Convergence of He's Variational Iteration Method for Nonlinear Oscillators

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Abstract

This paper deals with a new proof of convergence of He's variational iteration method applied to nonlinear oscillators. One example is given to verify convergence hypothesis and to illustrate the efficiency and simplicity of the method.

Keywords: Variational iteration method; Convergence theorem; Nonlinear oscillators

Biographical Notes: A. Ghorbani is a PhD student at Ferdowsi University of Mashhad. He has published more than 10 articles in ISI-listed journals; the total citation is more than 100. His current research interest mainly covers in approximate solution of nonlinear problems.

1 Introduction

Recently He [1] published a survey article. Some new asymptotic techniques with numerous examples were reviewed, the limitations of traditional perturbation procedures were illustrated, and various modified perturbation techniques were introduced, furthermore some mathematical tools such as variational theory, homotopy technology, and iteration technique were suggested to overcome the shortcomings arising in classical perturbation methods. For the nonlinear oscillators, all the reviewed schemes produced high approximate periods, but the accuracy of the amplitudes cannot be ameliorated by iteration. This review article [1] was on the variational approaches, parameter-expanding method, parameterized perturbation technique, homotopy perturbation method, iteration perturbation procedure and ancient Chinese methods. Variational approaches to soliton solution, bifurcation, limit cycle, and period solutions of nonlinear equations including the Ritz method, energy technique, variational iteration method were systemtically discussed in Ref.[1].
The variational iteration method (VIM) plays an important role in both mathematics and engineering. This method was proposed by He [2-5] as a modification of a general Lagrange multiplier method [6]. It has been shown that this procedure is a powerful tool for solving various kinds of problems (e.g., see [7-10]).
In this work, we adapt the technique to nonlinear oscillator problems and we prove the convergence of this method by proposing a new formulation of the method. An example is presented to show convergence assumption.

2 The variational iteration method

The idea of VIM is very simple and straightforward. To explain the basic idea of VIM, we consider a general nonlinear oscillator with specified initial conditions (i.e., $u(0) = A$ and $u'(0) = 0$) as follows (more general form can be considered without loss of generality):

$$\mathcal{F}(u,u',u'') := u'' + f(u)u' + g(u,u',u'')u = 0, \tag{1}$$

In the equation (1), f and g are continuous nonlinear operators with respect to their arguments, and $u(t)$ is an unknown variable. We first consider Eq. (1) as:

$$\mathcal{L}[u(t)] + \mathcal{N}[u(t)] = 0, \tag{2}$$

with, for example,

$$\mathcal{L}[u] = u'' + \omega^2 u, \text{ and } \mathcal{N}[u] = f(u)u' + g(u,u',u'')u - \omega^2 u, \tag{3}$$

where, as shown above, $\mathcal{L}$ with the property $\mathcal{L}f \equiv 0$ when $f \equiv 0$ denotes the linear operator with respect to u and $\mathcal{N}$ is a nonlinear operator with respect to u. We then construct a correction functional for Eq. (2) as [5]:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_{(t,s)}(u_n''(s) + \omega^2 u_n(s) + \mathcal{N}[\tilde{u}_n(s)])\,\mathrm{d}s, \tag{4}$$

where $u_0(t)$ is the initial guess and the subscript n denotes the n-th iteration, and $\lambda_{(t,s)} \neq 0$ denote the Lagrange multiplier, which can be identified efficiently via the variational theory, and $\tilde{u}_n$ is considered as a restricted variation [5], i.e., $\delta\tilde{u}_n = 0$.

Taking the variation with respect to the independent variable u_n, we notice that $\delta u_n(0) = 0$. Afterward, we make the correction functional stationary, and we obtain $\delta u_{n+1}(t) = 0$; therefore, we have

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta\int_0^t \lambda_{(t,s)}(u_n''(s) + \omega^2 u_n(s) + \mathcal{N}[\tilde{u}_n(s)])\mathrm{d}s \\ &= \delta u_n(t) + \lambda\delta u_n'(s)\big|_{s=t} - \frac{\partial\lambda}{\partial s}\delta u_n(s)\Big|_{s=t} + \int_0^t (\frac{\partial^2\lambda}{\partial s^2} + \omega^2\lambda)\delta u_n(s)\mathrm{d}s, \\ &= (1 - \frac{\partial\lambda}{\partial s})\delta u_n(s)\big|_{s=t} + \lambda\delta u_n'(s)\big|_{s=t} + \int_0^t (\frac{\partial^2\lambda}{\partial s^2} + \omega^2\lambda)\delta u_n(s)\mathrm{d}s = 0. \end{aligned} \tag{5}$$

