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Corrected fundamental solution for numerical solution of elliptic PDEs

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Abstract

Corrected fundamental solution (CFS) is a meshless method for homogeneous elliptic problems that corrects the density function in a simple layer potential integral. In the CFS method, we apply a new expansion of density function with variable coefficients which are approximated in a finite subspace of a complete space. These coefficients are determined by the moving least square method (MLS), using a suitable weight function that its support is in the real and artificial domain.

Keywords: Fundamental solution (FS); Method of fundamental solution (MFS); Corrected fundamental solution (CFS); Simple layer potential integral; Moving least square method (MLS); Complete bases; Homogenous and elliptic problems

1. Introduction

The method of fundamental solution (MFS) is a numerical method for homogeneous elliptic problems that needs knowing and having related fundamental solutions [6,3,8]. It is a boundary meshless method. This method originally presented by Kupradze and Aleksidze [6]. First, simple and double layer integrals is approximated on artificial boundary and then by expanding its density function in a finite subspace of a complete basis, one have an approximation solution depend on the related problem. Unknown coefficients of the density function must be found by collocating the approximate solution on the boundary. By selecting some boundary points equal to the coefficients, one can find the coefficients of the density function using the collocation method on the real boundary. Instead of the collocation method, many boundary points may be chosen, more than the unknown density coefficients and so, by using the least square method the unknown density coefficients can be found.

In 1968, Shepard [9] presented the moving least square method (MLS) in a simple and low order form entitled Shepard Interpolant. Then in 1981, Lancaster and Salkauskas [7] generalized the MLS method and extended it to higher order [4].

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In this paper, we change the density function expansion or series, and correcting it similar to the MLS approximation together with the variable coefficients. These expansions need a finite subspace of a complete basis such as polynomial and trigonometric functions.

The rest of this paper is structured as follows: Section 2 introduces the MFS. Section 3 is a review on the MLS method and in Section 4, we explained our method, the CFS. In Section 5, we presented a 2–D numerical example. Section 6 introduces a typical adaptivity. Section 7 gives our concluding remarks.

2. Method of fundamental solution

Fundamental solution of a problem is potential of a point source charge. In the MFS, one knows FS previously. This method is a boundary meshless method and needs an artificial domain, which is greater than real domain.

This method is as follows:

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3 be an open domain and $\partial\Omega = \Gamma$ be its boundary.

Consider the following homogeneous elliptic boundary value problem:

$$\begin{aligned} &\mathscr{L}u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \\ &u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \end{aligned} \tag{1}$$

Fundamental solution of the model problem (1) for a fixed point $\mathbf{y} \in \Omega$ is the solution of

$$\mathscr{L}\mathbf{H}(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \overline{\Omega}, \tag{2}$$

where δ is the Dirac delta distribution that shows point source charge at the fixed point $\mathbf{y}$ and the fundamental solution $\mathbf{H}(\mathbf{x}, \mathbf{y})$ means amount of potential at the point $\mathbf{x}$ with respect to the point charge at $\mathbf{y}$. In this method, we suppose that the FS of the problem are known.

Let $\sigma(\mathbf{y})$ be a density or a correction function and $\{\phi_j(\mathbf{y})\}_{j=1}^n$ be a finite subspace of a complete space. Then, the density function is approximated in the following form:

$$\sigma(\mathbf{y}) = \sum_{j=1}^{n} c_j \phi_j(\mathbf{y}), \quad \mathbf{y} \in \Omega. \tag{3}$$

The solution of the problem (1) in a simple layer potential integral lies in the following form:

$$\tilde{u}(\mathbf{x}) = \int_{\widehat{\Gamma}} \sigma(\mathbf{y}) \mathbf{H}(\mathbf{x}, \mathbf{y}) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \overline{\Omega}. \tag{4}$$

