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Adaptive Mesh Redistribution with Upwinding Scheme for Convection-Diffusion Equation

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Abstract

It is well known that moving mesh and upwinding schemes are two kinds of techniques for tracking the shock or wave front in the solution of PDEs. It is expected that their combination should produce more robust methods. Several upwinding scheme are considered for non-uniform meshes. Adaptive Mesh Redistribution method is also described. Numerical examples are given to illustrate the accuracy and effectiveness of the proposed algorithm.

Mathematics Subject Classification: 65M60

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1 Introduction

In this paper we consider a simple one-dimensional convection-diffusion equation:

$$u_t + f(u)_x = \epsilon(\sigma(u)u_x)_x,$$

where $0 < \epsilon \ll 1$ is a (small) viscosity. Adaptive Mesh Redistribution (AMR) methods have important application in a variety of physical and engineering areas. The physical phenomena in these areas develop dynamically singular or nearly singular solutions in fairly localized region. The numerical investigation of these physical problems may require extremely fine meshes over a small portion of the physical domain to resolve the large solution variations. In this work, we will develop an efficient AMR algorithm to solve convection-diffusion problems. Although the moving mesh method can place enough nodes in the wave front and works very well for a convex flux function and sufficiently smooth initial conditions, for nonconvex flux functions or piecewise initial conditions, there are still some oscillations appearing in the solution. High-order upwinding schemes are needed in these cases.

2 Adaptive Mesh Redistribution method

The Adaptive Mesh Redistribution(AMR) scheme consists of two independent part: a PDE evolution and a mesh-redistribution. We have following (explicit)finite volume method:

$$u^{n+1}_{i+\frac{1}{2}} = u^n_{i+\frac{1}{2}} - \frac{\Delta t^n}{|I_{i+\frac{1}{2}}|}(F^n_{i+1} - F^n_i) + \epsilon((\sigma(u)u_x)^n_{i+1} - (\sigma(u)u_x)^n_i)), \tag{1}$$

where F_i is the numerical flux. We further describe the mesh redistribution at each time step. The mesh generation equation, based on the standard equidistribution principle, is

$$(Mx_\xi)_\xi = 0, \qquad \xi \in [0,1]$$

where the function M is called monitor function. After obtaining the new grid $\{\tilde{x}_i\}$, we need to update u at the grid point $\tilde{x}_{i+\frac{1}{2}} = (\tilde{x}_i + \tilde{x}_{i+1})/2$ based on the knowledge of $\{x_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\}$. We use the second-order conservation interpolation formula in the following:

$$\Delta\tilde{x}_{i+\frac{1}{2}}\tilde{u}_{i+\frac{1}{2}} = \Delta x_{i+\frac{1}{2}} u_{i+\frac{1}{2}} - ((cu)_{i+1} - (cu)_i),$$

where $\Delta\tilde{x}_{i+\frac{1}{2}} = \tilde{x}_{i+1} - \tilde{x}_i, \quad c_i = x_i - \tilde{x}_i$. In the actual computation,the linear flux cu is approximated by an upwinding scheme, see [2]:

$$(\hat{cu})_i = \frac{c_i}{2}(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}) - \frac{|c_i|}{2}(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}).$$

3 Second-Order MUSCL Scheme

In the section , we consider the second-order MUSCL (Monotonic Upstream-Centered Scheme for Conservation Laws) method to reconstruct the numerical fluxes[3]. Thus

$$F_{i+1} = \frac{f(u_{i+\frac{1}{2}}) + f(u_{i+\frac{3}{2}})}{2} - \lambda_{i+\frac{1}{2}}\frac{u_{i+\frac{3}{2}} - u_{i+\frac{1}{2}}}{2} + \frac{\sigma^+_{i+\frac{1}{2}} - \sigma^-_{i+\frac{3}{2}}}{4}, \tag{2}$$

$$F_i = \frac{f(u_{i-\frac{1}{2}}) + f(u_{i+\frac{1}{2}})}{2} - \lambda_{i+\frac{1}{2}}\frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{2} + \frac{\sigma^+_{i+\frac{1}{2}} - \sigma^-_{i-\frac{1}{2}}}{4}, \tag{3}$$

