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Strong Runge–Kutta Methods With order one for Numerical Solution of Itô Stochastic Differential Equations

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In this paper, order conditions for coefficients of a class of stochastic Runge–Kutta (SRK) methods with strong global order 1, which applied for solving Itô stochastic differential equations (SDEs) with a single noise process, are presented. In particular, explicit two-stage and three-stage SRK methods of this class with minimum principal error constants are constructed. Numerical results with two test problems of our methods, the Itô method and Milstein method will be compared.

1 Introduction

The mathematical modeling of many real-life phenomena by reason of random noisy perturbation are not possible by ordinary differential equations (ODEs), and hence are often modeled by using SDEs in order for the model to become more realistic (see, for example, [8], [3]). Because such differential equations cannot usually be solved analytically, the study of numerical methods is required and these must be designed to perform with a certain order of accuracy. Consider the autonomous Itô SDE given by

$$dy(t) = g_0(y(t))\,dt + g_1(y(t))\,dW(t)\,, \qquad y(t_0) = y_0, \qquad t \in [t_0, T], \tag{1.1}$$

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where g_0 and g_1 are real–valued functions and $W(t)$ is a one-dimensional standard Wiener process and the solution $y(t)$ is an Itô process. The integral formulation of (1.1) can be written as

$$y(t) = y_0 + \int_{t_0}^{t} g_0(y(s))\, ds + \int_{t_0}^{t} g_1(y(s))\, dW(s)\,,$$

where the second term is a stochastic integral with respect to the Wiener process $W(t)$, which can be interpreted in many ways. The two most studied interpretations are by Itô and Stratonovich (see [3]), which depend both on modeling considerations and the choice of calculus. It is always possible to convert an Itô SDE to the Stratonovich from or vice versa by means of the following formula:

$$\bar{g}_0(y) = g_0(y) - \frac{1}{2}\, g_1'(y)\, g_1(y),$$

where the equation (1.1) is in the Stratonovich form when $\bar{g}_0$ is used in place of g_0. There are different numerical methods to solve these types of differential equations (see, for example, [1], [4]). In this paper we will present four new classes of methods for solving the SDE problem (1.1). For any approximation method to be derived, it is important that trajectories, that is, the sample paths, of the approximation be close to those of the Itô process and this leads to the concept of a strong solution of an SDE. This is the case when the solution is required for a given path, which is called pathwise solution. In [7], we have analyzed an estimate of the absolute error for SDEs by pathwise approximations. In order to evaluate the efficiency of a method for computing strong solutions on average, the order of strong convergence of a method is used as discussed in [5]. The outline of this paper is as follows: In the next section, Runge–Kutta methods for SDEs that have been proposed by Burrage and Burrage (see [1]) are introduced. In section 3, order conditions for a class of SRK methods with order 1 are stated. In particular, the new classes of explicit two-stage and three-stage Runge–Kutta methods for SDEs is constructed. Also we show that the 1-norm of principal error coefficients of our three-stage SRK methods are less than the 1-norm of principal error coefficients of two-stage SRK methods and also of the Itô method. Moreover, it is shown that the 1–norm of principal error coefficients of our two-stage methods are less than the 1–norm of principal error coefficients of the Itô method. Finally, some numerical results that show the efficiency of our methods will be presented in the last section.

2 Runge–Kutta methods for SDEs

In analogy to the deterministic setting, a family of numerical methods for solving SDEs can be obtained from a stochastic Taylor expansion. There are several possibilities to obtain a stochastic Taylor expansion; the most notable is the Itô–Taylor expansion, which is based on the iterated application of the Ito formula. By truncating different terms of this expansion, the different numerical methods can be constructed, but this technique involves considerable complexities in implementation because of the approximation of higher-order stochastic integrals and the evaluation of higher-order derivatives of the functions defining the SDE. The simplest of these methods is the Euler–Maruyama method, which is derived by truncating the Itô–Taylor expansion after one deterministic and one stochastic term. More accurate methods (which require derivative evaluations) can be obtained by using truncated forms of the stochastic Taylor series expansion but at the cost of derivative evaluations. The most famous of these methods is the Milstein method, which is derived by truncating the Itô–Taylor expansion after one deterministic and two stochastic terms. This method for the SDE problem (1.1) is given by

$$y_{n+1} = y_n + h\, g_0(y_n) + J_1\, g_1(y_n) + \frac{1}{2}({J_1}^2 - h)\, g_1'(y_n)\, g_1(y_n),$$

with initial value y_0, such that $J_1 = W(\tau_{n+1}) - W(\tau_n)$, for equidistant discretization times $\tau_n = t_0 + nh$ with $h = \frac{T-t_0}{N}$ for some integer N to be large enough so that $h \in (0,1)$. To avoid this computational cost, a great deal of attention has been paid to developing derivative-free numerical methods and this leads to SRK methods. In [1] the general family of s–stage SRK methods for the SDE problem (1.1) is given by

$$\begin{cases} Y_i & = y_n + \sum_{j=1}^{s} Z_{ij}^{(0)}\, g_0(Y_j) + \sum_{j=1}^{s} Z_{ij}^{(1)}\, g_1(Y_j), \qquad i = 1,2,\ldots,s \\ y_{n+1} & = y_n + \sum_{j=1}^{s} z_j^{(0)}\, g_0(Y_j) + \sum_{j=1}^{s} z_j^{(1)}\, g_1(Y_j), \end{cases} \tag{2.1}$$

which can be represented in tableau form as

$$\begin{array}{c|c|c} & Z^{(0)} & Z^{(1)} \\ \hline & {z^{(0)}}^T & {z^{(1)}}^T \end{array},$$

where $Z^{(k)} = (Z_{ij}^{(k)})$ for $i,j = 1,2,\ldots,s$ and ${z^{(k)}}^T = (z_1^{(k)},\ldots,z_s^{(k)})$ for $k = 0,1$. Since (2.1) is a generalization of the Runge–Kutta methods in the deterministic case, for consistency the stepsize will be included in the parameter matrix associated with the deterministic

components; so $Z^{(0)} = hA$ and ${z^{(0)}}^T = h\alpha^T$, while $Z^{(1)}$ and ${z^{(1)}}^T$ have elements that are arbitrary random variables. In order to derive SRK methods with strong global order 1, the Itô–Taylor series expansion of the exact solution and the Itô–Taylor series expansion of SRK method (2.1) are necessary. By these two expansions, the local truncation error over one step with an exact initial value can be written as (see [6] for further details)

$$L_1 \equiv y(t) - Y(t) = \sum_{t \in T^\star} e(t)\, F(t) y_0,$$

where $y(t)$ and $Y(t)$ denote the Itô–Taylor series expansion (1.1) and the Taylor series expansion of the SRK method in the Itô case, respectively. Here $F(t)(y_0)$ is the elementary differential associated with the rooted two-colored tree t, $e(t)$ is the coefficient of local truncation error for tree t and $T^\star$ is the set of all rooted two-colored trees. Assuming certain conditions on the coefficients of method and satisfying Lipschitz condition for the drift and diffusion coefficients SDE, a method will have strong global convergence of order p if it has strong local order p and mean local order p (see [2] for further details).