As a result, we have the following stationary conditions:

$$\lambda_{(t,s)}\big|_{s=t} = 0, \tag{6}$$

$$\frac{\partial\lambda_{(t,s)}}{\partial s}\Big|_{s=t} = 1, \tag{7}$$

$$\frac{\partial^2\lambda_{(t,s)}}{\partial s^2} + \omega^2\lambda_{(t,s)} = 0. \tag{8}$$

The Lagrange multiplier can be readily identified as

$$\lambda_{(t,s)} = \frac{1}{\omega}\sin\omega(s - t). \tag{9}$$

Moreover, we have the following variational iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_{(t,s)} \mathcal{F}(u_n(s), u_n'(s), u_n''(s)) \mathrm{d}s. \tag{10}$$

Accordingly, the successive approximations $u_n(t), n \geq 0$ of VIM will be readily obtained by choosing all the above-mentioned parameters. Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n \to \infty} u_n(t). \tag{11}$$

The initial guess can be freely chosen with possible unknown constants; it can also be solved from its corresponding linear homogeneous equation $\mathcal{L}[u_0(t)] = 0$. It is important to note that for linear problems, its exact solutions can be obtained easily by only one iteration step due to the fact that the auxiliary function can be suitably identified [11]. For nonlinear problems, in general, one iteration leads to highly accurate solution by VIM if the initial solution is carefully chosen with some unknown parameters.

3 Convergence theorem

The variational iteration formula makes a recurrence sequence $\{u_n(t)\}$. Obviously, the limit of the sequence will be the solution of Eq. (1) if the sequence is convergent. In this section, we give a new proof of convergence of VIM in details by introducing a new iterative formulation of this procedure. Here, $C^n[0,T]$ denotes the class of all real valued functions defined on $[0,T]$, which have continuous n th order derivatives.

Lemma 3.1 *If for any n, $u_n \in C^2[0,T]$, then the variational iteration formula (10) is equivalent to the following iterative relation:*

$$\mathcal{L}[u_{n+1}(t) - u_n(t)] = -\mathcal{F}(u_n, u_n', u_n''), \tag{12}$$

where $\mathcal{L}$ is as noted in (3).

Proof Suppose u_n and u_{n+1} satisfies the variational iteration formula (10). Applying d^2/dt^2 to both sides of (10) results in

$$\frac{d^2}{dt^2}[u_{n+1}(t) - u_n(t)] = \int_0^t \frac{\partial^2 \lambda_{(t,s)}}{\partial t^2} \mathcal{F} \mathrm{d}s + \left.\frac{\partial \lambda_{(t,s)}}{\partial t}\right|_{s=t} \mathcal{F} + \frac{d}{dt}[\lambda_{(t,s)}\big|_{s=t} \mathcal{F}]. \tag{13}$$

Now, using the conditions (6)-(8) and $\left.\frac{\partial \lambda_{(t,s)}}{\partial t}\right|_{s=t} = -1$, we will have

$$\frac{d^2}{dt^2}[u_{n+1}(t) - u_n(t)] + \omega^2[u_{n+1}(t) - u_n(t)] = -\mathcal{F}(u_n, u_n', u_n''). \tag{14}$$

From the definition (3) of $\mathcal{L}$, we obtain

$$\mathcal{L}[u_{n+1}(t) - u_n(t)] = -\mathcal{F}(u_n, u_n', u_n''). \tag{15}$$

Conversely, suppose u_n and u_{n+1} satisfies (12). In view of the definition $\mathcal{L}$ and $\lambda_{(t,s)} \neq 0$, multiplying (12) by $\lambda_{(t,s)}$ and then integrating from both sides of the resulted term from 0 to 1 yield

$$\int_0^t \lambda_{(t,s)}[u''_{n+1}(s) - u''_n(s)]\mathrm{d}s + \int_0^t \omega^2 \lambda_{(t,s)}[u_{n+1}(s) - u_n(s)]\mathrm{d}s = -\int_0^t \lambda_{(t,s)}\mathcal{F}\mathrm{d}s. \tag{16}$$