For preventing the singularity points of the fundamental solution $\mathbf{H}(\mathbf{x}, \mathbf{y})$, and having a better condition number for immediate linear systems, this boundary integral is defined on an artificial boundary. So, instead of the real domain Ω and its boundary Γ, a greater boundary $\widehat{\Omega}$ where $\Omega \subset \widehat{\Omega}$ and $\widehat{\Gamma} = \partial\widehat{\Omega}$ is applied. Therefore $\mathbf{x}$ and $\mathbf{y}$ can never be equal (singularity of the fundamental solution is the case when $\mathbf{x} = \mathbf{y}$) to each other and the boundary integral passes over its singularity. Consider quadrature points $\{\mathbf{y}_k\}_{k=1}^M$ on the artificial boundary $\widehat{\Gamma}$ (or in the region $\widehat{\Omega} \setminus \overline{\Omega}$). Using a suitable quadrature rule, the boundary integral (4) can be approximated in the form

$$\tilde{u}(\mathbf{x}) = \sum_{k=1}^{M} \sigma(\mathbf{y}_k) \mathbf{H}(\mathbf{x}, \mathbf{y}_k) w_k, \quad \mathbf{x} \in \overline{\Omega}, \tag{5}$$

where, w_k is the quadrature weight. Then, by substituting the density function series (3) in (5), the approximate solution

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{n} c_j \psi_j(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega} \tag{6}$$

as a linear combination of a basis functions

$$\psi_j(\mathbf{x}) = \sum_{k=1}^{M} \phi_j(\mathbf{y}_k) \mathbf{H}(\mathbf{x}, \mathbf{y}_k) w_k, \quad j = 1, 2, \ldots, n, \ \ \mathbf{x} \in \overline{\Omega} \tag{7}$$

constructs an approximation basis for the model problem (1) (see [5]). Therefore, $\tilde{u} \in \mathrm{span}(\{\psi_j(\mathbf{x})\}_{j=1}^n)$. Now, by applying the boundary conditions of the problem (1), the constant coefficients c_j, $j = 1,2,\ldots,n$, can be obtained by the weighted residual methods such as collocation or least square method. By selecting n suitable points $\{\mathbf{x}_i\}_{i=1}^n \subset \Gamma$ on the real boundary and solving the following system

$$\delta_{\mathbf{x}_i}(\tilde{u}) = \delta_{\mathbf{x}_i}(g), \quad i = 1,2,\ldots,n, \tag{8}$$

the coefficients $\{c_i\}_{i=1}^n$, can be found. In (8) which is distributional form of the collocation method, weight is the Dirac delta generalized function and pushes to have an exact solution for the selected boundary points. Unique solution of this system depends on the invertibility of the coefficient matrix, which is not symmetric because, $\psi_j(\mathbf{x}_i) \neq \psi_i(\mathbf{x}_j)$ for the $i,j = 1,2,\ldots,n$. Linear independency of the base functions $\{\psi_j\}_{j=1}^n$ is an open problem. As another method, the least square method can be used to calculate these coefficients. Let the points $\{\mathbf{x}_i\}_{i=1}^N$ be selected on the boundary Γ, such that $N \gg n$. The following functional is

$$\boldsymbol{J}(c_1,c_2,\ldots,c_n) = \sum_{k=1}^{N} (g(\mathbf{x}_k) - \tilde{u}(\mathbf{x}_k))^2, \tag{9}$$

which is convex and non-negative. By minimizing this functional with respect to the coefficients c_i, for $i = 1,2,\ldots,n$ the following system will be found

$$\partial \boldsymbol{J}/\partial c_i = \sum_{k=1}^{N} \psi_i(\mathbf{x}_k)(g(\mathbf{x}_k) - \tilde{u}(\mathbf{x}_k)) = 0, \quad i = 1,2,\ldots,n. \tag{10}$$

Suppose

$$\tilde{b}_i = \sum_{k=1}^{N} \psi_i(\mathbf{x}_k) g(\mathbf{x}_k) \quad \text{for} \quad i = 1,2,\ldots,n, \tag{11}$$

$$\tilde{a}_{ij} = \sum_{k=1}^{N} \psi_i(\mathbf{x}_k)\psi_j(\mathbf{x}_k) \quad \text{for} \quad i,j = 1,2,\ldots,n, \tag{12}$$

then the system (10) can be rewritten in the following simple form:

$$\sum_{j=1}^{n} \tilde{a}_{ij} c_j = \tilde{b}_i, \quad i = 1,2,\ldots,n. \tag{13}$$