3 Order conditions for a class of SRK methods with strong order 1.

In [6], the sufficient conditions for SRK methods with strong global order 1 for the SDE problem (1.1) are stated. In fact an SRK method will have strong global order 1 if

$${z^{(0)}}^T e = h, \qquad {z^{(1)}}^T e = J_1, \qquad {z^{(1)}}^T Z^{(1)} e = I_{11}, \tag{3.1}$$

$$E[{z^{(0)}}^T Z^{(1)} e] = E[{z^{(1)}}^T Z^{(0)} e] = E[{z^{(1)}}^T (Z^{(1)} e)^2] = E[{z^{(1)}}^T {Z^{(1)}}^2 e] = 0, \tag{3.2}$$

where $e = (1, \ldots, 1)^T \in \mathbb{R}^s$ and multiplication of vectors are considered componentwise. Note that the conditions (3.1) and (3.2) arise from strong local order 1 and mean local order 1, respectively. We will now study the order conditions in a class of methods given by (2.1) in which

$$Z^{(0)} = hA, \qquad {z^{(0)}}^T = h\alpha^T, \qquad Z^{(1)} = \sqrt{h} B^{(1)} + J_1 B^{(2)}, \qquad {z^{(1)}}^T = \sqrt{h} {\gamma^{(1)}}^T + J_1 {\gamma^{(2)}}^T, \tag{3.3}$$

where $A = (a_{ij})$ and $B^{(1)} = (b_{ij}^{(1)})$ and $B^{(2)} = (b_{ij}^{(2)})$ are $s \times s$ real matrices while $\alpha^T = (\alpha_1, \ldots, \alpha_s)$ and ${\gamma^{(1)}}^T = (\gamma_1^{(1)}, \gamma_2^{(1)}, \ldots, \gamma_s^{(1)})$ and ${\gamma^{(2)}}^T = (\gamma_1^{(2)}, \gamma_2^{(2)}, \ldots, \gamma_s^{(2)})$ are row s–dimensional vectors. If the matrices A, $B^{(1)}$ and $B^{(2)}$ are strictly lower triangular, then the method (3.3) is said to be explicit, but otherwise it is implicit. In order to construct a

method of the form (3.3) with strong global order 1, the conditions (3.1) and (3.2) must be considered. Since

$$E[J_1] = E[{J_1}^3] = 0, \qquad E[{J_1}^2] = h,$$

the left-hand side of each of these conditions is given by

$$\begin{cases} {z^{(0)}}^T e = (\alpha^T e)h \\ {z^{(1)}}^T e = ({\gamma^{(1)}}^T e)\sqrt{h} + ({\gamma^{(2)}}^T e)J_1 \\ {z^{(1)}}^T Z^{(1)} e = ({\gamma^{(1)}}^T B^{(1)} e)h + ({\gamma^{(1)}}^T B^{(2)} e + {\gamma^{(2)}}^T B^{(1)} e)\sqrt{h}J_1 + ({\gamma^{(2)}}^T B^{(2)} e)J_1^2 \\ E[{z^{(0)}}^T Z^{(1)} e] = (\alpha^T B^{(1)} e)h\sqrt{h} \\ E[{z^{(1)}}^T Z^{(0)} e] = ({\gamma^{(1)}}^T A e)h\sqrt{h} \\ E[{z^{(1)}}^T (Z^{(1)} e)^2] = ({\gamma^{(1)}}^T (B^{(1)} e)^2 + {\gamma^{(1)}}^T (B^{(2)} e)^2 + 2{\gamma^{(2)}}^T (B^{(1)} e)(B^{(2)} e))h\sqrt{h} \\ E[{z^{(1)}}^T {Z^{(1)}}^2 e] = ({\gamma^{(1)}}^T {B^{(1)}}^2 e + {\gamma^{(1)}}^T {B^{(2)}}^2 e + {\gamma^{(2)}}^T (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e))h\sqrt{h}. \end{cases} \tag{3.4}$$

Now from (3.1) and (3.2) and using (3.4) and the relation

$$I_{11} = \frac{1}{2}({J_1}^2 - h),$$

the sufficient conditions for a method of the form given in (3.3) to have strong global order 1 are given by

$$\begin{cases} \alpha^T e = 1 \\ {\gamma^{(1)}}^T e = 0 \\ {\gamma^{(2)}}^T e = 1 \\ {\gamma^{(1)}}^T B^{(1)} e = -\frac{1}{2} \\ {\gamma^{(1)}}^T B^{(2)} e + {\gamma^{(2)}}^T B^{(1)} e = 0 \\ {\gamma^{(2)}}^T B^{(2)} e = \frac{1}{2} \\ \alpha^T B^{(1)} e = 0 \\ {\gamma^{(1)}}^T A e = 0 \\ {\gamma^{(1)}}^T (B^{(1)} e)^2 + {\gamma^{(1)}}^T (B^{(2)} e)^2 + 2{\gamma^{(2)}}^T (B^{(1)} e)(B^{(2)} e) = 0 \\ {\gamma^{(1)}}^T {B^{(1)}}^2 e + {\gamma^{(1)}}^T {B^{(2)}}^2 e + {\gamma^{(2)}}^T (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e) = 0. \end{cases} \tag{3.5}$$

