c_2 & c_2 & 0 \\ \hline & 1 - \dfrac{1}{2c_2} & \dfrac{1}{2c_2} \end{array}$$

which satisfies the conditions $\alpha^T e = 1$ and $\alpha^T C = \dfrac{1}{2}$. One of most well–known this family is the modified Euler method that is given by the tableau:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \tag{8}$$

For a three–stage explicit Runge–Kutta method of order 3, we must satisfy the following conditions:

$$\alpha^T e = 1, \quad \alpha^T C = \frac{1}{2}, \quad \alpha^T C^2 = \frac{1}{3}, \quad \alpha^T AC = \frac{1}{6}$$

Note that the multiplication of vectors are componentwise. These equations yield to one two–parameter family and two one–parameter families of solutions (see [4], for further details). One example is given in the following tableau (see [4]):

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \hline & 0 & \frac{3}{4} & \frac{1}{4} \end{array} \tag{9}$$

Also for a three–stage implicit Runge–Kutta method of order 4, the order conditions are

$$\begin{cases} \alpha^T e = 1, \quad \alpha^T C = \frac{1}{2}, \quad \alpha^T C^2 = \frac{1}{3}, \quad \alpha^T AC = \frac{1}{6}, \quad \alpha^T C^3 = \frac{1}{4} \\ \alpha^T CAC = \frac{1}{8}, \quad \alpha^T AC^2 = \frac{1}{12}, \quad \alpha^T A^2 C = \frac{1}{24}. \end{cases} \tag{10}$$

One family of methods that satisfying conditions (10) is Lobatto III method. This class of methods in general have order $2s-2$ such that $c_1, c_2, \ldots, c_s$ are the distinct roots of $(\frac{d}{dx})^{s-2} x^{s-1}(x-1)^{s-1} = 0$, with $c_1 = 0$ and $c_s = 1$ (see [4]). One well–known example is given in the following tableau (see [4]):

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 1 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \tag{11}$$

3 Stochastic Runge–Kutta methods

Consider the scalar autonomous Stratonovich SDE given by

$$dy(t) = g_0(y(t))\, dt + g_1(y(t))\, odW(t)\,, \quad y(t_0) = y_0, \quad t \in [t_0, T]. \tag{12}$$

General form of s–stage stochastic Runge–Kutta(SRK) methods for solving SDE (12) given by

$$\begin{cases} Y_i & = y_n + \sum_{j=1}^{s} Z_{ij}^{(0)}\, g_0(Y_j) + \sum_{j=1}^{s} Z_{ij}^{(1)}\, g_1(Y_j), \quad i = 1, 2, \ldots, s \\ y_{n+1} & = y_n + \sum_{j=1}^{s} z_j^{(0)}\, g_0(Y_j) + \sum_{j=1}^{s} z_j^{(1)}\, g_1(Y_j), \end{cases} \tag{13}$$

which can be represented in tableau form :

$$\begin{array}{c|c|c} & Z^{(0)} & Z^{(1)} \\ \hline & {z^{(0)}}^T & {z^{(1)}}^T \end{array},$$

where $Z^{(k)} = (Z_{ij}^{(k)})_{s\times s}$ and ${z^{(k)}}^T = (z_1^{(k)}, \ldots, z_s^{(k)})$ for $k = 0, 1$. This class of methods was introduced by Burrage and Burrage (see [1]). In fact, they have extended rooted trees theory to the stochastic setting. Let τ_k $(k = 0, 1)$ be the tree with one vertex with colour k. Then other trees can be built up recursively by defining a new tree t which is formed by joining trees $t_1, \ldots, t_m$ to a new root τ_k as $t = [t_1, \ldots, t_m]_k$. Each tree t corresponds to an elementary differential $F(t)(y)$, defined by (see [2])

$$F(t)(y) = \begin{cases} g_k(y), & t = \tau_k, \\ {g_k}^{(m)}(F(t_1)(y), \ldots, F(t_m)(y)), & t = [t_1, \ldots, t_m]_k. \end{cases}$$

