

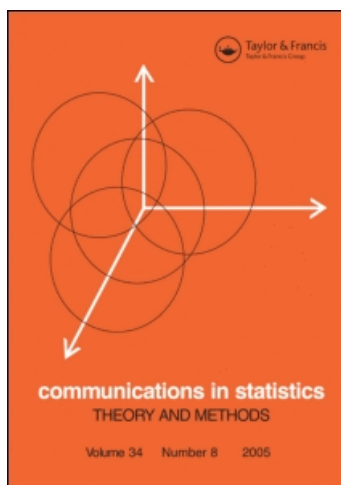
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Strong Gaussian Approximations of Product-Limit and Quantile Processes for Strong Mixing and Censored Data

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In this article, we consider the product-limit quantile estimator of an unknown quantile function under a censored dependent model. This is a parallel problem to the estimation of the unknown distribution function by the product-limit estimator under the same model. Simultaneous strong Gaussian approximations of the product-limit process and product-limit quantile process are constructed with rate $O[(\log n)^{-\lambda}]$ for some $\lambda > 0$. The strong Gaussian approximation of the product-limit process is then applied to derive the laws of the iterated logarithm for product-limit process.

Keywords Censored dependent data; Kaplan–Meier estimator; Kiefer process; Law of the iterated logarithm; α -Mixing; Strong Gaussian approximation.

Mathematics Subject Classification 62G20; 62G30.

1. Introduction and Preliminaries

In medical follow-up or in engineering life testing studies, the lifetime variable may not be observable. Let X_1, \dots, X_n be a sequence of lifetimes, having a common unknown continuous marginal distribution function F , with a density function $f = F'$ and hazard rate $\lambda = f/(1 - F)$. The random variables are not assumed to be mutually independent (see assumption (1) for the kind of dependence stipulated). Let the random variable X_i be censored on the right by the random variable Y_i , so that one observes only

$$Z_i = X_i \wedge Y_i \quad \text{and} \quad \delta_i = I(X_i \leq Y_i),$$

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where \wedge denotes minimum and $I(\cdot)$ is the indicator of the event specified in parentheses. In this random censorship model, we assume that the censoring random variables Y_1, \dots, Y_n are not mutually independent (see assumption (2) for the kind of dependence stipulated), having a common unknown continuous d.f. G , and that they are independent of the X_i s. Because censored data traditionally occur in lifetime analysis, we assume that X_i and Y_i are nonnegative. The actually observed Z_i s have a distribution function H satisfying

$$\bar{H}(t) = 1 - H(t) = [1 - F(t)][1 - G(t)].$$

Denote by

$$F_*(t) = P(Z \leq t, \delta = 1),$$

the subdistribution function for the uncensored observations. Define

$$N_n(t) = \sum_{i=1}^n I(Z_i \leq t, \delta = 1) = \sum_{i=1}^n I(X_i \leq t \wedge Y_i),$$

the number of uncensored observations less than or equal to t , and

$$Y_n(t) = \sum_{i=1}^n I(Z_i \geq t),$$

the number of censored or uncensored observations greater than or equal to t and also the empirical distribution functions of $\bar{H}(t)$ and $F_*(t)$ are respectively defined as

$$\bar{Y}_n(t) = n^{-1}Y_n(t), \quad \bar{N}_n(t) = n^{-1}N_n(t).$$

Then the Kaplan–Meier estimator for $1 - F(t)$, based on the censored data is

$$1 - \hat{F}_n(t) = \prod_{s \leq t} \left(1 - \frac{dN_n(s)}{Y_n(s)} \right), \quad t < Z_{(n)}, \quad (1.1)$$

where $Z_{(i)}$ are the order statistics of Z_i and $dN_n(t) = N_n(t) - N_n(t-)$.

As is known (see, e.g., Gill, 1980), for a d.f. F on $[0, \infty)$, the cumulative hazard function Λ is defined by

$$\Lambda(t) = \int_0^t \frac{dF(s)}{1 - F(s^-)},$$

and $\Lambda(t) = -\log[1 - F(t)]$ for the case that F is continuous. The empirical cumulative hazard function $\hat{\Lambda}_n(t)$ is given by

$$\hat{\Lambda}_n(t) = \int_0^t \frac{dN_n(s)}{Y_n(s)},$$

which is called the Nelson–Aalen estimator of $\Lambda(t)$ in the literature.