In order to construct a class of explicit SRK methods of the form (3.3) with $s = 2$, we consider the matrices A, $B^{(1)}$ and $B^{(2)}$ with the following forms:

$$A = \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} 0 & 0 \\ b_{21}^{(1)} & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & 0 \\ b_{21}^{(2)} & 0 \end{pmatrix},$$

and the row vectors α^T, ${\gamma^{(1)}}^T$ and ${\gamma^{(2)}}^T$ are defined as follows:

$$\alpha^T = \begin{pmatrix} \alpha_1 \ , \ \alpha_2 \end{pmatrix}, \quad {\gamma^{(1)}}^T = \begin{pmatrix} \gamma_1^{(1)} \ , \ \gamma_2^{(1)} \end{pmatrix}, \quad {\gamma^{(2)}}^T = \begin{pmatrix} \gamma_1^{(2)} \ , \ \gamma_2^{(2)} \end{pmatrix}.$$

Thus by the system (3.5), we have the following system of nine equations with nine unknowns:

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \gamma_1^{(1)} + \gamma_2^{(1)} = 0 \\ \gamma_1^{(2)} + \gamma_2^{(2)} = 1 \\ \gamma_2^{(1)} b_{21}^{(1)} = -\frac{1}{2} \\ \gamma_2^{(1)} b_{21}^{(2)} + \gamma_2^{(2)} b_{21}^{(1)} = 0 \\ \gamma_2^{(2)} b_{21}^{(2)} = \frac{1}{2} \\ \alpha_2 b_{21}^{(1)} = 0 \\ \gamma_2^{(1)} a_{21} = 0 \\ \gamma_2^{(1)} {b_{21}^{(1)}}^2 + \gamma_2^{(1)} {b_{21}^{(2)}}^2 + 2\gamma_2^{(2)} b_{21}^{(1)} b_{21}^{(2)} = 0. \end{cases} \tag{3.6}$$

Note that in this case, according to structure the matrices $B^{(1)}$ and $B^{(2)}$, the equation

$${\gamma^{(1)}}^T {B^{(1)}}^2 e + {\gamma^{(1)}}^T {B^{(2)}}^2 e + {\gamma^{(2)}}^T (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e) = 0,$$

from system (3.5) always holds, and hence the number of equations are reduced to nine. The system (3.6) in MAPLE environment may be solved and it is observed that the system has two classes of one-parameter solution in the following forms:

$$\begin{cases} a_{21} = 0, \quad \alpha_1 = 1, \quad \alpha_2 = 0 \\ b_{21}^{(1)} = -\dfrac{1}{2\gamma_2^{(1)}}, \quad \gamma_1^{(1)} = -\gamma_2^{(1)}, \quad \gamma_2^{(1)} \neq 0 \\ b_{21}^{(2)} = \dfrac{1}{2\gamma_2^{(1)}}, \quad \gamma_1^{(2)} = -\gamma_2^{(1)} + 1, \quad \gamma_2^{(2)} = \gamma_2^{(1)} \end{cases} \tag{3.7}$$

$$\begin{cases} a_{21} = 0, \quad \alpha_1 = 1, \quad \alpha_2 = 0 \\ b_{21}^{(1)} = -\dfrac{1}{2\gamma_2^{(1)}}, \quad \gamma_1^{(1)} = -\gamma_2^{(1)}, \quad \gamma_2^{(1)} \neq 0 \\ b_{21}^{(2)} = -\dfrac{1}{2\gamma_2^{(1)}}, \quad \gamma_1^{(2)} = \gamma_2^{(1)} + 1, \quad \gamma_2^{(2)} = -\gamma_2^{(1)}, \end{cases} \tag{3.8}$$

the answers to which we discuss.

Case 1: From (3.7), the one-parameter solution has the following tableau:

$$\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\gamma_2^{(1)}}(\sqrt{h}-J_1) & 0 \\ \hline h & 0 & -\gamma_2^{(1)}\sqrt{h}+(1-\gamma_2^{(1)})J_1 & \gamma_2^{(1)}(\sqrt{h}+J_1) \end{array} \tag{3.9}$$

Case 2: From (3.8), the other one-parameter solution has the following tableau:

$$\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\gamma_2^{(1)}}(\sqrt{h}+J_1) & 0 \\ \hline h & 0 & -\gamma_2^{(1)}\sqrt{h}+(1+\gamma_2^{(1)})J_1 & \gamma_2^{(1)}(\sqrt{h}-J_1) \end{array} \tag{3.10}$$

Note that for each case the condition $\alpha_2 a_{21} = \frac{1}{2}$ does not hold and consequently the deterministic components (3.9) and (3.10) are not a Runge–Kutta method of order greater than or equal to 2. In principle, it is possible to choose the free parameter $\gamma_2^{(1)}$ for each case by minimizing the principal error constants method. By using the relations

$$E[I_{10}]^2 = E[I_{01}]^2 = \frac{1}{3}h^3, \quad I_{111} = \frac{1}{6}{J_1}^3 - \frac{1}{2}hJ_1, \quad E[I_{111}]^2 = \frac{1}{6}h^3,$$

these error constants for the first case are given by

$$\begin{cases} E[I_{10} - {z^{(0)}}^T Z^{(1)} e]^2 = \dfrac{1}{3}h^3, \\ E[I_{01} - {z^{(1)}}^T Z^{(0)} e]^2 = \dfrac{1}{3}h^3, \\ E[I_{111} - {z^{(1)}}^T {Z^{(1)}}^2 e]^2 = \dfrac{1}{6}h^3, \\ E[I_{111} + \dfrac{1}{2}I_{01} - \dfrac{1}{2}{z^{(1)}}^T (Z^{(1)} e)^2 - \dfrac{1}{2}{z^{(1)}}^T Z^{(0)} e]^2 = \left(\dfrac{1}{4} - \dfrac{3}{8\gamma_2^{(1)}} + \dfrac{3}{16{\gamma_2^{(1)}}^2}\right) h^3. \end{cases} \tag{3.11}$$

Note that condition (3.11) arises from trees $[\tau_1]_0$, $[\tau_0]_1$, $[[\tau_1]_1]_1$ and $[\tau_1, \tau_1]_1$ (see [6] for further details). In fact, the minimum of the last equation occurs when $\gamma_2^{(1)} = 1$, in which

case the minimum value is $\frac{1}{16}$. Consequently a family of methods satisfying (3.5) with minimum principal error constants can be presented by the tableau:

