Extended Abstarct Of
4th Iranian Conference on Applied Mathematics
10-12 March 2010

University Of
Sistan & Balochestan

A study of blowup in reaction diffusion equations with adaptive mesh method

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Abstract

A new concept called the dominance of equidistribution is introduced for analyzing moving mesh partial differential equations for numerical simulation of blowup in reaction diffusion. Theoretical and numerical results show that a moving mesh works successfully when the employed moving mesh equation has the dominance of equidistribution.

Keywords: Moving mesh, Blowup problem , Dominance of equidistribution .

1 Introduction

Moving mesh methods have been proved to be very efficient in resolving singular solutions to reaction-diffusion equations. In this paper, we are interested in the solution of blowup problems and focus particularly on the MMPDE (moving mesh PDE) developed in [1].
An outline of the paper is as follows: In the section 2, we describe the MMPDE method for classic problem with blowup solutions. Theoretical and numerical analysis of MMPDE5 for constant τ is given in section 3. Additional conclusions are contained in the final section.

2 Moving mesh PDE method

We study the moving mesh method for classic a blowup problem:

$$u_t = u_{xx} + \exp(u), \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x) > 0. \tag{1}$$

Theorem 2.1 *Let u(x,t) be the solution of (1); then*

$$\lim_{t\to T}\left[u\left(x^* + \eta\left[(T-t)(\alpha - \log(T-t))\right]^{1/2}, t\right) + \log(T-t)\right] = -\log\left(1 + \frac{|\eta|^2}{4}\right) \tag{2}$$

uniformly on compacts in η, where α is a constant depending only the initial solution.

The theorem shows that the blowup profile can best be shown in the so-called kernel coordinate $\eta = \eta(x,t)$, which is fixed as t→ T and defined as

$$\eta = (x - x^*)\left[(T-t)(\alpha - \log(T-t))\right]^{-1/2}. \tag{3}$$

Define a one-to-one coordinate transformation by

$$x = x(\xi,t), \quad \xi \in [0,1], \quad x(0,t) = 0, \quad x(1,t) = 1. \tag{4}$$

Transforming (1) from the physical coordinates (x,t) to the computational ones (ξ,t), we have

$$\dot{u} - \frac{u_\xi}{x_\xi}\dot{x} = \frac{1}{x_\xi}\frac{\partial}{\partial\xi}\left(\frac{u_\xi}{x_\xi}\right) + \exp(u)\ . \tag{5}$$

The MMPDE5, which is considered in this paper, as

$$\tau\dot{x} = \frac{\partial}{\partial\xi}\left(M\frac{\partial x}{\partial\xi}\right), \tag{6}$$

We consider the monitor function in the form

$$M(x,t) = \exp(u). \tag{7}$$

MMPDE5 is by adding a mesh speed term to the equidistribution principle that takes the form

$$\frac{\partial}{\partial\xi}\left(M\frac{\partial x}{\partial\xi}\right) = 0, \tag{8}$$

subject to the boundary conditions (4). The resulting coordinate transformation takes the form

$$x(\xi,t) = x^* + (T-t)^{1/2}\left[\alpha - \log(T-t)\right]^{1/2} z(\xi,t), \tag{9}$$

with the property

$$z(\xi,t) = z_0(\xi) + O(\tau) + o(1). \tag{10}$$

3 Moving mesh PDEs with constant τ

As mentioned in Section 2, the solution profile in the peak region of blowup can be properly resolved in the computational coordinate ξ when the coordinate transformation is of form (9) with property (10). The MMPDE5 has the form (6). Expanding the derivative on the right-hand side gives

$$\tau\dot{x} = M\frac{\partial^2 x}{\partial\xi^2} + \frac{\partial M}{\partial\xi}\frac{\partial x}{\partial\xi}. \tag{11}$$