For a censored model with $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ independent and identically distributed (i.i.d) sequences and mutually independent, Burke, Csörgő,

and Horváth (1981, 1988) established strong Gaussian approximation of the product-limit (PL) process $\sqrt{n}[\widehat{F}_n(t) - F(t)]$ by a two-parameter Gaussian process at the almost sure rate of $O[n^{-1/2}(\log n)^2]$. In left truncation and right censorship (LTRC) model, Zhou and Yip (1999) initiated and Tse (2003, 2005) established strong Gaussian approximation of the PL-process by a two-parameter Gaussian process at the almost sure rate of $O[(\log n)^{3/2}n^{-1/8}]$, a rate that reflects the two-dimensional nature of the LTRC model.

The quantile function Q and its sample estimator Q_n are defined by

$$Q(p) := \inf\{t : F(t) \geq p\}, \quad Q_n(p) := \inf\{t : \widehat{F}_n(t) \geq p\}$$

for $0 < p < 1$. The role of the quantile function in statistical data modeling was emphasized by Parzen (1979). In econometrics, Gastwirth (1971) used the quantile function to give a succinct definition of the Lorenz curve, which measures inequality in distribution of resources and in size distribution.

In the independent framework with no censoring, the properties of estimator Q_n (where \widehat{F}_n is replaced by the empirical d.f. F_n) have been extensively studied (see, e.g., Csörgő, 1983; Shorack and Wellner, 1986). Under ϕ -mixing condition (for the definition see Doukhan, 1996), the Bahadur representation was obtained by Sen (1972) and the extension to the α -mixing case was obtained by Yoshihara (1995). Under α -mixing condition (see definition below), the strong approximation of the normed quantile process $\rho_n(p) := \sqrt{n}f(Q(p))[Q(p) - Q_n(p)]$ by a two-parameter Gaussian process at the rate $O[(\log n)^{-\lambda}]$ for some $\lambda > 0$ was obtained by Fotopoulos, Ahn, and Cho (1994) and was later improved by Yu (1996).

For a censored model with X_i and Y_i s, independent and identically distributed sequences and mutually independent, Padgett and Thombs (1989) stated the strong consistency and asymptotic normality for a smooth estimator of $Q(p)$. Sander (1975) obtained some asymptotic properties, and Csörgő (1983) and Cheng (1984) discussed strong approximation results with some applications for $Q_n(p)$. In left truncation and right censorship model, Tse (2005) obtained strong Gaussian approximations of the PL-quantile process by a two-parameter Kiefer type process at the rate $O[(\log n)^{3/2}n^{-1/8}]$. Ould-Saïd and Sadki (2005) established the strong consistency and a Bahadur-type representation of K-M quantile function $Q_n(\cdot)$ under a strong mixing hypothesis.

The main aim of this article is to derive strong Gaussian approximations of the PL-process and PL-quantile process, for the case in which the underlying lifetime are assumed to be α -mixing whose definition is given below. As a result, we obtain the law of the iterated logarithm for PL-process.

For easy reference, let us recall the following definition.

Definition 1.1. Let $\{X_i, i \geq 1\}$ denote a sequence of random variables. Given a positive integer m , set

$$\alpha(m) = \sup_{k \geq 1} \{ |P(A \cap B) - P(A)P(B)|; A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+m}^\infty \}, \quad (1.2)$$

where \mathcal{F}_i^k denote the σ -field of events generated by $\{X_j; i \leq j \leq k\}$. The sequence is said to be α -mixing (strongly mixing) if the mixing coefficient $\alpha(m) \rightarrow 0$ as $m \rightarrow \infty$.

Among various mixing conditions used in the literature, α -mixing is reasonably weak and has many practical applications. There exist many processes and time series fulfilling the strong mixing condition. In particular, the stationary autoregressive-moving average (ARMA) processes, which are widely applied in time series analysis, are α -mixing with exponential mixing coefficient; i.e., $\alpha(n) = e^{-\nu n}$ for some $\nu > 0$. The threshold models, the EXPAR models (see Ozaki, 1979), the simple ARCH models (see Engle, 1982; Masry and Tjostheim, 1995, 1997), and their extensions (see Diebolt and Guégan, 1993) and the bilinear Markovian models are geometrically strongly mixing under some general ergodicity conditions. Auestad and Tjostheim (1990) provided excellent discussions on the role of α -mixing for model identification in nonlinear time series analysis.

Now, for the sake of simplicity, the assumptions used in this article are as follows:

Assumptions.