Let T denote the set of all rooted trees with colour k $(k = 0, 1)$ then for any $t \in T$ the J–integral associated with tree t defined recursively by (see [2])

$$J(t)(h) = \begin{cases} W_k(h), & t = \tau_k, \\ \int_0^h \prod_{j=1}^{m} J(t_j)(s)\, odW_k(s), & t = [t_1, \ldots, t_m]_k, \end{cases}$$

where $W_0(s) = s$. The Stratonovich Taylor expansion for the exact solution of the SDE (12) is (see [2])

$$y(t) = \sum_{t \in T} \frac{\gamma(t)}{\rho(t)!} J(t)\alpha(t)F(t)(y_0), \tag{14}$$

where $\alpha(t)$ represents the number of possible different monotonic labellings associated with tree t. Let $a(t) = z^{(k)}\Phi(t)$ where Φ defined recursively by

$$\Phi(t) = \begin{cases} e, & t = \tau_k, \\ (Z^{(k)}\Phi(t_1)) * (Z^{(k)}\Phi(t_2)) * \ldots * (Z^{(k)}\Phi(t_m)), & t = [t_1, \ldots, t_m]_k, \end{cases}$$

where $*$ denote the component–by component product, then the Stratonovich Taylor series expansion of the numerical method by (13), given by (see [2])

$$Y(t) = \sum_{t \in T} \frac{\gamma(t)}{\rho(t)!} a(t)\alpha(t)F(t)(y_0). \tag{15}$$

From (14) and (15) local truncation error over one step with an exact initial value can be written

$$L_1 \equiv y(t) - Y(t) = \sum_{t \in T^\star} e(t)\, F(t)(y_0),$$

where for tree t term $e(t) = \dfrac{\gamma(t)}{\rho(t)!}(J(t) - a(t))\alpha(t)$ is the coefficient of local truncation error. In [1], all coefficient of local truncation error for all two–coloured trees up to three nodes are listed.

4 SRK methods with strong order 1.0

For solving the Stratonovich SDE (12), a class of SRK methods given by (13) can also be characterized by

$$Z^{(0)} = hA, \quad {z^{(0)}}^T = h\alpha^T, \quad Z^{(1)} = J_1 B, \quad {z^{(1)}}^T = J_1\gamma^T, \tag{16}$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are $s \times s$ real matrices and $\alpha^T = (\alpha_1, \ldots, \alpha_s)$ and $\gamma^T = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ are row s–dimensional vectors. If the matrices A and B are strictly lower triangular, then the method (16) is said to be explicit, otherwise it is implicit. The convergence Theorem in [2] shows that the SRK method of the form (16) will converge to the exact solution of SDE (12) with strong order 1.0 if the local truncation error satisfies

$$(E(y(t) - Y(t))^2)^{\frac{1}{2}} = O(h^{1.5}), \quad E(y(t) - Y(t)) = O(h^2). \tag{17}$$

Hence a SRK method of the form (16) will satisfy the mean–square condition in (17) if

$$E(h - h\alpha^T e)^2 = 0, \quad E(J_1 - J_1\gamma^T e)^2 = 0, \quad E(J_{11} - {J_1}^2\gamma^T Be)^2 = 0 \tag{18}$$

Note that conditions (18) arise from trees τ_0, τ_1 and $[\tau_1]_1$ (see [1]). These conditions are equivalent to

$$\alpha^T e = 1, \quad \gamma^T e = 1, \quad \gamma^T Be = \frac{1}{2} \tag{19}$$

On the other hand, the method will satisfy mean condition in (17) if:

$$\begin{cases} E(J_{10} - J_1 h\alpha^T Be) = 0 \\ E(J_{01} - J_1 h\gamma^T Ae) = 0 \\ E(J_{111} - {J_1}^3\gamma^T B(Be)) = 0 \\ E(J_{111} - \frac{1}{2}{J_1}^3\gamma^T (Be)^2) = 0. \end{cases} \tag{20}$$

Note that conditions (20) arise from trees $[\tau_1]_0$, $[\tau_0]_1$, $[[\tau_1]_1]_1$ and $[\tau_1, \tau_1]_1$ (see [1] for further details). It can be shown that the mean conditions (20) are all satisfied and hence a SRK method of the form given in (16) will be of strong order 1.0 if the order conditions (19) are satisfied. In order to construct a class of explicit SRK methods of the form (16) with $s = 2$, we try to find a method with coefficients:

$$\begin{array}{c|cc|cc} & 0 & 0 & 0 & 0 \\ & ha_{21} & 0 & J_1 b_{21} & 0 \\ \hline & h\alpha_1 & h\alpha_2 & J_1\gamma_1 & J_1\gamma_2 \end{array} \tag{21}$$