Now, if we set

$$\begin{aligned} &\boldsymbol{A} = (\tilde{a}_{ij})_{i,j=1}^n, \\ &\boldsymbol{b} = [\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n]^{\mathrm{T}}, \\ &\boldsymbol{c} = [c_1, c_2, \ldots, c_n]^{\mathrm{T}}, \end{aligned} \tag{14}$$

then, we have a linear system

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{b}. \tag{15}$$

For the least square method, the coefficient matrix $\mathbf{A}$ is invertible because it is symmetric and positive definite. After finding the coefficients c_j for $j = 1,2,\ldots,n$, approximate solution (6) of the model problem (1) are known. This approximate solution is not a typical interpolation but it is an approximation of the model problem.

3. Moving least square method (MLS)

Let $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^d$, $d = 1,2$ or 3 be an unknown continuous function that we try to approximate it by having some data point. Given $\mathbf{x}_j \in \Omega$, $j = 1,2,\ldots,n$, an irregular distribution of nodes in the

domain and $u_j = u(\mathbf{x}_j)$, $j = 1, 2, \ldots, n$. Let $\boldsymbol{P}(\mathbf{x})$ be a given m-dimensional base, for example, in 1-D case, let $\boldsymbol{P}^{\mathrm{T}}(x) = \{1, x, \ldots, x^{m-1}\}$. Define local approximation

$$\tilde{u}_{\mathbf{y}}(\mathbf{x}) = \boldsymbol{P}^{\mathrm{T}}(\mathbf{x})\boldsymbol{a}(\mathbf{y}), \tag{16}$$

where $\mathbf{y} \in \Omega$ is fixed and the coefficient vector $\boldsymbol{a}(\mathbf{y}) = [a_1(\mathbf{y}), a_2(\mathbf{y}), \ldots, a_m(\mathbf{y})]^{\mathrm{T}}$ should be found. Let $w_i(\mathbf{x})$, $i = 1, 2, \ldots, n$ be a suitable weight. By minimizing the weighted discrete square of the local error functional

$$\mathbf{J}(\mathbf{a}(\mathbf{y})) = \|u(\cdot) - \tilde{u}_{\mathbf{y}}(\cdot)\|_w^2 = \sum_{j=1}^{n} w(\mathbf{y} - \mathbf{x}_j)(u_j - \tilde{u}_{\mathbf{y}}(\mathbf{x}_j))^2 = \sum_{j=1}^{n} w(\mathbf{y} - \mathbf{x}_j)(u_j - \boldsymbol{P}^{\mathrm{T}}(\mathbf{x}_j)\boldsymbol{a}(\mathbf{y}))^2, \tag{17}$$

with respect to the coefficient vector $\mathbf{a}(\mathbf{y})$, we will have the following system:

$$\boldsymbol{A}(\mathbf{y})\boldsymbol{a}(\mathbf{y}) = \boldsymbol{F}(\mathbf{y})\boldsymbol{U}, \tag{18}$$

where

$$\begin{aligned}
&\boldsymbol{A}(\mathbf{y}) = \boldsymbol{B}\boldsymbol{W}(\mathbf{y})\boldsymbol{B}^{\mathrm{T}},\\
&\boldsymbol{F}(\mathbf{y}) = \boldsymbol{B}\boldsymbol{W}(\mathbf{y}),\\
&\boldsymbol{B} = \{\mathbf{x}_j^{i-1}\}, \quad i = 1, \ldots, m, \;\; j = 1, \ldots, n,\\
&\boldsymbol{W}(\mathbf{y}) = \mathrm{diag}(w(\mathbf{y} - \mathbf{x}_1), \ldots, w(\mathbf{y} - \mathbf{x}_n)),\\
&\boldsymbol{U} = [u(\mathbf{x}_1), u(\mathbf{x}_2), \ldots, u(\mathbf{x}_n)]^{\mathrm{T}}.
\end{aligned}$$