- (1) Suppose that $\{X_i, i \geq 1\}$ is a sequence of stationary α -mixing random variables with continuous distribution function F and mixing coefficient $\alpha_1(n)$.
- (2) Suppose that $\{Y_i, i \geq 1\}$ is a sequence of stationary α -mixing random variables with continuous distribution function G and mixing coefficient $\alpha_2(n)$. Moreover, we assume the censoring times are independent of $\{X_i, i \geq 1\}$.
- (3) $\alpha(n) = O(e^{-(\log n)^{1+\zeta}})$ for some $\zeta > 0$, with $\alpha(n) = \max[\alpha_1(n), \alpha_2(n)]$ (see Remark 2.1. in Ould-Saïd and Sadki, 2005).

Cai (1998, Lemma 1) showed that the Z_i s are α -mixing random variables (with appropriate coefficient α).

The layout of the article is as follows. Section 2 contains main results. The proofs of the main results are relegated to Sec. 3.

2. Main Results

In the first theorem, we construct a two-parameter mean zero Gaussian process that strongly uniformly approximate the empirical processes $Z_{n1}(t) = \sqrt{n}[\widehat{\Lambda}_n(t) - \Lambda(t)]$ and $Z_{n2}(t) = \sqrt{n}[\widehat{F}_n(t) - F(t)]$.

Theorem 2.1. *Suppose that assumptions (1)–(3) are satisfied. On a rich probability space, there exists a two-parameter mean zero Gaussian process $\{B(u, v) \mid u, v \geq 0\}$ such that,*

$$\sup_{t \geq 0} |Z_{n1}(t) - B(t, n)| = O[(\log n)^{-\lambda}] \quad a.s., \quad (2.1)$$

$$\sup_{t \geq 0} |Z_{n2}(t) - [1 - F(t)]B(t, n)| = O[(\log n)^{-\lambda}] \quad a.s. \quad (2.2)$$

Remark 2.1. In the α -mixing case, we cannot achieve the same rate as in the iid case; i.e., $O[n^{-1/2}(\log n)^2]$ (see Burke, Csörgő, and Horváth, 1988, Theorem 2.1). The main reason is that our approach utilizes the strong approximation introduced by Dhompongsa (1984) as a Kiefer process with a negligible reminder term of order $O[n^{-1/2}(\log n)^{-\lambda}]$. This is not as sharp as in iid case.

Corollary 2.1. *Under assumptions (1)–(3), we have,*

$$\sup_{t \in \mathbb{R}} |\widehat{\Lambda}_n(t) - \Lambda(t)| = O\left(\frac{\log \log n}{n}\right)^{1/2} \quad \text{a.s.}, \tag{2.3}$$

$$\sup_{t \in \mathbb{R}} |\widehat{F}_n(t) - F(t)| = O\left(\frac{\log \log n}{n}\right)^{1/2} \quad \text{a.s.} \tag{2.4}$$

In the next theorem, we construct a two-parameter mean zero Gaussian process that strongly uniformly approximates the empirical process $\rho_n(p)$.

Theorem 2.2. *Let $0 < p_0 \leq p_1 < 1$. Under assumptions (1)–(3), assume that F is Lipschitz continuous and that F is twice continuously differentiable on $[Q(p_0) - \delta, Q(p_1) + \delta]$ for some $\delta > 0$ such that f is bounded away from zero. Then there exists a two-parameter mean zero Gaussian process $B(t, u)$ for $t, u \geq 0$ such that,*

$$\sup_{p_0 \leq p \leq p_1} |\rho_n(p) - (1 - p)B[Q(p), n]| = O[(\log n)^{-\lambda}] \quad \text{a.s.}, \tag{2.5}$$

with $\lambda > 0$.

3. Proofs

In order to prove Theorem 2.1, we need the following lemma.

Lemma 3.1 (Theorem 3 in Dhompongsa, 1984). *Under assumptions (1) and (3), there exists a Kiefer process $\{K(s, t), s \in \mathbb{R}, t \geq 0\}$ with covariance function*

$$E[K(s, t)K(s', t')] = \Gamma(s, s') \min(t, t')$$

and $\Gamma(s, s')$ is defined by

$$\Gamma(s, s') = \text{Cov}[g_1(s), g_1(s')] + \sum_{k=2}^{\infty} \{\text{Cov}[g_1(s), g_k(s')] + \text{Cov}[g_1(s'), g_k(s)]\},$$

where $g_k(s) = I(Z_k \leq s) - H(s)$, such that, for some $\lambda > 0$ depending only on v , given in assumption (3),

$$\sup_{t \in \mathbb{R}} |\overline{Y}_n(t) - \overline{H}(t) - K(t, n)/n| = O(b_n), \quad \text{a.s.}$$

where

$$b_n = n^{-1/2}(\log n)^{-\lambda}.$$

Proof of Theorem 2.1. We start with the usual decomposition of $Z_{n1}(t)$.