Using integration by parts, the expression (16) becomes

$$\begin{aligned} &\lambda_{(t,s)}\Big|_{s=t} [u'_{n+1}(t) - u'_n(t)] - \frac{\partial \lambda_{(t,s)}}{\partial s}\Bigg|_{s=t} [u_{n+1}(t) - u_n(t)] \\ &\quad + \int_0^t \Big(\frac{\partial^2 \lambda_{(t,s)}}{\partial s^2} + \omega^2 \lambda_{(t,s)}\Big)[u_{n+1}(s) - u_n(s)]\mathrm{d}s = -\int_0^t \lambda_{(t,s)}\mathcal{F}\mathrm{d}s, \end{aligned} \tag{17}$$

which exactly results in (10) upon imposing the conditions (6)-(8), i.e.

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_{(t,s)}\mathcal{F}(u_n(s), u'_n(s), u''_n(s))\mathrm{d}s, \tag{18}$$

and this ends the proof. □

Theorem 3.2 *If the sequence (11) converges, where $u_n(t)$ is produced by the variational iteration formulation of (10), then it must be the exact solution of the equation (1).*

Proof If the sequence $\{u_n(t)\}$ converges, we can write

$$v(t) = \lim_{n\to\infty} u_n(t), \tag{19}$$

and it holds

$$v(t) = \lim_{n\to\infty} u_{n+1}(t), \tag{20}$$

Using the expressions (19) and (20), and the definition of $\mathcal{L}$ in (3), we can easily gain

$$\lim_{n\to\infty} \mathcal{L}[u_{n+1}(t) - u_n(t)] = \mathcal{L} \lim_{n\to\infty} [u_{n+1}(t) - u_n(t)] = 0. \tag{21}$$

From (21) and according to the Lemma 3.1, we obtain

$$\mathcal{L} \lim_{n\to\infty} [u_{n+1}(t) - u_n(t)] = -\lim_{n\to\infty} \mathcal{F}(u_n, u'_n, u''_n) = 0, \tag{22}$$

which gives us

$$\lim_{n\to\infty} \mathcal{F}(u_n, u'_n, u''_n) = 0. \tag{23}$$

From (23) and continuity of f and g operators, it holds

$$\begin{aligned} \lim_{n\to\infty} \mathcal{F}(u_n, u'_n, u''_n) &= \lim_{n\to\infty} (u''_n + f(u_n)u'_n + g(u_n, u'_n, u''_n)u_n) \\ &= (\lim_{n\to\infty} u_n)'' + f(\lim_{n\to\infty} u_n) \lim_{n\to\infty} u_n \\ &\quad + g(\lim_{n\to\infty} u_n, (\lim_{n\to\infty} u_n)', (\lim_{n\to\infty} u_n)'') \lim_{n\to\infty} u_n \end{aligned} \tag{24}$$

$$= v'' + f(v)v' + g(v, v', v'')v.$$

From the equations (23) and (24), we have

$$v'' + f(v)v' + g(v, v', v'')v = 0, \quad t \geq 0. \tag{25}$$

On the other hand, using the specified initial conditions and the definition of the initial guess, we have

$$v(0) = \lim_{n\to\infty} u_n(0) = A, \text{ since } u_n(0) = A, \quad n \geq 0, \tag{26}$$

$$v'(0) = \lim_{n\to\infty} u_n'(0) = 0, \text{ since } u_n'(0) = 0, \quad n \geq 0. \tag{27}$$

Therefore, according to the above three expressions, (25)-(27), $v(t)$ must be the exact solution of the equation (1). This ends the proof. □

Note that the above theorem is valid for the linear operator $\mathcal{L}$ defined by (3). This convergence theorem is important. It is because of this theorem that we can focus on ensuring that the approximation sequence converges. It is clear that the convergence of the sequence (11) depends upon the initial guess $u_0(t)$ and the linear operator $\mathcal{L}$. Fortunately, VIM provides us with great freedom of choosing them. Thus, as long as $u_0(t)$ and $\mathcal{L}$ are so properly chosen that the sequence (11) converges in a region $0 \leq t \leq T$, it must converge to the exact solution in this region. Therefore, the combination of the convergence theorem and the freedom of the choice of the initial guess $u_0(t)$ and the linear operator $\mathcal{L}$ establish the cornerstone of the validity and flexibility of VIM.