Then the local approximation (16) becomes

$$\tilde{u}_{\mathbf{y}}(\mathbf{x}) = \Phi_{\mathbf{y}}^{\mathrm{T}}(\mathbf{x})\boldsymbol{U} \tag{19}$$

and the global approximation will be

$$\tilde{u}(\mathbf{x}) = \Phi^{\mathrm{T}}(\mathbf{x})\boldsymbol{U} = \sum_{j=1}^{n} \phi_j(\mathbf{x})u_j, \tag{20}$$

where the vector base function in the global form with components $\{\phi_j(\mathbf{x})\}_{j=1}^{n}$ is

$$\Phi^{\mathrm{T}}(\mathbf{x}) = \boldsymbol{P}^{\mathrm{T}}(\mathbf{x})\boldsymbol{A}^{-1}(\mathbf{x})\boldsymbol{F}(\mathbf{x}). \tag{21}$$

4. Corrected fundamental solution (CFS)

Referring to the Section 3, for a fixed point $\mathbf{y} \in \Omega$, let $\sigma(\mathbf{x}, \mathbf{y})$ be a corrected density function emerged of the MLS method and $\{\phi_j(\mathbf{y})\}_{j=1}^{n}$ be a finite subspace of a complete space. Then, the density function can be approximated in the following form:

$$\sigma(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} c_j(\mathbf{x})\phi_j(\mathbf{y}), \quad \mathbf{x} \in \overline{\Omega}. \tag{22}$$

The solution of the problem (1) based on the single layer potential by this density function, becomes

$$\tilde{u}(\mathbf{x}) = \int_{\widehat{\Gamma}} \sigma(\mathbf{x}, \mathbf{y})\mathbf{H}(\mathbf{x}, \mathbf{y})\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \overline{\Omega}, \tag{23}$$

where $\widehat{\Gamma}$ is the artificial boundary and $\mathbf{H}(\mathbf{x}, \mathbf{y})$ is the fundamental solution of related problem. The function $\tilde{u}(\mathbf{x})$ is the solution of the model problem (1) if the following conditions satisfy:

$$\mathscr{L}c_j(\mathbf{x}) = 0, \quad \mathbf{x} \in \overline{\Omega} \quad j = 1, 2, \ldots, n. \tag{24}$$

If we get quadrature points $\{\mathbf{y}_k\}_{k=1}^{M}$ on the artificial boundary $\widehat{\Gamma}$, by a suitable quadrature rule, the boundary integral (23) can be approximated in the form

$$\tilde{u}(\mathbf{x}) = \sum_{k=1}^{M} \sigma(\mathbf{x}, \mathbf{y}_k)\mathbf{H}(\mathbf{x}, \mathbf{y}_k)w_k, \quad \mathbf{x} \in \overline{\Omega}, \tag{25}$$

where, w_k is the quadrature weight. The approximate solution of the model problem, based on its basis becomes

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{n} c_j(\mathbf{x})\psi_j(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega}, \tag{26}$$

where the functions $\{\psi_j\}_{j=1}^{n}$ are defined in (7). The coefficient functions $c_j(\mathbf{x})$, $j = 1, 2, \ldots, n$ can be found by the MLS method, by selecting the points $\{\mathbf{x}_k\}_{k=1}^{N}$ on the boundary Γ such that $N \gg n$ and a weight function with smoothing degree more than order of the operator $\mathscr{L}$. Then, the following functional will be constructed:

$$\boldsymbol{J}(c_1(\mathbf{x}), \ldots, c_n(\mathbf{x})) = \sum_{k=1}^{N} w_k(\mathbf{x})(g(\mathbf{x}_k) - \tilde{u}_{\mathbf{x}}(\mathbf{x}_k))^2, \tag{27}$$

which is a weighted discrete square $\mathbf{L}^2(\overline{\Omega})$-norm of local error and

$$\tilde{u}_{\mathbf{x}}(\mathbf{y}) = \sum_{j=1}^{n} c_j(\mathbf{x})\psi_j(\mathbf{y}), \quad \mathbf{y} \in \{\mathbf{x}_k\}_{k=1}^{N}, \ \ \mathbf{x} \in \overline{\Omega} \tag{28}$$