$$\begin{aligned} Z_{n1}(t) &= \sqrt{n}[\widehat{\Lambda}_n(t) - \Lambda(t)] = \sqrt{n} \left[\int_0^t \frac{d\overline{N}_n(x)}{\overline{Y}_n(x)} - \int_0^t \frac{dF_*(x)}{\overline{H}(x)} \right] \\ &= \int_0^t \frac{\sqrt{n}[\overline{H}(x) - \overline{Y}_n(x)]}{(\overline{H}(x))^2} dF_*(x) + R_{n1}(t) \end{aligned}$$

where

$$\begin{aligned} n^{-1/2}R_{n1}(t) &= \int_0^t \frac{[\bar{Y}_n(x) - \bar{H}(x)]^2}{\bar{Y}_n(x)[\bar{H}(x)]^2} dF_*(x) + \int_0^t \frac{d[\bar{N}_n(x) - F_*(x)]}{\bar{H}(x)} \\ &\quad + \int_0^t \left(\frac{1}{\bar{Y}_n(x)} - \frac{1}{\bar{H}(x)} \right) d[\bar{N}_n(x) - F_*(x)] \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Define, for $t \geq 0$ the sequence of Gaussian processes

$$B(t, n) = \int_0^t \frac{K(x, n)/\sqrt{n}}{[\bar{H}(x)]^2} dF_*(x), \tag{3.1}$$

where $K(s, t)$ is the Kiefer process in Lemma 3.1. Clearly, $E[B(t, n)] = 0$,

$$Cov[B(s, m), B(t, n)] = \sqrt{\frac{m}{n}} \int_0^t \int_0^s \frac{\Gamma(x, y)}{\bar{H}(x)^2 \bar{H}(y)^2} dF_*(x) dF_*(y).$$

where $\Gamma(s, t)$ is defined in Lemma 3.1. Let

$$\beta(t, n) = -\sqrt{n}[\bar{Y}_n(t) - \bar{H}(t)] - K(t, n)/\sqrt{n}.$$

Theorem 2.1 is about the order

$$\sup_{t \geq 0} |\widehat{Z}_{n1}(t) - B(t, n)| = \sup_{t \geq 0} |R_{n1}(t) + R_{n2}(t)|, \tag{3.2}$$

where

$$R_{n2}(t) = \int_0^t \frac{\beta(x, n)}{[\bar{H}(x)]^2} dF_*(x).$$

To deal with $R_{n1}(t)$, we treat $\bar{Y}_n(x)$ as an empirical d.f. associated to Z_i 's and from Theorem 1 in Cai (1998), we have

$$I_1 = O(a_n^2) \quad \text{a.s.}, \tag{3.3}$$

where

$$a_n = \left(\frac{\log \log n}{n} \right)^{1/2}.$$

To estimates of I_2 , divide the interval $[0, t]$ into subintervals $[x_i, x_{i+1}]$, $i = 1, \dots, k_n$ where $k_n = O(a_n^{-1})$, and $0 = x_1 < x_2 < \dots < x_{k_n+1} = t$ are such that $x_{i+1} - x_i = O(b_n)$. Then

$$|I_2| = \left| \int_0^t \frac{d[\bar{N}_n(x) - F_*(x)]}{\bar{H}(x)} \right| \leq \sum_{i=0}^{k_n+1} \left| \int_{x_i}^{x_{i+1}} \frac{d[\bar{N}_n(x) - F_*(x)]}{\bar{H}(x)} \right|.$$

The integral on the right-hand side of the latter inequality is bounded above by $(\log n)^{-\lambda-\beta}/\sqrt{n}$, almost surely (the proof of this can be done using similar arguments to A in Lemma 3.4 in Ould-Saïd and Sadki (2005)). Therefore,

$$I_2 = O[(\log n)^{-\lambda-\beta}] \text{ a.s.} \tag{3.4}$$

The estimate of I_3 is similar to the estimate of I_3 in the proof of Theorem 2 in Cai (1998). Hence,

$$I_3 = O(b_n) \text{ a.s.} \tag{3.5}$$

Therefore, by combining (3.3)–(3.5), we have

$$\sup_{t \geq 0} |R_{n1}(t)| = O[(\log n)^{-\lambda-\beta}] \text{ a.s.} \tag{3.6}$$