where we used the Sweby's notation to define slopes of the solution as

$$\sigma^+_{i+\frac{1}{2}} = (f(u_{i+\frac{3}{2}}) + \lambda_{i+\frac{1}{2}} u_{i+\frac{3}{2}} - f(u_{i+\frac{1}{2}}) - \lambda_{i+\frac{1}{2}} u_{i+\frac{1}{2}})\phi(\theta^+_{i+\frac{1}{2}}),$$

$$\sigma^-_{i+\frac{1}{2}} = (f(u_{i+\frac{3}{2}}) - \lambda_{i+\frac{1}{2}} u_{i+\frac{3}{2}} - f(u_{i+\frac{1}{2}}) + \lambda_{i+\frac{1}{2}} u_{i+\frac{1}{2}})\phi(\theta^-_{i+\frac{1}{2}}),$$

where

$$\theta^+_{i+\frac{1}{2}} = \frac{f(u_{i+\frac{1}{2}}) + \lambda_{i+\frac{1}{2}} u_{i+\frac{1}{2}} - f(u_{i-\frac{1}{2}}) - \lambda_{i+\frac{1}{2}} u_{i-\frac{1}{2}}}{f(u_{i+\frac{3}{2}}) + \lambda_{i+\frac{1}{2}} u_{i+\frac{3}{2}} - f(u_{i+\frac{1}{2}}) - \lambda_{i+\frac{1}{2}} u_{i+\frac{1}{2}}}$$

$$\theta^-_{i+\frac{1}{2}} = \frac{f(u_{i+\frac{1}{2}}) - \lambda_{i+\frac{1}{2}} u_{i+\frac{1}{2}} - f(u_{i-\frac{1}{2}}) + \lambda_{i+\frac{1}{2}} u_{i-\frac{1}{2}}}{f(u_{i+\frac{3}{2}}) - \lambda_{i+\frac{1}{2}} u_{i+\frac{3}{2}} - f(u_{i+\frac{1}{2}}) + \lambda_{i+\frac{1}{2}} u_{i+\frac{1}{2}}}$$

and ϕ represents a slop limiter function chosen to be the Van-Leer limiter

$$\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}.$$

That λ_i is the characteristic speed as $\lambda_{i+\frac{1}{2}} = \max_i |f'(u_{i+\frac{1}{2}})|$,

4 Second-Order Lax-Friedrich Scheme

second-order Lax-Friedrich method to reconstruct the numerical fluxes is:

$$F(u_i^-, u_i^+) = \frac{1}{2}[f(u_i^-) + f(u_i^+) - max|f_u|.(u_i^+ - u_i^-)], \tag{4}$$

where the maximum is taken between u_i^- and u_i^+. in order to approximate the flux, we reconstruct a linear approximation in each cell:

$$u_i^- = u_{i-\frac{1}{2}} + \frac{1}{2}s_{_{i-\frac{1}{2}}}(x_i - x_{_{i-\frac{1}{2}}}), \quad u_i^+ = u_{i+\frac{1}{2}} + \frac{1}{2}s_{_{i+\frac{1}{2}}}(x_i - x_{_{i+\frac{1}{2}}})$$

where

$$s^+_{i+\frac{1}{2}} = \frac{u_{i+\frac{3}{2}} - u_{i+\frac{1}{2}}}{x_{_{i+\frac{3}{2}}} - x_{i+\frac{1}{2}}}, \quad s^-_{i+\frac{1}{2}} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{x_{_{i+\frac{1}{2}}} - x_{i-\frac{1}{2}}},$$

$$s_{i+\frac{1}{2}} = (sign(s^-_{i+\frac{1}{2}}) + sign(s^+_{i+\frac{1}{2}}))\frac{|s^-_{i+\frac{1}{2}}.s^+_{i+\frac{1}{2}}|}{|s^-_{i+\frac{1}{2}}| + |s^+_{i+\frac{1}{2}}|}.$$

5 Third-Order Piecewise Hyperbolic Method

The PHM (Piecewise Hyperbolic Method) constructs numerical flux function as

$$F_{i+\frac{1}{2}} = f_{i+\frac{1}{2}}(u_L, u_R) = \frac{1}{2}(f(u_L) + f(u_R)). \tag{5}$$

let $h_i = x_i - x_{i-1}$. then we have

$$u_L = u_i + \frac{h_{i+1}}{2}\frac{\partial u}{\partial x}|_{x=x_i} = u_i + \frac{h_{i+1}}{2}(\frac{u_i - u_{i-1}}{h_i}).B(r_i),$$

where $B(r_i)$ is a flux limiter defined by

$$B(r_i) = \frac{r_i + |r_i|}{1 + |r_i|}$$

and

$$r_i = \frac{u^c_{i+\frac{1}{2}} - u_i}{u^u_{i+\frac{1}{2}} - u_i} = \frac{(u_{i+1} - u_i)/h_{i+1}}{(u_i - u_{i-1})/h_i},$$