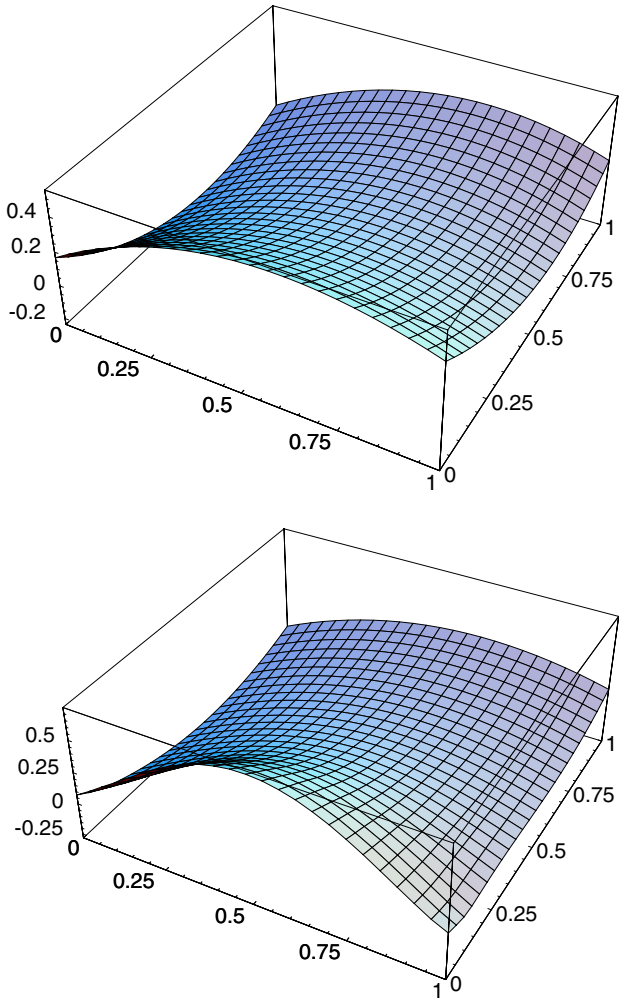

Fig. 1. Approximation for the CFS (down) and the MFS (up) under similar conditions.

is the local approximation introduced in (16). By minimizing the functional $\boldsymbol{J}(c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots, c_n(\mathbf{x}))$ with respect to the $c_j(\mathbf{x})$, $j = 1, 2, \ldots, n$ the following system will be found:

$$\partial \boldsymbol{J}/\partial c_i(\mathbf{x}) = \sum_{k=1}^{N} w_k(\mathbf{x})\psi_i(\mathbf{x}_k)(g(\mathbf{x}_k) - \tilde{u}_{\mathbf{x}}(\mathbf{x}_k)) = 0, \quad i = 1, 2, \ldots, n. \tag{29}$$

Let

$$\tilde{b}_i(\mathbf{x}) = \sum_{k=1}^{N} w_k(\mathbf{x})\psi_i(\mathbf{x}_k)g(\mathbf{x}_k), \quad i = 1, 2, \ldots, n, \tag{30}$$

$$\tilde{a}_{ij}(\mathbf{x}) = \sum_{k=1}^{N} w_k(\mathbf{x})\psi_i(\mathbf{x}_k)\psi_j(\mathbf{x}_k), \quad i, j = 1, 2, \ldots, n \tag{31}$$

then the system (29) can be rewritten by the following simple form:

$$\sum_{j=1}^{n} \tilde{a}_{ij}(\mathbf{x})c_j(\mathbf{x}) = \tilde{b}_i(\mathbf{x}), \quad i = 1, 2, \ldots, n. \tag{32}$$