Next, by applying Lemma 3.1, we have

$$\sup_{t \geq 0} |R_{n2}(t)| = O[(\log n)^{-\lambda}] \text{ a.s.} \tag{3.7}$$

Combining (3.2), (3.6), and (3.7) we obtain (2.1). It can be shown that

$$\widehat{F}_n(t) - F(t) = [1 - F(t)][\widehat{\Lambda}_n(t) - \Lambda(t)] + O\left(\frac{\log \log n}{n}\right) \text{ a.s.} \tag{3.8}$$

Therefore, (2.2) is proved via (3.8). □

Proof of Corollary 2.1. By the law of the iterated logarithm for Kiefer processes (see, Theorem A in Berkes and Philipp, 1977), and (3.1) we have,

$$\sup_{t \in \mathbb{R}} |B(t, n)| \leq C \sup_{t \in \mathbb{R}} |K(t, n)|/\sqrt{n} = O\left(\frac{\log \log n}{n}\right)^{1/2} \text{ a.s.} \tag{3.9}$$

where C is a positive constant. From (2.1), (2.2), and (3.9) we obtain the results. □

The proof of Theorem 2.2 is mainly based on the following lemmas of Ould-Saïd and Sadki (2005). Lemma 3.2 shows that \widehat{F}_n composed with Q_n is an approximate identity up to order $O(b_n)$. Lemmas 3.3 and 3.4 give global and local bounds for the deviation between Q_n and Q .

Lemma 3.2. *Let $0 < p_0 \leq p_1 < 1$. Under assumptions (1)–(3), assuming that F is continuous, then*

$$\sup_{p_0 \leq p \leq p_1} |\widehat{F}_n[Q_n(p)] - p| = O(b_n) \text{ a.s.}$$

where $b_n = (\log n)^{-\lambda}/\sqrt{n}$ with $\lambda > 0$.

Lemma 3.3. Under assumptions (1)–(3), assuming that $F' = f$ is continuous and strictly positive on $[Q(p_0) - \delta, Q(p_1) + \delta]$ for some $\delta > 0$. Then,

$$\sup_{p_0 \leq p \leq p_1} \sqrt{n} |Q_n(p) - Q(p)| = O(\sqrt{\log \log n}) \quad \text{a.s.}$$

Lemma 3.4. Let $\lambda_n = \text{const.} \cdot b_n$. Under (1)–(3) and assuming that F is Lipschitz continuous on $[0, \tau]$. Then,

$$\sup_{|s-t| \leq \lambda_n} \sqrt{n} |Z_{n2}(s) - Z_{n2}(t)| = O[(\log n)^{-\beta-\lambda}] \quad \text{a.s.},$$

where $\beta > 0$ and $\lambda > 0$.

Proof of Theorem 2.2. We continue to use the notation λ_n as in Lemma 3.4. Let $s = Q_n(p)$ and $t = Q(p)$, $p_0 \leq p \leq p_1$, Lemma 3.3 yields $\sqrt{n}|s - t| = O(\sqrt{\log \log n})$ a.s. Applying Lemma 3.4 gives,

$$\widehat{F}_n[Q_n(p)] - \widehat{F}_n[Q(p)] = F[Q_n(p)] - F[Q(p)] + O[(\log n)^{-\beta-\lambda}/\sqrt{n}] \quad \text{a.s.} \quad (3.10)$$

By Lemma 3.2, $\widehat{F}_n[Q_n(p)]$ can be replaced by p up to $O(b_n)$. For the right-hand side, a Taylor expansion of the first term about $Q(p)$ up to second-order term gives,

$$f[Q(p)][Q(p) - Q_n(p)] + O[(Q_n(p) - Q(p))^2] + O[(\log n)^{-\beta-\lambda}/\sqrt{n}] \quad \text{a.s.},$$

for $p_0 \leq p \leq p_1$. Invoking Lemma 3.3 and rearranging terms in (3.10), we have,

$$\sqrt{n}f[Q(p)][Q_n(p) - Q(p)] = \sqrt{n}\{p - \widehat{F}_n[Q(p)]\} + O[(\log n)^{-\beta-\lambda}] \quad \text{a.s.},$$

for $p_0 \leq p \leq p_1$. Since F is continuous, $F[Q(p)] = p$. Recalling the definitions of the PL process Z_n and PL-quantile process ρ_n , we have,

$$\rho_n(p) = \widehat{Z}_n[Q(p)] + O[(\log n)^{-\beta-\lambda}/\sqrt{n}] \quad \text{a.s.} \quad (3.11)$$

for $p_0 \leq p \leq p_1$. By using Theorem 2.1 and (3.11), theorem is proved. \square

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References

- Auestad, B., Tjøstheim, D. (1990). Identification of nonlinear time series: First order characterisation and order determination. *Biometrika* 77:669–687.
- Berkes, I., Philipp, W. (1977). An almost sure invariance principle for the empirical distribution function of mixing random variables. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 41:115–137.