We seek a coordinate transformation in form (9). By differentiating (9) with respect to t and ξ, we have

$$\dot{x} = \tfrac{1}{2}(T-t)^{-1/2}\left[\alpha - \log(T-t)\right]^{1/2}\left[-1 + \left[\alpha - \log(T-t)\right]^{-1}\right] z + (T-t)^{1/2}\left[\alpha - \log(T-t)\right]^{1/2}\dot{z}, \tag{12}$$

$$x_\xi = (T-t)^{1/2}\left[\alpha - \log(T-t)\right]^{1/2} z_\xi, \quad x_{\xi\xi} = (T-t)^{1/2}\left[\alpha - \log(T-t)\right]^{1/2} z_{\xi\xi}. \tag{13}$$

Using the exact form (2) for the solution $u(x,t)$, from (9) and (7) we have

$$u(x,t) = -\log(1 + \frac{|z|^2}{4}) - \log(T-t), \tag{14}$$

$$M = \frac{1}{(T-t)(1+\frac{|z|^2}{4})}, \quad M_\xi = \frac{-2zz_\xi}{4(T-t)(1+\frac{|z|^2}{4})^2}. \tag{15}$$

Inserting these results into (11) and after simplification yields

$$\frac{\tau}{2}\left[-1 + \left[\alpha - \log(T-t)\right]^{-1}\right] z + \tau(T-t)\dot{z} = \frac{-2zz_\xi^2}{4(1+\frac{|z|^2}{4})^2} + \frac{z_{\xi\xi}}{(1+\frac{|z|^2}{4})} + o(1), \tag{16}$$

It is not difficult to see that (16) permits a formal expansion for $z_0(\xi)$ as

$$z(\xi,t) = z_0(\xi) + o(1), \tag{17}$$

where $z_0(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2 z_0}{d\xi^2} = -\frac{\tau}{2}(1+\frac{z_0^2}{4})z_0 + \frac{2z_0}{4(1+\frac{z_0^2}{4})}\left(\frac{dz_0}{d\xi}\right)^2. \tag{18}$$

This, combined with expansion (17) and the fact that (9) is valid only in the blowup region, suggests

$$z_0(\xi^L) = z_0^L, \quad z_0(\xi^R) = z_0^R, \tag{19}$$

where ξ^L, ξ^R, z_0^L and z_0^R are bounded constants with $-\infty < z_0^L,\ z_0^R < \infty$ and $0 < \xi^L < \xi^R < 1$. Hence, ξ^l is close to zero and ξ^R close to 1, i.e.,

$$\xi^L \approx 0, \quad \xi^R \approx 1. \tag{20}$$

In form (9), they must correspond to the limits of large $|z|$:$z(0,t) = -\infty$ and $z(1,t) = \infty$. From (17) and (20) z_0^L and z_0^R , although bounded, should be very large, viz.,

$$z_0^L \approx -\infty, \quad z_0^R \approx \infty. \tag{21}$$

Letting $v = \frac{dz_0}{d\xi}$, (18) becomes

$$\frac{dv}{dz_0} - \frac{2zv}{4+z_0^2} = -\frac{\tau}{2}(1+\frac{z_0^2}{4})z_0\frac{1}{v},$$

This gives

$$(\frac{dz_0}{d\xi})^2 = (4+z_0^2)^2(-\frac{\tau}{8}\log(|4+z_0^2|)+C), \tag{22}$$

where C is an integration constant. For large τ, we consider the ODE (22) with boundary conditions (19). Since the approximation (21) is valid, the constant C must be positive and large to keep the right-hand side term of ODE (22) positive. Therefore, the ODE (22) with boundary conditions (21) has no solutions. Next, we consider the situation when τ is sufficiently small. The boundary conditions (19) then imply $z_0^L \le z_0 \le z_0^R$. Thus, when τ is sufficiently small such that

$$\frac{\tau}{8}\log(|4+(z_0^L)^2|) \ll 1, \quad \frac{\tau}{8}\log(|4+(z_0^R)^2|) \ll 1,$$

(22) can be written as

$$\frac{d\xi}{dz_0} = \left[C + O(\tau)\right]^{-1/2}(4+z_0^2)^{-1} \tag{23}$$

Integrating this equation about z_0 from z_0^L to z_0^R and applying the boundary conditions (19) give

$$\frac{\xi-\xi^L}{\xi^R-\xi^L} = \frac{\arctan\frac{z_0}{2} - \arctan\frac{z_0^L}{2}}{\arctan\frac{z_0^R}{2} - \arctan\frac{z_0^L}{2}} + O(\tau) \tag{24}$$