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\sqrt{h}-J_1) & 0 & \\ \hline h & 0 & -\sqrt{h} & \sqrt{h}+J_1 & \end{array}$$

which is named "*EM*1," and has the principal error constants

$$\frac{1}{3}h^3,\ \frac{1}{3}h^3,\ \frac{1}{6}h^3,\ \frac{1}{16}h^3.$$

Similarly for the second case, the error constants are given by:

$$\begin{cases} E[I_{10}-z^{(0)^T}Z^{(1)}e]^2=\dfrac{1}{3}h^3, \\ E[I_{01}-z^{(1)^T}Z^{(0)}e]^2=\dfrac{1}{3}h^3, \\ E[I_{111}-z^{(1)^T}Z^{(1)^2}e]^2=\dfrac{1}{6}h^3, \\ E[I_{111}+\dfrac{1}{2}I_{01}-\dfrac{1}{2}z^{(1)^T}(Z^{(1)}e)^2-\dfrac{1}{2}z^{(1)^T}Z^{(0)}e]^2=\left(\dfrac{1}{4}+\dfrac{3}{8\gamma_2^{(1)}}+\dfrac{3}{16\gamma_2^{(1)^2}}\right)h^3. \end{cases}$$

Now the minimum of the last equation occurs when $\gamma_2^{(1)}=-1$, in which case the minimum value is $\frac{1}{16}$. Consequently the other family of methods satisfying (3.5) with minimum principal error terms can be presented by the tableau:

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & \\ 0 & 0 & \frac{1}{2}(\sqrt{h}+J_1) & 0 & \\ \hline h & 0 & \sqrt{h} & -\sqrt{h}+J_1 & \end{array}$$

which is named "*EM*2," and has the principal error constants

$$\frac{1}{3}h^3,\ \frac{1}{3}h^3,\ \frac{1}{6}h^3,\ \frac{1}{16}h^3.$$

Also, the Itô method (see [6]), which is the derivatives free version of the Milstein method with strong global order 1, can be presented by tableau:

$$\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & \\ 0 & 0 & \sqrt{h} & 0 & \\ \hline h & 0 & J_1-\frac{\sqrt{h}}{2}((\frac{J_1}{\sqrt{h}})^2-1) & \frac{\sqrt{h}}{2}((\frac{J_1}{\sqrt{h}})^2-1) & \end{array}$$

which is named "*IRK*," and has the principal error constants

$$\frac{1}{3}h^3,\ \frac{1}{3}h^3,\ \frac{1}{6}h^3,\ \frac{3}{8}h^3.$$

In the explicit case with $s = 3$, we consider the matrices $A, B^{(1)}$ and $B^{(2)}$ with the following forms:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ b_{21}^{(1)} & 0 & 0 \\ b_{31}^{(1)} & b_{32}^{(1)} & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ b_{21}^{(2)} & 0 & 0 \\ b_{31}^{(2)} & b_{32}^{(2)} & 0 \end{pmatrix},$$

such that the row vectors $\alpha^T, {\gamma^{(1)}}^T$ and ${\gamma^{(2)}}^T$ are defined as follows:

$$\alpha^T = \begin{pmatrix} \alpha_1 \ , & \alpha_2 \ , & \alpha_3 \end{pmatrix}, \quad {\gamma^{(1)}}^T = \begin{pmatrix} \gamma_1^{(1)} \ , & \gamma_2^{(1)} \ , & \gamma_3^{(1)} \end{pmatrix}, \quad {\gamma^{(2)}}^T = \begin{pmatrix} \gamma_1^{(2)} \ , & \gamma_2^{(2)} \ , & \gamma_3^{(2)} \end{pmatrix}.$$

Hence by Equations (3.5), we have the following system of ten equations with eighteen unknowns:

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \gamma_1^{(1)} + \gamma_2^{(1)} + \gamma_3^{(1)} = 0 \\ \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_3^{(2)} = 1 \\ \gamma_2^{(1)} b_{21}^{(1)} + \gamma_3^{(1)} (b_{31}^{(1)} + b_{32}^{(1)}) = -\dfrac{1}{2} \\ \gamma_2^{(1)} b_{21}^{(2)} + \gamma_3^{(1)} (b_{31}^{(2)} + b_{32}^{(2)}) + \gamma_2^{(2)} b_{21}^{(1)} + \gamma_3^{(2)} (b_{31}^{(1)} + b_{32}^{(1)}) = 0 \\ \gamma_2^{(2)} b_{21}^{(2)} + \gamma_3^{(2)} (b_{31}^{(2)} + b_{32}^{(2)}) = \dfrac{1}{2} \\ \alpha_2 b_{21}^{(1)} + \alpha_3 (b_{31}^{(1)} + b_{32}^{(1)}) = 0 \\ \gamma_2^{(1)} a_{21} + \gamma_3^{(1)} (a_{31} + a_{32}) = 0 \\ \gamma_2^{(1)} ({b_{21}^{(1)}}^2 + {b_{21}^{(2)}}^2) + \gamma_3^{(1)} ((b_{31}^{(1)} + b_{32}^{(1)})^2 + (b_{31}^{(2)} + b_{32}^{(2)})^2) + 2\gamma_2^{(2)} b_{21}^{(1)} b_{21}^{(2)} \\ +2\gamma_3^{(2)} (b_{31}^{(1)} + b_{32}^{(1)})(b_{31}^{(2)} + b_{32}^{(2)}) = 0 \\ \gamma_3^{(1)} (b_{32}^{(1)} b_{21}^{(1)} + b_{32}^{(2)} b_{21}^{(2)}) + \gamma_3^{(2)} (b_{32}^{(1)} b_{21}^{(2)} + b_{32}^{(2)} b_{21}^{(1)}) = 0. \end{cases} \tag{3.12}$$