- Burke, M. D., Csörgő, S., Horváth, L. (1981). Strong approximations of some biometric estimates under random censorship. *Z. Wahrsch. Verw. Gebiete* 56:87–112.
- Burke, M. D., Csörgő, S., Horváth, L. (1988). A correction to and improvement of “Strong Approximations of Some Biometric Estimates under Random Censorship.” *Probab. Theor. Relat. Field.* 79:51–57.
- Cai, Z. (1998). Asymptotic properties of Kaplan–Meier estimator for censored dependent data. *Stat. Probab. Lett.* 37:381–389.
- Cheng, K. F. (1984). On almost sure representation for quantiles of the product limit estimator with applications. *Sankhya* 46:426–443.
- Csörgő, M. (1983). *Quantile Processes with Statistical Applications*. Philadelphia: SIAM.
- Dhompongsa, S. (1984). A note on the almost sure approximation of the empirical process of weakly dependent random variables. *Yokohama Mathematical Journal* 32:113–121.
- Diebolt, J., Guégan, D. (1993). Tail behaviour of the stationary density of general nonlinear autoregressive processes of order 1. *J. Appl. Probab.* 30:315–329.
- Doukhan, P. (1996). *Mixing: Properties and Examples*. Lecture Notes in Statistics, Vol. 61. New York: Springer Verlag.
- Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrics* 50(4):987–1007.
- Fotopoulos, S., Ahn, S. K., Cho, S. (1994). Strong approximation of the quantile processes and its applications under strong mixing properties. *J. Multivariate Anal.* 51:17–47.
- Gastwirth, J. L. (1971). A general definition of the Lorenz curve. *Econometrica* 39:1037–1039.
- Gill, R. D. (1980). Censoring and stochastic integrals. *Mathematical Centre Tracts*. Vol. 124. Amsterdam: Mathematisch Centrum.
- Masry, E., Tjøstheim, D. (1995). Nonparametric estimation and identification of nonlinear ARCH time series: Strong convergence and asymptotic normality. *Econ. Theor.* 11:258–289.
- Masry, E., Tjøstheim, D. (1997). Additive nonlinear ARX time series and projection estimates. *Econ. Theor.* 13:214–256.
- Ould-Saïd, E., Sadki, O. (2005). Strong approximation of quantile function for strong mixing and censored processes. *Comm. Stat. Theor. Meth.* 34:1449–1459.
- Ozaki, T. (1979). Nonlinear time series models for nonlinear random vibrations. Technical Report. Manchester, UK: University of Manchester.
- Padgett, W. T., Thombs, L. A. (1989). A smooth nonparametric quantile estimation from right censored data. *Stat. Probab. Lett.* 7:113–121.
- Parzen, E. (1979). Nonparametric statistical data modeling. *J. Am. Stat. Assoc.* 74:105–131.
- Sander, J. (1975). The weak convergence of quantiles of the product limit estimator. Technical Report Number 5. Palo Alto, CA: Stanford University, Department of Statistics.
- Sen, P. K. (1972). On the Bahadur representation of samples quantiles for sequences of α -mixing random variables. *J. Multivariate Anal.* 2:77–95.
- Shorack, G. R., Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics*. New York: Wiley-Interscience.
- Tse, S. M. (2003). Strong Gaussian approximations in the left truncated and right censored model. *Statistica Sinica* 13:275–282.
- Tse, S. M. (2005). Quantile processes for left truncated and right censored data. *Ann. Inst. Stat. Math.* 57:61–69.
- Yoshihara, K. I. (1995). The Bahadur representation of sample quantiles for sequences of strongly mixing random variables. *Stat. Probab. Lett.* 24:299–304.
- Yu, H. (1996). A note on strong approximation for quantile processes of strong mixing sequences. *Stat. Probab. Lett.* 30, 1–7.
- Zhou, Y., Yip, S. F. (1999). A strong representation of the product-limit estimator for left truncated and right censored data. *J. Multivariate Anal.* 69:261–280.