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Method of Lines for Stochastic Boundary-Value Problems

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Abstract

In this article, we use the method of lines for solving stochastic boundary value problems. This method is actually the development of the deterministic theory of method of lines applying to the time, space and randomness separately. In this paper, a specific second order finite-volume scheme is implemented for spatial discretization and then the stochastic differential system is solved by a class of Stochastic Rung-Kutta methods (SRK).

Keywords: Stochastic partial differential equations , Method of lines, finite volume methods, Wiener process.

1 Introduction

Stochastic partial differential equations have been extensively used to model random effects in engineering and mathematical sciences. So, many numerical methods have been extended to solve stochastic partial differential equations.
Let (Ω, F, P) be a probability space, where Ω is the space of basic outcomes, F is the $\sigma-$ algebra associated with Ω, and P is the (probability) measure on F.
Consider the stochastic boundary-value problem of the form

$$\partial_t u + \partial_x f(u) - \nu \partial^2_{xx} u = \sigma(x)\zeta(t,x) \tag{1}$$

where $(t,x) \in (0,T] \times D$, t and x are time and space variables, ∂_t and ∂_x denote derivatives with respect to t and x, respectively. Here $f(u)$ is a linear or nonlinear flux function and $\zeta(t,x)$ is a random noise assumed to be either time-dependent or space- dependent and also is Gaussian with zero mean. The equation (1) is solved in a bounded spatial domain D with smooth boundary, For a time interval $(0,T)$, and equipped with the initial condition $u(0,x) = u_0(x), \quad x \in D$, where u_0 is a given initial data. The two-dimensional boundary- value problems, the equation (1) can be developed in the form as

$$\partial_t u + \partial_x f(u) + \partial_y g(u) - \nu(\partial^2_{xx} u + \partial^2_{yy} u) = \sigma(x,y)\zeta(t,x,y), \tag{2}$$

with g is the flux function in y- direction and the random forcing term depends also on the spatial coordinate y.

2 Formulation of Method of Lines

Method of lines (MOL) is a two step numerical procedure that is used for solving deterministic partial differential equations and in this paper is extended to the stochastic case. The first step in the method of lines involve discretizing the spatial dimension of the stochastic partial differential equations (SPDEs). This transform the SPDE into a system of stochastic differential equations (SDEs). The second stage of the algorithm uses one of the many numerical methods available for solving SDEs to provide a numerical solution for the transformed SPDE. In this article, a class of stochastic Rung-Kutta methods has been applied for solving the attained SDE.
The spatial discretization of the equation $(1)-(2)$ is carried out using a second - order finite volume method. Hence, the spatial domain D is discretizated into control volumes $[x_{i+\frac{1}{2}}, x_{i-\frac{1}{2}}]$ with uniform dimension $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$. Integrating (1) with respect to x over the control volume and keeping the time t continuous, the following semi-discrete system be obtained:

$$\frac{dU_i}{dt} + \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta x} - \nu \frac{U_{i+1} - 2U_i + U_{i-1}}{(\Delta x)^2} = \sigma_i \zeta(t, x_i) \tag{3}$$

where U_i is the space average of the solution u in the cell $[x_{i+\frac{1}{2}}, x_{i-\frac{1}{2}}]$ at time t,

$$U_i(t) = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(t, x) dx$$

where $F_{i\pm 1/2} = f((U_{i\pm 1/2}, t))$ are the numerical fluxes at $x = x_{i\pm 1/2}$. In this paper, we consider the second-order MUSCL method [1] to reconstruct the numerical fluxes in (3). Thus,

$$F_{i+1/2} = \frac{f(u_i) + f(u_{i+1})}{2} - \lambda_i \frac{u_{i+1} - u_i}{2} + \frac{\sigma_i^+ - \sigma_{i+1}^-}{4}, \tag{4}$$

$$F_{i-1/2} = \frac{f(u_{i-1}) + f(u_i)}{2} - \lambda_i \frac{u_i - u_{i-1}}{2} + \frac{\sigma_{i-1}^- - \sigma_i^+ -}{4}, \tag{5}$$

where we use the Sweby's notation [1], [3] to define slopes of the solution in the cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ as

$$\begin{aligned} \sigma_i^+ &= (f(u_{i+1}) + \lambda_i u_{i+1} - f(u_i) - \lambda_i u_i)\phi(\theta_i^+), \\ \sigma_i^- &= (f(u_{i+1}) - \lambda_i u_{i+1} - f(u_i) + \lambda_i u_i)\phi(\theta_i^-), \end{aligned}$$

where

$$\begin{aligned} \theta_i^+ &= \frac{f(u_i) + \lambda_i u_i - f(u_{i-1}) - \lambda_i u_{i-1}}{f(u_{i+1}) + \lambda_i u_{i+1} - f(u_i) - \lambda_i u_i}, \\ \theta_i^- &= \frac{f(u_i) - \lambda_i u_i - f(u_{i-1}) + \lambda_i u_{i-1}}{f(u_{i+1}) - \lambda_i u_{i+1} - f(u_i) + \lambda_i u_i}, \end{aligned}$$

and ϕ represents a slope limiter function chosen to be the Van-Leer limiter $\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}$, [1]. In $(4)-(5)$, λ_i are the characteristic speeds is to set $\lambda_i = max_i(|f'(u_i)|)$.