Moreover by using (3.5), we can minimize the error constants corresponding to trees $[\tau_1]_0$, $[\tau_0]_1$ and $[[\tau_1]_1]_1$, which are given by:

$$
\begin{cases}
E[I_{10} - z^{(0)^T} Z^{(1)} e]^2 &= \left(\frac{1}{3} - (\alpha^T B^{(2)} e) + (\alpha^T B^{(2)} e)^2 + (\alpha^T B^{(1)} e)^2\right) h^3 \\
&= \left(\frac{1}{3} - (\alpha^T B^{(2)} e) + (\alpha^T B^{(2)} e)^2\right) h^3 \\
E[I_{01} - z^{(1)^T} Z^{(0)} e]^2 &= \left(\frac{1}{3} - (\gamma^{(2)^T} A e) + (\gamma^{(2)^T} A e)^2 + (\gamma^{(1)^T} A e)^2\right) h^3 \\
&= \left(\frac{1}{3} - (\gamma^{(2)^T} A e) + (\gamma^{(2)^T} A e)^2\right) h^3 \\
E[I_{111} - z^{(1)^T} Z^{(1)^2} e]^2 &= \Big(\frac{1}{6} - 2(\gamma^{(2)^T} B^{(2)^2} e) + (\gamma^{(1)^T} (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e))^2 \\
&+ 2(\gamma^{(1)^T} (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e))(\gamma^{(2)^T} B^{(1)^2} e) + (\gamma^{(2)^T} B^{(1)^2} e)^2 \\
&+ 6(\gamma^{(2)^T} B^{(2)^2} e)(\gamma^{(1)^T} (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e)) + 2(\gamma^{(1)^T} B^{(1)^2} e)^2 \\
&+ 6(\gamma^{(2)^T} B^{(2)^2} e)(\gamma^{(2)^T} B^{(1)^2} e) + 15(\gamma^{(2)^T} B^{(2)^2} e)^2\Big) h^3
\end{cases}
$$

These three equations are minimized with the minimum value $\frac{1}{12}$, $\frac{1}{12}$ and 0 respectively, if the following conditions hold:

$$\alpha^T B^{(2)} e = \frac{1}{2}, \quad \gamma^{(2)^T} A e = \frac{1}{2}, \quad \gamma^{(2)^T} B^{(2)^2} e = \frac{1}{6}, \quad \gamma^{(1)^T} B^{(1)^2} e = 0$$

$$\gamma^{(1)^T} (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e) + \gamma^{(2)^T} B^{(1)^2} e = -\frac{1}{2},$$

or equivalently:

$$
\begin{cases}
\alpha_2 b_{21}^{(2)} + \alpha_3 (b_{31}^{(2)} + b_{32}^{(2)}) = \frac{1}{2} \\
\gamma_2^{(2)} a_{21} + \gamma_3^{(2)} (a_{31} + a_{32}) = \frac{1}{2} \\
\gamma_3^{(2)} b_{32}^{(2)} b_{21}^{(2)} = \frac{1}{6} \\
\gamma_3^{(1)} b_{32}^{(1)} b_{21}^{(1)} = 0 \\
\gamma_3^{(1)} (b_{32}^{(1)} b_{21}^{(2)} + b_{32}^{(2)} b_{21}^{(1)}) + \gamma_3^{(2)} b_{32}^{(1)} b_{21}^{(1)} = -\frac{1}{2}.
\end{cases}
\tag{3.13}
$$

For $s = 2$, it was seen that:

$$\alpha^T B^{(2)} e = 0, \quad \gamma^{(2)^T} A e = 0, \quad \gamma^{(2)^T} B^{(2)^2} e = 0,$$

$$\gamma^{(1)^T} (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e) + \gamma^{(2)^T} B^{(1)^2} e = 0.$$

But if $s = 3$ then $\alpha^T B^{(2)} e$, $\gamma^{(2)^T} A e$, $\gamma^{(2)^T} B^{(2)^2} e$ and $\gamma^{(1)^T} (B^{(1)} B^{(2)} e + B^{(2)} B^{(1)} e) + \gamma^{(2)^T} B^{(1)^2} e$ are not zero and in order to have the minimum principal error they must be taken to be the

minimum values, which are $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{6}$ and $-\frac{1}{2}$ respectively. By adding the Equations (3.13) to the system (3.12) and solving the new system in the MAPLE environment, it is observed that the new system has two classes of two-parameters solution in the following forms:

$$\begin{cases} a_{21}=0,\ a_{31}=-\dfrac{-48a_{32}\gamma_2^{(1)^4}+8\sqrt{3}a_{32}\gamma_2^{(1)^3}+(8a_{32}-12)\gamma_2^{(1)^2}+1}{8\gamma_2^{(1)^2}(-6\gamma_2^{(1)^2}+\sqrt{3}\gamma_2^{(1)}+1)} \\ \alpha_1=-\dfrac{6\sqrt{3}\gamma_2^{(1)^3}-9\gamma_2^{(1)^2}+1}{12\gamma_2^{(1)^2}-1},\ \alpha_2=3\gamma_2^{(1)^2},\ \alpha_3=\dfrac{-36\gamma_2^{(1)^4}+6\sqrt{3}\gamma_2^{(1)^3}+6\gamma_2^{(1)^2}}{12\gamma_2^{(1)^2}-1} \\ b_{21}^{(1)}=-\dfrac{1}{2\gamma_2^{(1)}},\ b_{31}^{(1)}=0,\ b_{32}^{(1)}=\dfrac{12\gamma_2^{(1)^2}-1}{4\gamma_2^{(1)}(-6\gamma_2^{(1)^2}+\sqrt{3}\gamma_2^{(1)}+1)},\ \gamma_1^{(1)}=-\gamma_2^{(1)},\ \gamma_3^{(1)}=0 \\ b_{21}^{(2)}=\dfrac{\sqrt{3}}{6\gamma_2^{(1)}},\ b_{31}^{(2)}=-\dfrac{24\sqrt{3}\gamma_2^{(1)^3}-12\gamma_2^{(1)^2}-2\sqrt{3}\gamma_2^{(1)}+1}{12\gamma_2^{(1)^2}(-6\gamma_2^{(1)^2}+\sqrt{3}\gamma_2^{(1)}+1)}, \\ b_{32}^{(2)}=\dfrac{12\sqrt{3}\gamma_2^{(1)^2}-\sqrt{3}}{12\gamma_2^{(1)}(-6\gamma_2^{(1)^2}+\sqrt{3}\gamma_2^{(1)}+1)} \\ \gamma_1^{(2)}=-\dfrac{8\sqrt{3}\gamma_2^{(1)^3}-10\gamma_2^{(1)^2}-\frac{\sqrt{3}}{3}\gamma_2^{(1)}+1}{12\gamma_2^{(1)^2}-1},\ \gamma_2^{(2)}=2\gamma_2^{(1)^2}+\dfrac{\sqrt{3}}{3}\gamma_2^{(1)} \\ \gamma_3^{(2)}=\dfrac{-24\gamma_2^{(1)^4}+4\sqrt{3}\gamma_2^{(1)^3}+4\gamma_2^{(1)^2}}{12\gamma_2^{(1)^2}-1}, \end{cases}$$