which satisfies with the conditions in (19). We have six unknowns and there are three equations to be satisfied. We choose the deterministic part coefficients of (21) the Runge–Kutta method given in (8). This ensure that our method works well in the case of small stochastic influence. From (19) it is seen that we can assume $B = A$ and $\gamma = \alpha$, consequently we have the following method that is called 'EM1' and is presented by the tableau:

$$\begin{array}{c|cc|cc} & 0 & 0 & 0 & 0 \\ & \frac{1}{2}h & 0 & \frac{1}{2}J_1 & 0 \\ \hline & 0 & h & 0 & J_1 \end{array}$$

As there are six parameters to be determined, additional conditions can be considered, for example, the conditions for minimum principal error coefficients, namely the terms corresponding to $h^{1.5}$ have minimum coefficients. The mean square of the principal error coefficients are give by

$$\begin{cases} (\frac{1}{3} - \alpha^T Be + (\alpha^T Be)^2)h^3, \\ (\frac{1}{3} - \gamma^T Ae + (\gamma^T Ae)^2)h^3, \\ (\frac{1}{36} - \frac{1}{3}\gamma^T B(Be) + (\gamma^T B(Be))^2)15h^3, \\ (\frac{1}{9} - \frac{2}{3}\gamma^T (Be)^2 + (\gamma^T (Be)^2)^2)\frac{15}{4}h^3, \end{cases}$$

which can be derived from (20). These principal error coefficients are minimized if (see [1])

$$\alpha^T Be = \frac{1}{2}, \quad , \gamma^T Ae = \frac{1}{2}, \quad \gamma^T B(Be) = \frac{1}{6}, \quad \gamma^T (Be)^2 = \frac{1}{3}, \tag{22}$$

and so principal error coefficients respectively are given by

$$\frac{1}{12}h^3, \ \frac{1}{12}h^3, \ 0, \ 0.$$

Note that the 'EM1' method has principal error coefficients

$$\frac{1}{12}h^3,\ \frac{1}{12}h^3,\ \frac{5}{12}h^3,\ \frac{5}{192}h^3$$

and the Burrage method (see [1]) is presented by the tableau:

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3}h & 0 & \frac{2}{3}J_1 & 0 \\ \hline \frac{1}{4}h & \frac{3}{4}h & \frac{1}{4}J_1 & \frac{3}{4}J_1 \end{array}$$

which has minimum principal error coefficients

$$\frac{1}{12}h^3,\ \frac{1}{12}h^3,\ \frac{5}{12}h^3, 0.$$

Also the Platen method (see [1]) is

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 \\ h & 0 & J_1 & 0 \\ \hline h & 0 & \frac{1}{2}J_1 & \frac{1}{2}J_1 \end{array}$$

and has principal error coefficients

$$\frac{1}{3}h^3,\ \frac{1}{3}h^3,\ \frac{5}{12}h^3, \frac{1}{36}h^3.$$

With the restriction of $s = 2$, it was seen that $\gamma^T B(Be) = 0$, but if $s = 3$ then $\gamma^T B(Be)$ is not zero and in order to have the minimum principal error it must take its minimum value which is $\frac{1}{6}$. In order to construct a class of explicit SRK methods of the form (16) with $s = 3$, we consider the matrices A and B and the row vectors α^T and γ^T with the following forms:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix},\quad B = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & b_{32} & 0 \end{pmatrix},\quad \alpha^T = \begin{pmatrix} \alpha_1 , & \alpha_2 , & \alpha_3 \end{pmatrix},\quad \gamma^T = \begin{pmatrix} \gamma_1 , & \gamma_2 , & \gamma_3 \end{pmatrix}.$$

Hence by equations (19) and (22), we have the following system of seven equations with twelve unknowns:

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \gamma_1 + \gamma_2 + \gamma_3 = 1 \\ \gamma_2 b_{21} + \gamma_3(b_{31} + b_{32}) = \frac{1}{2} \\ \alpha_2 b_{21} + \alpha_3(b_{31} + b_{32}) = \frac{1}{2} \\ \gamma_2 a_{21} + \gamma_3(a_{31} + a_{32}) = \frac{1}{2} \\ \gamma_3 b_{32} b_{21} = \frac{1}{6} \\ \gamma_2 {b_{21}}^2 + \gamma_3(b_{31} + b_{32})^2 = \frac{1}{3}. \end{cases} \tag{23}$$