Where $u^c_{i+1} = \frac{1}{2}(u_i + u_{i+1})$ is the central difference term, and

$$u^u_{i+1} = u_i + (h_{i+1}/2)((u_{i+1} - u_i)/h_i)$$

is the left upwinding term. Similarly,

$$u_R = u_{i+1} - \frac{h_{i+1}}{2}(\frac{u_{i+2} - u_{i+1}}{h_{i+2}}).B(\frac{1}{r_{i+1}}).$$

in the next section, we are going to used AMR method and upwinding schemes which was introduced.

6 Numerical Results

Example 6.1

$$u_t + (u^2/2)_x = \epsilon u_{xx}, \qquad x \in (-2, 2), \quad t \in (0, 2.5)$$

with boundary conditions $u(2,t) = 0, u(-2,t) = 1$ and initial condition $u(x,0) = 1$ if $x \leq 0$ and $u(x,0) = 0$ if $x > 0$, and the monitor function used is $\sqrt{1 + 80u_\xi^2}$

Example 6.2

$$u_t + f(u)_x = \epsilon(\sigma(u)u_x)_x, \quad f(u) = \frac{u^2}{u^2 + (1-u)^2}, \quad \sigma(u) = 4u(1-u).$$

The initial function is

$$u(x,0) = \begin{cases} 1 - 3x & 0 \leq x \leq \frac{1}{3} \\ 0 & \frac{1}{3} < x \leq 1 \end{cases}$$

and the boundary value of $u(0,t) = 1$ is kept fixed.and the monitor function used is $\sqrt{1 + 50u_\xi^2}$.

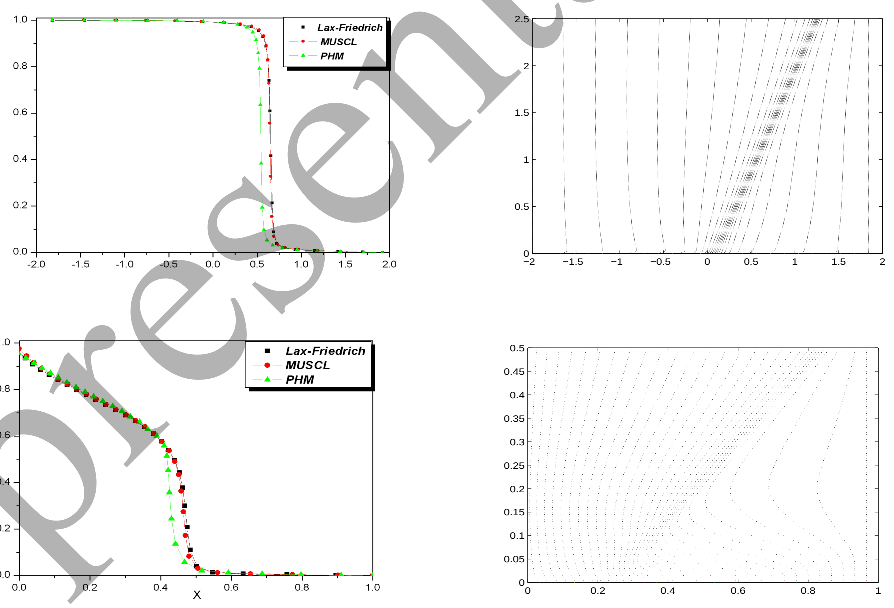

Figure 1: (Up) Example 6.1: adaptive mesh solutions at $t = 1.2$ with upwind schemes and mesh trajectory with $N = 25$, (Down) Example 6.2: adaptive mesh solution at $t = 0.2$ and mesh trajectory with $N = 30$.

As desired, considerable portion of grid points has been moved to steep gradient region, the ability of the AMR method to capture and follow the moving large gradient is clearly demonstrated. Also with comparison the obtained results from three schemes we notice, that third-order PHM can resolve the large gradient very good and the results of the Lax-friedrich method is better than the MUSCL method.

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