Now, assume

$$\begin{aligned} &\boldsymbol{A}(\mathbf{x}) = (\tilde{a}_{ij}(\mathbf{x}))_{i,j=1}^{n}, \\ &\boldsymbol{b}(\mathbf{x}) = [\tilde{b}_1(\mathbf{x}), \tilde{b}_2(\mathbf{x}), \ldots, \tilde{b}_n(\mathbf{x})]^{\mathrm{T}}, \\ &\boldsymbol{c}(\mathbf{x}) = [c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots, c_n(\mathbf{x})]^{\mathrm{T}} \end{aligned} \tag{33}$$

then, the compact form of the system (32), is in the following form:

$$\boldsymbol{A}(\mathbf{x})\boldsymbol{c}(\mathbf{x}) = \boldsymbol{b}(\mathbf{x}). \tag{34}$$

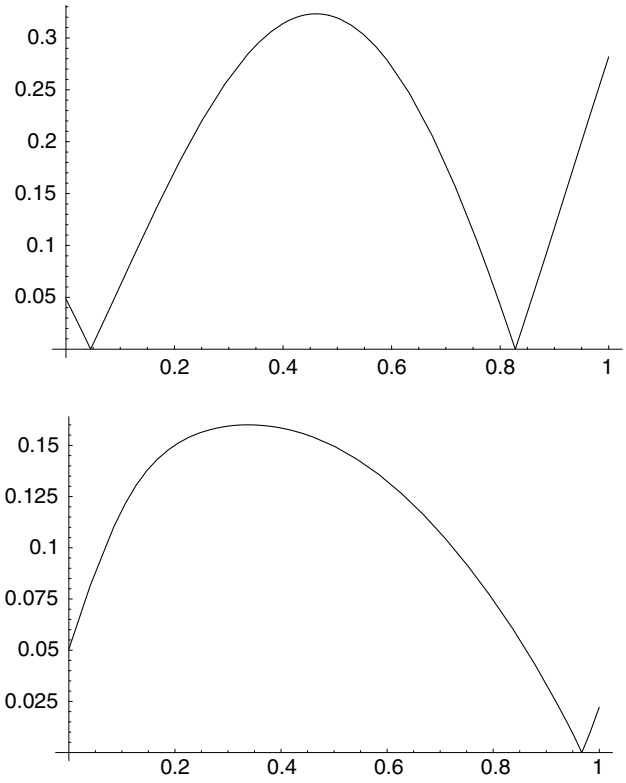

Fig. 2. Linear error of the Fig. 1 in the x-direction when $y = 0.1$ and for the CFS (down) and the MFS (up).

Support of each weight function is a subdomain in Ω which the weight function is nonnegative and its closure is compact. Here, center of the weights are the boundary points $\{\mathbf{x}_k\}_{k=1}^{N} \subset \Gamma$. So, one part of the supports are in the real domain and the other part are in the artificial domain. These supports are overlapping such that at least n number of the points $\{\mathbf{x}_k\}_{k=1}^{N}$ belong to each support. Furthermore, each arbitrary boundary point be in the intersection of at least n number of these supports. After applying the operator $\mathscr{L}$ on the system (32), and using the constraint (24), we will have a new system for the coefficients $c_j(\mathbf{x}), j = 1, 2, \ldots n$ in the following form:

$$\sum_{j=1}^{n} \mathscr{L}\tilde{a}_{ij}(\mathbf{x})c_j(\mathbf{x}) = \mathscr{L}\tilde{b}_i(\mathbf{x}), \quad i = 1, 2, \ldots, n. \tag{35}$$

It is important to know that the smoothing degree of the weight functions must be more than the order of derivative operator in $\mathscr{L}$.

5. A numerical example

Let $\overline{\Omega} = [0, 1] \times [0, 1]$ be closure of the real domain and $\overline{\widehat{\Omega}} = [-1.0, 2.0] \times [-1.0, 2.0]$ be closure of the artificial domain. The real and artificial domain are rectangular, and their boundary comprise of four lines.