4 An illustrative example

In order to illustrate the efficiency of the VIM described in this paper, we present one example.

Example Let us consider the nonlinear oscillator [12]

$$u'' + u + \varepsilon u u' u'' = 0, \; u(0) = A, \quad \text{and} \quad u'(0) = 0. \tag{28}$$

By expanding the exact solution $u(t)$ of (28) with the help of the Maple computational software, we can obtain

$$u(t) = A - \frac{1}{2}At^2 + \frac{1}{6}A^3\varepsilon t^3 + (\frac{1}{24}A - \frac{1}{8}A^5\varepsilon^2)t^4 + (-\frac{1}{15}A^3\varepsilon + \frac{1}{8}A^7\varepsilon^3)t^5 + \mathrm{O}(t^6). \tag{29}$$

Solution We shall apply the method for solving (28). If we choose $u_0(t) = A$, then we obtain

$$u_1(t) = A - \frac{1}{2}At^2 + \mathrm{O}(t^3), \tag{30}$$

$$u_2(t) = A - \frac{1}{2}At^2 + \frac{1}{6}A^3\varepsilon t^3 + \mathrm{O}(t^4), \tag{31}$$

$$u_3(t) = A - \frac{1}{2}At^2 + \frac{1}{6}A^3\varepsilon t^3 + (\frac{1}{24}A - \frac{1}{8}A^5\varepsilon^2)t^4 + \mathrm{O}(t^5), \tag{32}$$

$$\begin{aligned} u_4(t) = {} & A - \frac{1}{2}At^2 + \frac{1}{6}A^3\varepsilon t^3 + (\frac{1}{24}A - \frac{1}{8}A^5\varepsilon^2)t^4 \\ & + (-\frac{1}{15}A^3\varepsilon + \frac{1}{8}A^7\varepsilon^3)t^5 + \mathrm{O}(t^6), \end{aligned} \tag{33}$$

$\vdots$

It is easy to see that

$$\lim_{n \to \infty} u_n(t) = u(t), \tag{34}$$

which is the solution of (28).

5 Conclusion

In this work, we have given a new proof of convergence of He's variational iteration method by presenting a new formulation of He's method. The main property of this method is in its flexibility and ability to solve nonlinear equations accurately and conveniently without decomposing the nonlinear terms, which is very complex. This technique is a very powerful tool for solving nonlinear problems. Furthermore, it gives an accurate and easily computable solution by means of a truncated series whose convergence is fast.

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References

[1] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141 − 1199.

[2] J.H. He, Variational iteration method for delay differential equations, Comm. Non-Linear. Sci. Numer. Simulation 2 (1997) 235 − 236.

[3] J.H. He, Approximate solution of nonlinear differential equations with convolution product non-linearities, Comput. Methods. Appl. Mech. Engng. 167 (1998) 69 − 73.

[4] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods. Appl. Mech. Engng. 167 (1998) 57 − 68.

[5] J.H. He, Variational iteration method- a kind of non-linear analytical technique: some examples, Internat. J. Non-Linear Mech. 34 (1999) 699 − 708.

[6] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in non-linear mathematical physics, in: Variational Methods in the Mechanics of Solids, Pergamon Press, NewYork, 1978, pp. 156 − 162.

[7] S. Momani, S. Abuasad, Application of He's variational iteration method to Helmholtz equation, Chaos Solitons Fractals 27 (2006) 1119 − 1123.

[8] M.A. Abdou, A.A. Soliman, Variational iteration method for solving Burger's and coupled Burger's equations, J. Comput. Appl. Math. 181 (2005) 245 − 251.

[9] M. Tatari, M. Dehghan, On the convergence of He's variational iteration method, J. Comput. Appl. Math. 207 (2007) 121 − 128.

[10] M. Dehghan, M. Tatari, The use of He's variational iteration method for solving the Fokker–Planck equation, Phys. Scripta 74 (2006) 310 − 316.

[11] J.H. He, Variational iteration method- some recent results and new interpretations, J. Comput. Appl. Math., 207 (2007) 3-17.

[12] H. Kojouharov, B. Welfert, A nonstandard Euler scheme for $y'' + g(y)y' + f(y) = 0$, J. Comput. Appl. Math. 151 (2003) 335 − 353.