In order to reduce the free parameters, we choose the deterministic components of SRK method in (13) the Runge–Kutta method given in (9). From equations (23) it is seen that we can assume $B = A$ and $\gamma = \alpha$, consequently we have the following three–stage explicit SRK method with minimum principal error coefficients, that is presented by the tableau:

$$\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3}h & 0 & 0 & \frac{2}{3}J_1 & 0 & 0 \\ -h & h & 0 & -J_1 & J_1 & 0 \\ \hline & 0 & \frac{3}{4}h & \frac{1}{4}h & 0 & \frac{3}{4}J_1 & \frac{1}{4}J_1 \end{array}$$

where is called 'EM2', and has principal error coefficients:

$$\frac{1}{12}h^3,\ \frac{1}{12}h^3,\ 0,\ 0.$$

Now we will generalize the above explicit SRK method to implicit method for SDEs. In the implicit case with $s = 3$, we will use the matrices:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix},\quad B = \begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

which using conditions (19) and (22). We will have the following system of seven equations with eighteen unknowns:

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \gamma_1 + \gamma_2 + \gamma_3 = 1 \\ \gamma_1 b_{11} + \gamma_2(b_{21} + b_{22}) + \gamma_3(b_{31} + b_{32} + b_{33}) = \frac{1}{2} \\ \alpha_1 b_{11} + \alpha_2(b_{21} + b_{22}) + \alpha_3(b_{31} + b_{32} + b_{33}) = \frac{1}{2} \\ \gamma_1 a_{11} + \gamma_2(a_{21} + a_{22}) + \gamma_3(a_{31} + a_{32} + a_{33}) = \frac{1}{2} \\ \gamma_1 {b_{11}}^2 + \gamma_2(b_{21}b_{11} + b_{22}(b_{21} + b_{22})) + \gamma_3(b_{31}b_{11} + b_{32}(b_{21} + b_{22}) + b_{33}(b_{31} + b_{32} + b_{33})) = \frac{1}{6} \\ \gamma_1 b_{11}^2 + \gamma_2(b_{21} + b_{22})^2 + \gamma_3(b_{31} + b_{32} + b_{33})^2 = \frac{1}{3}. \end{cases}$$

Again we choose the deterministic part coefficients of (13) the Runge–Kutta method given in (11). Consequently with assuming $B = A$ and $\gamma = \alpha$ a family of three–stage implicit SRK methods with minimum principal error coefficients can be presented by the tableau:

$$\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}h & \frac{1}{4}h & 0 & \frac{1}{4}J_1 & \frac{1}{4}J_1 & 0 \\ 0 & h & 0 & 0 & J_1 & 0 \\ \hline & \frac{1}{6}h & \frac{2}{3}h & \frac{1}{6}h & \frac{1}{6}J_1 & \frac{2}{3}J_1 & \frac{1}{6}J_1 \end{array}$$

where is referred to 'IM', and has principal error coefficients:

$$\frac{1}{12}h^3,\ \frac{1}{12}h^3,\ 0,\ 0.$$

If we use the 1–norm to estimate the contribution of all error terms to the principal error term then Tab. 1 presents these values for methods '$Platen$' , '$EM1$' , '$Burrage$' , '$EM2$' and 'IM'. We observe, in Tab. 1,

Table 1. Norm of Principal Error Coefficients

	$Platen$	$EM1$	$Burrage$	$EM2$	IM
$\|principal\ \ error\|_1$	1.1111	0.6094	0.5833	0.1667	0.1667

that the 1–norm principal error coefficients '$EM2$' and 'IM' methods are less than the 1–norm of principal error coefficients '$Platen$', '$EM1$' and '$Burrage$' methods. Also the difference between the 1–norm of principal error coefficients the '$EM1$' and '$Burrage$' methods is very small, and less than the '$Platen$' method. In order to improve the results of employing the 'IM' method at each step, the stage–variable Y_2 will be solved by the fixed–point iteration scheme with starting value for this variable comes from the '$EM2$' method. Since $J_1 \sim N(0,h)$, so $J_1 = \sqrt{h}R_n$ where $R_n \sim N(0,1)$. Hence for the stage–variable Y_2 in the 'IM' method let