and

$$\begin{cases} a_{21}=0,\ a_{31}=-\dfrac{-48a_{32}\gamma_2^{(1)^4}-8\sqrt{3}a_{32}\gamma_2^{(1)^3}+(8a_{32}-12)\gamma_2^{(1)^2}+1}{8\gamma_2^{(1)^2}(-6\gamma_2^{(1)^2}-\sqrt{3}\gamma_2^{(1)}+1)} \\ \alpha_1=-\dfrac{-6\sqrt{3}\gamma_2^{(1)^3}-9\gamma_2^{(1)^2}+1}{12\gamma_2^{(1)^2}-1},\ \alpha_2=3\gamma_2^{(1)^2},\ \alpha_3=\dfrac{-36\gamma_2^{(1)^4}-6\sqrt{3}\gamma_2^{(1)^3}+6\gamma_2^{(1)^2}}{12\gamma_2^{(1)^2}-1} \\ b_{21}^{(1)}=-\dfrac{1}{2\gamma_2^{(1)}},\ b_{31}^{(1)}=0,\ b_{32}^{(1)}=\dfrac{12\gamma_2^{(1)^2}-1}{4\gamma_2^{(1)}(-6\gamma_2^{(1)^2}-\sqrt{3}\gamma_2^{(1)}+1)},\ \gamma_1^{(1)}=-\gamma_2^{(1)},\ \gamma_3^{(1)}=0 \\ b_{21}^{(2)}=-\dfrac{\sqrt{3}}{6\gamma_2^{(1)}},\ b_{31}^{(2)}=-\dfrac{-24\sqrt{3}\gamma_2^{(1)^3}-12\gamma_2^{(1)^2}+2\sqrt{3}\gamma_2^{(1)}+1}{12\gamma_2^{(1)^2}(-6\gamma_2^{(1)^2}-\sqrt{3}\gamma_2^{(1)}+1)}, \\ b_{32}^{(2)}=-\dfrac{12\sqrt{3}\gamma_2^{(1)^2}-\sqrt{3}}{12\gamma_2^{(1)}(-6\gamma_2^{(1)^2}-\sqrt{3}\gamma_2^{(1)}+1)} \\ \gamma_1^{(2)}=-\dfrac{-8\sqrt{3}\gamma_2^{(1)^3}-10\gamma_2^{(1)^2}+\frac{\sqrt{3}}{3}\gamma_2^{(1)}+1}{12\gamma_2^{(1)^2}-1},\ \gamma_2^{(2)}=2\gamma_2^{(1)^2}-\dfrac{\sqrt{3}}{3}\gamma_2^{(1)} \\ \gamma_3^{(2)}=\dfrac{-24\gamma_2^{(1)^4}-4\sqrt{3}\gamma_2^{(1)^3}+4\gamma_2^{(1)^2}}{12\gamma_2^{(1)^2}-1}. \end{cases}$$

It is seen that for each set of solutions $\alpha_2 a_{21} + \alpha_3(a_{31} + a_{32}) = \frac{3}{4}$ and consequently the deterministic part corresponding to each solution cannot choose from a Runge–Kutta method with an order greater than or equal to 2. We will try to choose $\gamma_2^{(1)}$ for each solution by minimizing the error constant corresponding to the tree $[\tau_1, \tau_1]_1$. Hence for the first set of solutions, this error constant will be:

$$\left\{ E\left[I_{111} + \frac{1}{2} I_{01} - \frac{1}{2} z^{(1)^T} (Z^{(1)} e)^2 - \frac{1}{2} z^{(1)^T} Z^{(0)} e \right]^2 \right. = \left(\frac{11664\gamma_2^{(1)^8} - 10800\sqrt{3}\gamma_2^{(1)^7} + 12420\gamma_2^{(1)^6}}{1728\gamma_2^{(1)^4}(6\gamma_2^{(1)^2} - \sqrt{3}\gamma_2^{(1)} - 1)^2} + \frac{-1872\sqrt{3}\gamma_2^{(1)^5} - 1422\gamma_2^{(1)^4} + 612\sqrt{3}\gamma_2^{(1)^3}}{1728\gamma_2^{(1)^4}(6\gamma_2^{(1)^2} - \sqrt{3}\gamma_2^{(1)} - 1)^2} + \frac{12\gamma_2^{(1)^2} - 24\sqrt{3}\gamma_2^{(1)} + 5}{1728\gamma_2^{(1)^4}(6\gamma_2^{(1)^2} - \sqrt{3}\gamma_2^{(1)} - 1)^2} \right) h^3.$$

The minimum occurs when $\gamma_2^{(1)} = \frac{662}{681}$, in which case the minimum value is $\frac{607}{6414}$. Since there is only one free parameter, this leads to a number of possible methods, and the following method is selected:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.4080024374 & -0.8660254040 & 0 \end{pmatrix},$$

$$B^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ -0.5143504532 & 0 & 0 \\ 0 & -0.8904881170 & 0 \end{pmatrix}$$

$$B^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0.2969603727 & 0 & 0 \\ 0.7228984640 & -0.5141235541 & 0 \end{pmatrix},$$

$$\alpha^T = \begin{pmatrix} -0.1974618999 \ , & 2.834934374 \ , & -1.637472474 \end{pmatrix}$$

$$\gamma^{(1)^T} = \begin{pmatrix} -0.9720998532 \ , & 0.9720998532 \ , & 0 \end{pmatrix},$$

$$\gamma^{(2)^T} = \begin{pmatrix} -0.3595500450 \ , & 2.451198361 \ , & -1.091648316 \end{pmatrix}$$

which is referred to as "EM3" and has the principal error coefficients:

$$\frac{1}{12}h^3, \ \frac{1}{12}h^3, \ 0, \ \frac{607}{6414}h^3.$$

Note that for this method, the free parameter a_{32} was chosen to be $-\frac{\sqrt{3}}{2}$.