Using the approximations (20) and (21), we obtain

$$z_0(\xi) \approx 2\tan(\pi(\xi - \frac{1}{2})) + O(\tau). \tag{25}$$

Then the coordinate transformation and the physical solution in the peak region of blowup satisfy

$$x(\xi,t) = x^* + (T-t)^{1/2}\left[\alpha - \log(T-t)\right]^{1/2}\left(2\tan(\pi(\xi-\frac{1}{2})) + O(\tau)\right), \tag{26}$$

$$u(x(\xi,t),t) = -\log\left((T-t)(1+\tan^2(\pi(\xi-\frac{1}{2}))) + O(\tau)\right). \tag{27}$$

3.1 Numerical examples

The initial solution is taken as $u_0(x) = 5\sin(\pi x)$. We also plot $|x_i - x^*|$ against $e^{u_{max}}$ in logarithmic scale. To explain these functions, we take (26) and (27). Then we have $e^{u_{max}} \approx (T-t)^{-1}$ or $(T-t) \approx e^{-u_{max}}$. It follows that, as $t \to T$,

$$e^{u-u_{\max}} \to \cos^2(\pi(\xi - \frac{1}{2})), \quad \log|x_i - x^*| \to -\frac{1}{2}\log(e^{u_{\max}}) + d_i,$$

where d_i is a constant depending on ξ. Thus, when MMPDE5 works satisfactorily, the computed solution $e^{(u-u_{max})}$ converges to a steady-state profile $\cos^2(\pi(\xi - \frac{1}{2}))$ while $\log|x_i - x^*|$ is becoming linear in $\log(e^{u_{max}})$ for most mesh points in the limit $t \to T$.

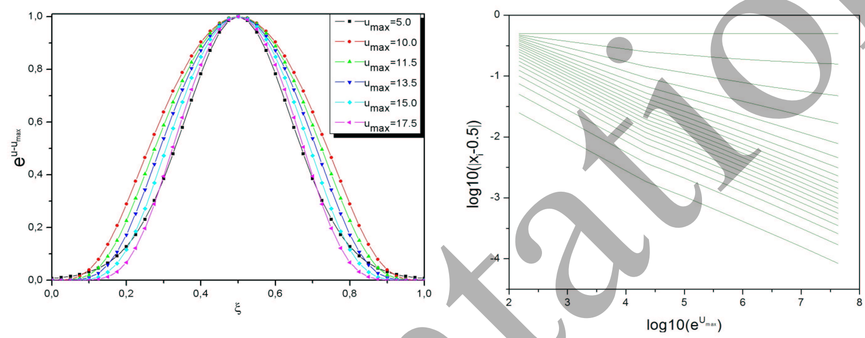

Figure 1: MMPDE5, $\tau = 10^{-2}$, M=exp(u)

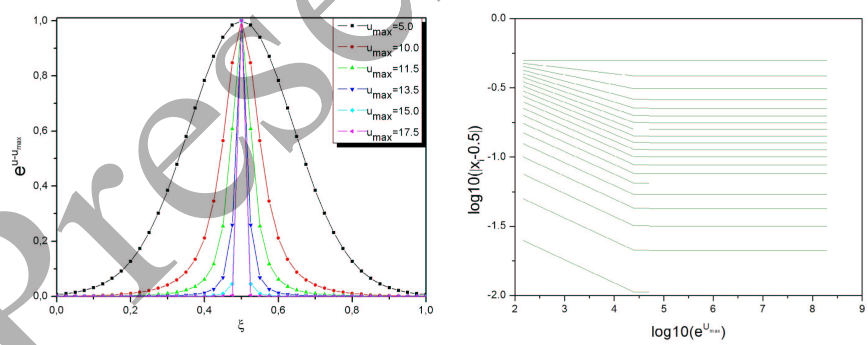

Figure 2: MMPDE5, $\tau = 10^2$, M=exp(u)

References

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