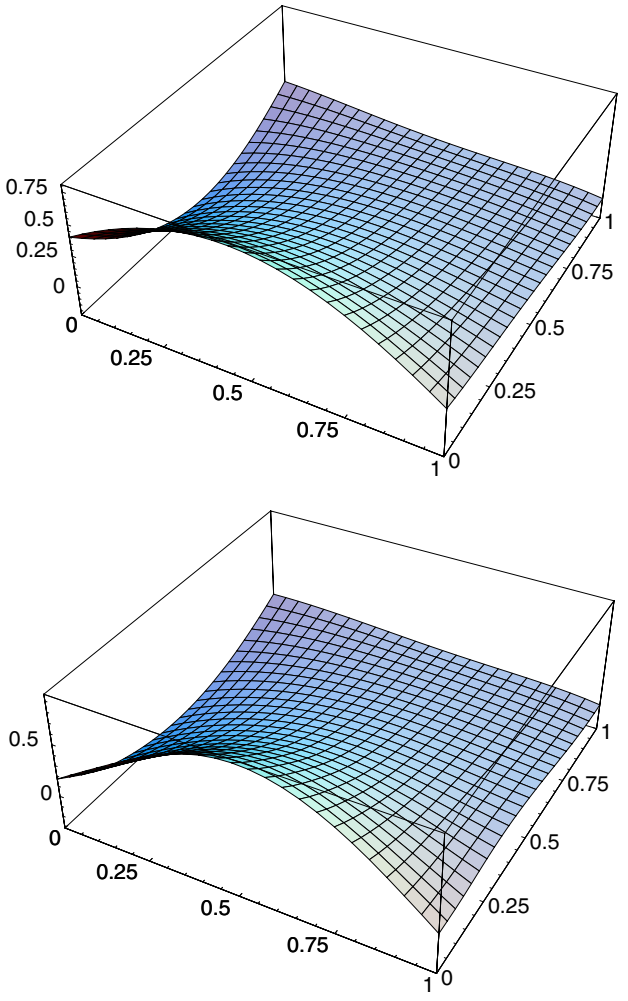

Fig. 3. Approximation for the CFS (down) and the MFS (up) under one condition.

Here, our model problem is

$$\begin{aligned} &\Delta u(x,y) = 0, \quad (x,y) \in \Omega, \\ &u(0,y) = u(1,y) = u(x,1) = 0, \\ &u(x,0) = \sin(\pi x), \quad x \in [0,1], \end{aligned} \tag{36}$$

which is the Laplace problem and Dirichlet boundary conditions. The exact solution of this problem is $u(x,y) = \sin(\pi x)(\cosh(\pi y) - \coth(\pi)\sinh(\pi y))$, where $(x,y) \in \overline{\Omega}$. The FS of the Laplace operator, is

$$\mathbf{H}(x,y,\xi,\eta) = \frac{-1}{4\pi} \ln((x-\xi)^2 + (y-\eta)^2), \quad (x,y) \in \overline{\Omega}, \tag{37}$$

which is independent of the boundary conditions and $(\xi,\eta) \in \Omega$ is a fixed point. The density function for the results shown in Fig. 1 is expanded in the finite subset of the polynomial $\{1,x,y\}$, for the results given in Fig. 3 is expanded in the higher order set $\{1,x,y,x^2,xy,y^2\}$.

The Gauss–Legendre rule is used for the numerical quadrature that applied for approximation of the boundary integral (4) and (23) into the (5) and (25) respectively. Here, in the MFS, the density constants be obtained by the least square method and in the CFS, the density coefficient functions are calculated by the MLS method. The coefficients in the MFS and the CFS, be obtained on each side of the real domain boundary. We consider 12 nodes with uniform distribution. In the CFS and the MLS method, the following inverse square singular weight function be used:

$$w(x,y,\xi,\eta) = 1/((x-\xi)^2 + (y-\eta)^2), \tag{38}$$

which have global support and is decreasing radially from its center. Figs. 1 and 3 show the graph of both the MFS (up) and the CFS (down), when the density function approximation is of linear and quadratic order