$$G(Y_2) \equiv y_n + \frac{1}{4}h(g_0(y_n) + g_0(Y_2)) + \frac{1}{4}\sqrt{h}R_n(g_1(y_n) + g_1(Y_2)),$$

and therefore the fixed–point iteration for solving Y_2 is given by

$$Y_2{}^{[s+1]} = G(Y_2{}^{[s]}), \qquad s = 0, 1, 2, \ldots$$

with stopping criteria

$$|Y_2^{[s+1]} - Y_2^{[s]}| < \epsilon, \tag{24}$$

where ϵ is a positive known tolerance number. If h is sufficiently small, the contraction mapping Theorem can be applied to prove existence and uniqueness of the solution of nonlinear equation $G(Y_2) = Y_2$ (see [12]). Consequently the stage–variable Y_3 is given by

$$Y_3 = y_n + hg_0(Y_2{}^{[s+1]}) + \sqrt{h}R_n g_1(Y_2{}^{[s+1]}),$$

such that $Y_2^{[s+1]}$ satisfy condition (24). Finally y_{n+1} for the 'IM' method will be evaluated by

$$y_{n+1} = y_n + h(\frac{1}{6}g_0(y_n) + \frac{2}{3}g_0(Y_2{}^{[s+1]}) + \frac{1}{6}g_0(Y_3)) + \sqrt{h}R_n(\frac{1}{6}g_1(y_n) + \frac{2}{3}g_1(Y_2{}^{[s+1]}) + \frac{1}{6}g_1(Y_3)),$$

where $Y_2^{[s+1]}$ satisfies condition (24).

5 Numerical results and conclusion

In this section, numerical results from the implementation of five methods are presented. These methods are '$Platen$' , '$EM1$' , '$Burrage$' , '$EM2$' and 'IM'. The above methods will be implemented in fixed step size mode on two different problems. In order to simulate the Gaussian variable J_1 with distribution $N(0,h)$ we have taken pseudo–random numbers generated by "randn" in $MATLAB$ 7.0. When these methods are simulated, the same sequence of random numbers for the Wiener increment J_1 are used for the step size under consideration. For each problem, it is necessary to simulate many trajectories of the Wiener process and we take, 1000, where K stands for the number of different realizations of the Wiener process. The implementation determines the average error for each step size at the end point of the interval $[0,T]$ is defined by

$$AE = \frac{1}{K}\sum_{i=1}^{K} \mid y_N^{(i)} - y^{(i)}(t_N) \mid,$$

where $y_N^{(i)}$ is the approximation solution and $y^{(i)}(t_N)$ is the exact solution of SDE at $t_N = T$ on the i–th path of the Wiener process. The results appear in Tab. 2 $\sim$ 5. In all tables the column 6 determines the average error for 'IM' method, while at each step starting value for the stage–variable Y_2 come from the '$EM2$' method with $\epsilon = 0.0001$.

Test Problem 1. Consider

$$\begin{cases} dy = -a^2y(1-y^2)dt + a(1-y^2)dW(t) = a(1-y^2)odW(t), & t \in [0,1] \\ y(0) = 0. \end{cases}$$

The exact solution of this equation is (see [6])

$$y(t) = \tan h(aW(t) + \arctan h(y_0)).$$

This problem is purely stochastic and was solved numerically twice, with two choices of parameters $a = 2$ and $a = 1$. **Test Problem 2.** Consider

Table 2. Global errors for test Problem 1, with $a = 2,\ K = 1000$ and $\epsilon = 0.0001$.

h	$Platen$	$Burrage$	$EM1$	$EM2$	IM
$\frac{1}{25}$	0.23379	0.17543	0.14732	0.90772e–1	0.49411e–2
$\frac{1}{50}$	0.11910	0.87367e–1	0.72644e–1	0.27840e–1	0.10981e–2
$\frac{1}{100}$	0.61476e–1	0.45824e–1	0.38650e–1	0.10742e–1	0.34410e–3
$\frac{1}{200}$	0.33242e–1	0.25185e–1	0.21597e–1	0.46055e–2	0.11119e–3
$\frac{1}{400}$	0.15803e–1	0.11847e–1	0.10035e–1	0.19898e–2	0.39186e–4

$$\begin{cases} dy = -(\alpha + \beta^2 y)(1 - y^2)dt + \beta(1 - y^2)dW(t) = -\alpha(1 - y^2)dt + \beta(1 - y^2)odW(t), & t \in [0, 1] \\ y(0) = 0. \end{cases}$$