Similarly, for the second set of solutions, the error constant corresponding to the tree $[\tau_1, \tau_1]_1$ is given by

$$\begin{cases} E\left[I_{111} + \frac{1}{2}I_{01} - \frac{1}{2}z^{(1)^T}(Z^{(1)}e)^2 - \frac{1}{2}z^{(1)^T}Z^{(0)}e\right]^2 = \left(\frac{11664\gamma_2^{(1)^8} + 10800\sqrt{3}\gamma_2^{(1)^7} + 12420\gamma_2^{(1)^6}}{1728\gamma_2^{(1)^4}(6\gamma_2^{(1)^2} + \sqrt{3}\gamma_2^{(1)} - 1)^2}\right. \\ \qquad + \frac{1872\sqrt{3}\gamma_2^{(1)^5} - 1422\gamma_2^{(1)^4} - 612\sqrt{3}\gamma_2^{(1)^3}}{1728\gamma_2^{(1)^4}(6\gamma_2^{(1)^2} + \sqrt{3}\gamma_2^{(1)} - 1)^2} \\ \qquad \left. + \frac{12\gamma_2^{(1)^2} + 24\sqrt{3}\gamma_2^{(1)} + 5}{1728\gamma_2^{(1)^4}(6\gamma_2^{(1)^2} + \sqrt{3}\gamma_2^{(1)} - 1)^2}\right) h^3. \end{cases}$$

This equation is minimized, if $\gamma_2^{(1)} = -\frac{662}{681}$, in which case the minimum value is $\frac{607}{6414}$. Consequently this class of method by choosing $a_{32} = -\frac{\sqrt{3}}{2}$ leads to the following method:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.4080024374 & -0.8660254040 & 0 \end{pmatrix},$$

$$B^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0.5143504532 & 0 & 0 \\ 0 & 0.8904881170 & 0 \end{pmatrix}$$

$$B^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0.2969603727 & 0 & 0 \\ 0.7228984640 & -0.5141235541 & 0 \end{pmatrix},$$

$$\alpha^T = \begin{pmatrix} -0.1974618999, 2.834934374, -1.637472474 \end{pmatrix}$$

$$\gamma^{(1)^T} = \begin{pmatrix} 0.9720998532, -0.9720998532\ , 0 \end{pmatrix},$$

$$\gamma^{(2)^T} = \begin{pmatrix} -0.3595500450, 2.451198361, -1.091648316 \end{pmatrix}$$

which is referred to as "EM4" and has the principal error coefficients

$$\frac{1}{12}h^3,\ \frac{1}{12}h^3,\ 0,\ \frac{607}{6414}h^3.$$

If we use the 1–norm to estimate the contribution of all error terms to the principal error term, Table 3.1 presents these values for methods "*IRK*," "*EM*1," "*EM*2," "*EM*3," and "*EM*4."

Table 3.1 Norm of principal error coefficients.

	IRK	*EM*1	*EM*2	*EM*3	*EM*4
$\|Principal\ \ error\|_1$	1.2083	0.89583	0.89583	0.26130	0.26130

We observe that the principal error of the methods "*EM*3" and "*EM*4" are less than that of "*IRK*," "*EM*1," and "*EM*2" methods, and the principal error of all "*EM*" methods is less than that of the "*IRK*" method.

4 Numerical results and Conclusion

In this section, numerical results of the six methods, "*IRK*," "*Milstein*," "*EM*1," "*EM*2," "*EM*3," and "*EM*4," are compared. These methods are implemented in a fixed, step-size mode on two test problems taken from [3], for which the exact solution in terms of a Wiener process is known. In order to simulate the Gaussian variable J_1 with distribution $N(0,h)$, we have taken pseudo–random numbers generated by the "randn" in MATLAB 7.0. When these methods are simulated, the same sequence of random numbers for the Wiener increment J_1 is used for the step size considered. The implementation determines the average error for each step size at the end of the interval of integration, and is defined by

$$AE = \frac{1}{K}\sum_{i=1}^{K} \mid y_N^{(i)} - y^{(i)}(t_N) \mid,$$

where $y_N^{(i)}$ is the numerical approximation and $y^{(i)}(t_N)$ is the exact solution of SDE at t_N in the *i–th* simulation over all K simulations. All the numerical results are based on 1000 simulated trajectories. The results appear in Tables 4.1, 4.2, and 4.3. The CPU times for computing the average error (produced on a Pentium IV, 2.8 GHZ) are plotted in Figures 4.1 and 4.2.

Table 4.1 Global errors for test Problem 1, with $a = 1,\ K = 1000$.

h	*IRK*	*Milstein*	*EM*1	*EM*2	*EM*3	*EM*4
$\frac{1}{25}$	0.21400e–1	0.16276e–1	0.12121e–1	0.12043e–1	0.10417e–1	0.10928e–1
$\frac{1}{50}$	0.10299e–1	0.82454e–2	0.59344e–2	0.57056e–2	0.50661e–2	0.49935e–2
$\frac{1}{100}$	0.51948e–2	0.42156e–2	0.30475e–2	0.29270e–2	0.24155e–2	0.24900e–2
$\frac{1}{200}$	0.24299e–2	0.19930e–2	0.14587e–2	0.13901e–2	0.11717e–2	0.12024e–2
$\frac{1}{400}$	0.12254e–2	0.10127e–2	0.70585e–3	0.71060e–3	0.59737e–3	0.60778e–3

Table 4.2 Global errors for test problem 2, with $\alpha = -1,\ \beta = 1,\ K = 1000$.

h	*IRK*	*Milstein*	*EM*1	*EM*2	*EM*3	*EM*4
$\frac{1}{25}$	0.12763e–1	0.11513e–1	0.96413e–2	0.93988e–2	0.89722e–2	0.52558e–2
$\frac{1}{50}$	0.58682e–2	0.51633e–2	0.41781e–2	0.42298e–2	0.37325e–2	0.23548e–2
$\frac{1}{100}$	0.29961e–2	0.27770e–2	0.21225e–2	0.20985e–2	0.18169e–2	0.13255e–2
$\frac{1}{200}$	0.15034e–2	0.13806e–2	0.10660e–2	0.10210e–2	0.88244e–3	0.68361e–3
$\frac{1}{400}$	0.74495e–3	0.68995e–3	0.54324e–3	0.52317e–3	0.42077e–3	0.34825e–3