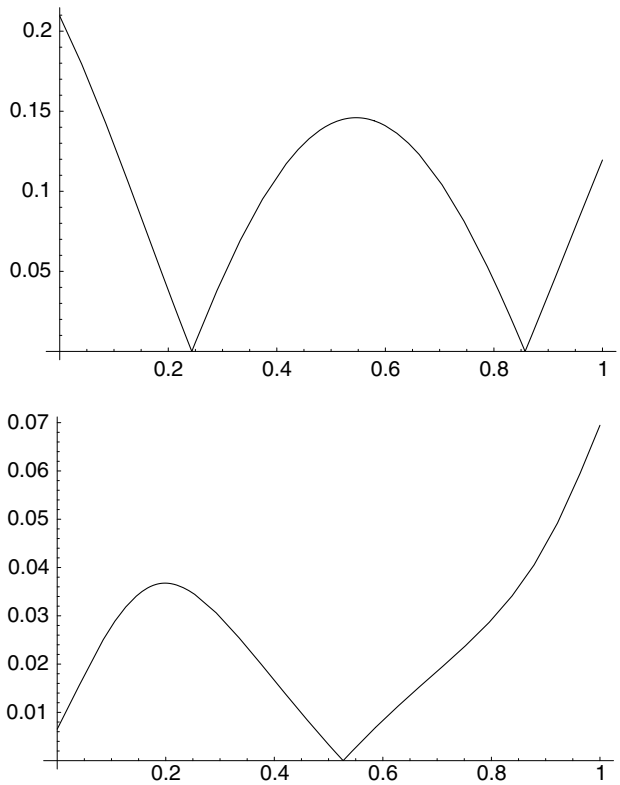

Fig. 4. Linear error of the Fig. 3 in x-direction for $y = 0.1$ and for the CFS (down) and the MFS (up).

respectively. Linear error $(\tilde{u}(x,0.1)-u(x,0.1))$ of both methods for $y=0.1$ and $x\in[0,1]$ (Figs. 2 and 4), shows the accuracy of the CFS with respect to the MFS.

6. Adaptivity

We can add another base function like $\{p_\ell\}_{\ell=1}^{n_e}$, such as polynomial base over another base $\{\psi_j\}_{j=1}^{n}$ that was introduced in (7), enrichment of the approximation (6) can be done. So, an alternative approximation becomes

$$\tilde{u}(\mathbf{x})=\sum_{j=1}^{n}c_j\psi_j(\mathbf{x})+\sum_{\ell=1}^{n_e}d_\ell p_\ell(\mathbf{x}),\quad \mathbf{x}\in\overline{\Omega}. \tag{39}$$

It is important to note that the above approximation is a solution of the model problem (1), if

$$\mathscr{L}p_\ell(\mathbf{x})=0,\quad \mathbf{x}\in\overline{\Omega},\quad \text{for } \ell=1,2,\ldots,n_e. \tag{40}$$

For example, if we want to enrich the bases of the model problem (36), a finite subset of the base $\{\Re z^\ell\}_{\ell=0}$ can be appended to the approximation space $\{\psi_j\}_{j=1}^{n}$. Because, the real and imaginary part of an analytic function is harmonic and satisfies in the Laplace equation [2].

7. Concluding remarks

- Computational task of the CFS is more than the MFS. Because, the inverse of $n\times n$ matrix for each evaluation point must be calculated.
- Both the MFS and the CFS require neither domain nor boundary discretization. They are boundary meshless methods (see [5]).
- Radial form of the MFS and the CFS is intensive to dimensionality of the problem and thus is very attractive to high-dimensional problems.
- The correction function in reproducing kernel particle method (RKPM) [1] can be considered for any change in the density function.
- Numerical quadrature error of the boundary integral must be balanced with another error sources.
- Relation between the real domain and the artificial domain and type of the artificial boundary have not any strict reply. For example, these two questions have not any strict reply: how much the artificial boundary should be greater than the real domain and what is its type?
- The MFS and CFS are applicable for homogeneous and elliptic problems when the fundamental solution are known. Otherwise, these methods cannot be applied.
- The approximate basis functions $\psi_j, j=1,2,\ldots,n$ have global support. So, local support form of it can be searched and, the MFS and the CFS can be localized.
- If one can expand the FS of a problem by a complete basis, computational effort will be more simple and efficient.
- Computational cost of the MFS and CFS is relatively inexpensive with respect to domain or mesh methods.
- The solution can be extended outside of the real domain and in the artificial domain.

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