3 Stochastic Rung-Kutta Schemes for Time Integration

The semi-discrete equation (3) can be formulated as a system of *Itô* stochastic differential equation (SDEs) which can be written in a form as

$$dX_t = F(t, X_t)dt + G(t, X_t)dW_t, \tag{6}$$

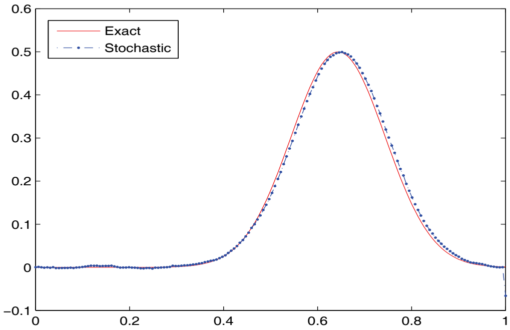

Figure 1: Results for the one-dimensional stochastic advection-diffusion problem for the case $\nu = 5 \times 10^{-3}$ using MUSCL scheme with 160 gridpoints

where X_t contains the unknown semi-discretized solution, $F(t, X_t)$ and $G(t, X_t)$ represent respectively, the semi-discrete form of the deterministic differential operator and the stochastic part in (3). If the spatial domain is discretized in M control volumes then, X_t , W_t and F are $M-$valued vectors with entries $U_i, W_{i,t}$ and $\frac{F_{i+1/2}-F_{i-1/2}}{\Delta x} + \nu \frac{U_{i+1}-2U_i+U_{i-1}}{(\Delta x)^2}$, respectively.
Let the time interval $[0, T]$ be divided into N subinterval $[t_n, t_{n+1}]$ of length Δt such that $t_n = n\Delta t$. The s-stage SRK method applied to the SDE (6) is given by $Y_0 = X_{t_0}$, and

$$\begin{aligned} Y_{n+1} &= Y_n + \sum_{j=1}^{s} \alpha_i F(t_n + c_i \Delta t, H_i)\Delta t + \sum_{k=1}^{M}\sum_{i=1}^{s} \beta_i G^k(t_n)\hat{I}_{(k)}, \\ H_i &= Y_n + \sum_{j=1}^{s} A_{ij} F(t_n + c_j \Delta t, H_j)\Delta t + \sum_{l=1}^{M}\sum_{j=1}^{s} B_{ij} G^l(t_n)\hat{I}_{(l)}. \end{aligned}$$

The random variables $\hat{I}_{(k)}$ used by SRK method, are independent identically $N(0, \Delta t)$ distributed and the coefficient in the SRK method can be given by extended Butcher tableaus [2], [3].

4 Numerical Results

In this section, we investigate the performance of the method of lines for stochastic advection-diffusion problem in one and two dimensions.
One-dimensional stochastic advection-diffusion equation: consider the stochastic advection-diffusion equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial W}{\partial x}, \qquad x \in [0, 1],$$

with the initial condition $u(0, x) = exp(-\frac{(x-0.2)^2}{\nu})$, and the boundary conditions

$$u(t, 0) = \frac{1}{\sqrt{4t+1}} exp(-\frac{(-0.2 - vt)^2}{\nu(4t+1)}) \qquad , \qquad u(t, 1) = \frac{1}{\sqrt{4t+1}} exp(-\frac{(-0.8 - vt)^2}{\nu(4t+1)}).$$

In this examples we used $v = 0.6, \sigma = 2.5$ and all the solutions are computed at time $t = 0.75$. In order to qualify the results for the stochastic advection-diffusion problem, we plot in Figure (1) the deterministic

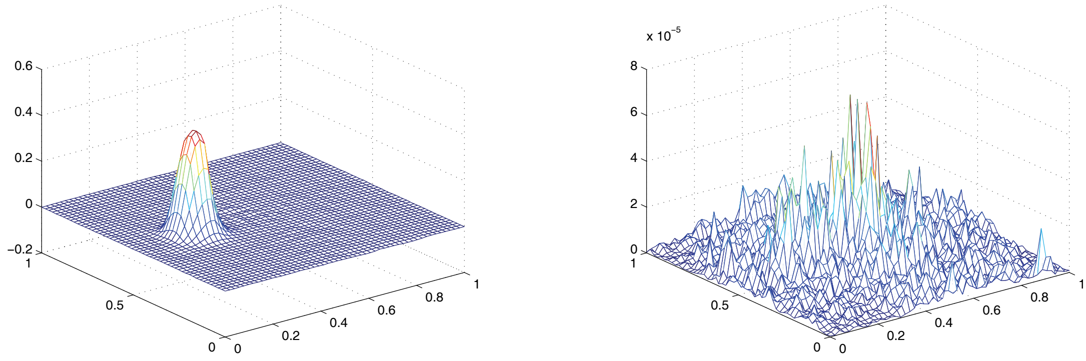

Figure 2: Mean solution (top) and variance (bottom) for the two- dimensional stochastic advection-diffusion equation at $t = T/4$

and stochastic solutions along with the analytical solution .
Two-dimensional advection-diffusion equation: Consider the stochastic advection-diffusion problem of a rotating Gaussian pulse:

$$\frac{\partial u}{\partial t} + v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \sigma \frac{\partial^2 W}{\partial x \partial y} \tag{7}$$

with $v_1 = -4y$ and $v_2 = 4x$. Initial and boundary conditions are selected according to analytical expected solution

$$\bar{u}(t, x, y) = \frac{\delta^2}{\delta^2 + 4\nu t} exp\left(-\frac{(\bar{x} - x_0)^2 + (\bar{y} - y_0)^2}{\delta^2 + 4\nu t}\right)$$

where $\bar{x} = xcos(4t) + ysin(4t)$, $\bar{y} = -xsin(4t) + ycos(4t)$, $x_0 = -0.25$, $y_0 = 0$ and $\delta^2 = 0.002$. We used $\nu = 10^{-4}$ and the amplitude of stochastic term is set to $\sigma = 1$. The computational domain is $[-0.5, 0.5] \times [-0.5, 0.5]$ covered by a uniform mesh with 50×50 control volumes, and the time period applied for one complete rotation $T = \frac{\pi}{2}$.

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