Table 4.3 Global errors for test problem 2, with $\alpha = -1,\ \beta = 0.01,\ K = 1000$.

h	*IRK*	*Milstein*	*EM*1	*EM*2	*EM*3	*EM*4
$\frac{1}{25}$	0.50778e–2	0.50778e–2	0.50778e–2	0.50778e–2	0.27025e–2	0.26712e–2
$\frac{1}{50}$	0.25193e–2	0.25193e–2	0.25193e–2	0.25193e–2	0.13011e–2	0.12901e–2
$\frac{1}{100}$	0.12544e–2	0.12544e–2	0.12544e–2	0.12544e–2	0.63851e–3	0.63466e–3
$\frac{1}{200}$	0.62592e–3	0.62592e–3	0.62592e–3	0.62592e–3	0.31617e–3	0.31481e–3
$\frac{1}{400}$	0.31264e–3	0.31264e–3	0.31264e–3	0.31264e–3	0.15730e–3	0.15682e–3

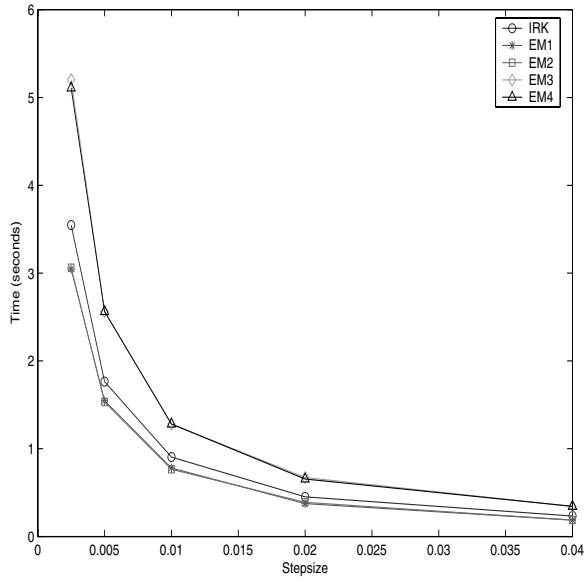

Figure 4.1 CPU time for computing global errors for test Problem 1 with $a = 1$, $K = 1000$.

Test Problem 1. Consider

$$dy = -a^2 y(1-y^2)dt + a(1-y^2)dW(t), \quad y(0) = 0, \quad t \in [0,1],$$

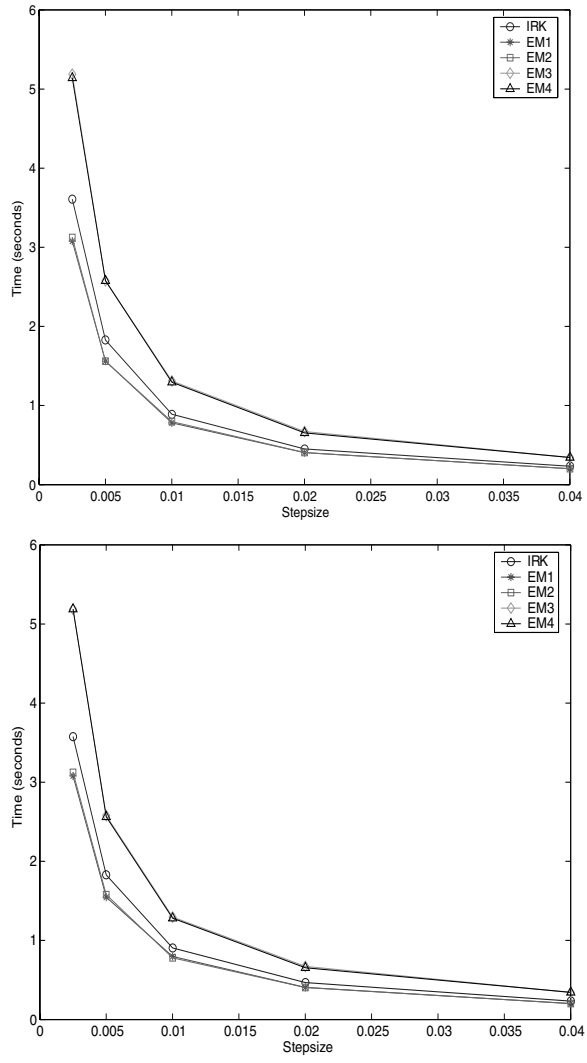

Figure 4.2 CPU time for computing global errors for test problem 2 with $\alpha = -1$, $\beta = 1$ (up), and $\alpha = -1$, $\beta = 0.01$ (down), for $K = 1000$.

with the exact solution

$$y(t) = \tan h(aW(t) + \arctan h(y_0)).$$

This problem was solved numerically with choice of parameter $a = 1$.

Test Problem 2. Consider

$$dy = -(\alpha + \beta^2 y)(1 - y^2)dt + \beta(1 - y^2)dW(t), \quad y(0) = 0.5, \quad t \in [0, 1],$$

with the exact solution

$$y(t) = \frac{(1+y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1+y_0)\exp(-2\alpha t + 2\beta W(t)) + 1 - y_0}.$$

This problem was solved numerically with $\alpha = -1$ and different values $\beta = 1$ and $\beta = 0.01$. These demonstrate the variation in emphasis of the stochastic and deterministic parts of the SDE. By comparing the numerical results in Tables 4.1, 4.2 and 4.3, we conclude that the "*EM*3" and "*EM*4" methods are more accurate than the "*IRK*," "*Milstein*," "*EM*1," and "*EM*2" methods. Also for two–stage SRK methods the "*EM*1" and "*EM*2" methods are more effective than "*IRK*" and "*Milstein*" methods. On the other hand, for problems in which the deterministic term dominates (test Problem 2 with $\beta = 0.01$), the error values for two–stage SRK methods are the same. This is because of the same deterministic components in the Butcher's array of the methods. Figures 4.1 and 4.2 show that the CPU times of "*EM*1" and "*EM*2" methods for calculating the average error is less than that for the "*IRK*" method. Moreover the CPU time of the "*EM*1" method is very close to that of the "*EM*2" method, and this is true for the "*EM*3" and "*EM*4" methods.

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