For this problem, the solution is known to be (see [6])

$$y(t) = \frac{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + 1 - y_0}.$$

This problem is solved with two different values of parameters where $\alpha = 1$ and $\beta = 2,\ 0.01$. In the first case, one has that the stochastic part is significant, whereas it is much smaller in the second case. For this problem,

Table 3. Global errors for test Problem 1, with $a = 1,\ K = 1000$ and $\epsilon = 0.0001$.

h	$Platen$	$Burrage$	$EM1$	$EM2$	IM
$\frac{1}{25}$	0.24542e–1	0.13005e–1	0.95001e–2	0.55887e–2	0.11868e–3
$\frac{1}{50}$	0.11982e–1	0.64091e–2	0.48965e–2	0.21210e–2	0.31872e–4
$\frac{1}{100}$	0.63255e–2	0.34932e–2	0.26472e–2	0.96237e–3	0.92418e–5
$\frac{1}{200}$	0.32726e–2	0.17777e–2	0.13249e–2	0.45166e–3	0.33780e–5
$\frac{1}{400}$	0.15283e–2	0.81432e–3	0.60395e–3	0.22187e–3	0.32333e–5

the solution is known to be (see [6])

$$y(t) = \frac{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + 1 - y_0}.$$

This problem be solved with two different values of parameters where $\alpha = 1$ and $\beta = 2,\ 0.01$. In the first case, one has that the stochastic part is significant, whereas it is much smaller in the second case. With comparing

Table 4. Global errors for test Problem 2, with $\alpha = 1,\ \beta = 2,\ K = 1000$ and $\epsilon = 0.0001$.

h	$Platen$	$Burrage$	$EM1$	$EM2$	IM
$\frac{1}{25}$	0.19910	0.15513	0.13032	0.75505e–1	0.40515e–2
$\frac{1}{50}$	0.10342	0.78391e–1	0.65485e–1	0.24255e–1	0.10537e–2
$\frac{1}{100}$	0.54342e–1	0.42437e–1	0.36003e–1	0.94268e–2	0.31865e–3
$\frac{1}{200}$	0.29818e–1	0.23167e–1	0.19813e–1	0.41433e–2	0.98992e–4
$\frac{1}{400}$	0.13516e–1	0.10575e–1	0.90643e–2	0.17987e–2	0.40232e–4

Table 5. Global errors for test Problem 2, with $\alpha = 1,\ \beta = 0.01,\ K = 1000$ and $\epsilon = 0.0001$.

h	$Platen$	$Burrage$	$EM1$	$EM2$	IM
$\frac{1}{25}$	0.73817e–2	0.11148e–3	0.66508e–4	0.23513e–5	0.23346e–6
$\frac{1}{50}$	0.36668e–2	0.27635e–4	0.16562e–4	0.28906e–6	0.28717e–7
$\frac{1}{100}$	0.18273e–2	0.69352e–5	0.41577e–5	0.36640e–7	0.36196e–8
$\frac{1}{200}$	0.91215e–3	0.17761e–5	0.10659e–5	0.48004e–8	0.47420e–9
$\frac{1}{400}$	0.45571e–3	0.46845e–6	0.28133e–6	0.66410e–9	0.48845e–10

the results in Tab. 2 $\sim$ 5, we conclude that the 'IM' and '$EM2$' methods are more accurate than the '$Platen$' , '$Burrage$' and '$EM1$' methods, as the error values for 'IM' method is less than '$EM2$' method. Moreover for two–stage SRK methods the '$EM1$' method is more effective '$Platen$' and '$Burrage$' methods. On the other hand for problems in which the deterministic term dominate (test problem 2 with $\beta = 0.01$) the improvement the '$Burrage$' , '$EM1$' , '$EM2$' and 'IM' methods becomes noticeable as the stepsize is reduced. This is because the deterministic component of the '$Burrage$' and '$EM1$' methods are the second order Runge–Kutta methods, while the deterministic component '$EM2$' and 'IM' methods are the third and fourth order Runge–Kutta methods, respectively.

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