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# HOMOTOPY PERTURBATION METHOD (HPM) FOR SOLVING HIGHER ORDER BOUNDARY VALUE PROBLEMS (BVP)

**JAFAR SABERI-NADJAFI and SHIRIN ZAHMATKESH**

Department of Applied Mathematics
School of Mathematical Sciences
Ferdowsi University of Mashhad
Mashhad, Iran
E-mails: najafi141@gmail.com
zahmatkesh77@yahoo.com

## Abstract

In this paper, we apply the homotopy perturbation method for solving the boundary value problems of higher order by reformulating them as an equivalent system of integral equations. One can obtain this equivalent formulation by using a suitable transformation. The analytical results of the integral equations have been obtained in terms of convergent series with easily computable components. Few examples are given to illustrate the efficiency and implementation of the homotopy perturbation method. Comparisons are made to confirm the reliability of the homotopy perturbation method.

## 1. Introduction

In recent years, much attention has been given to develop some analytical methods for solving integral equations including the perturbation methods and the Adomian decomposition method [19]. It is well known that perturbation methods [13, 14] provide the most versatile tools available in nonlinear analysis of engineering problems. The major drawback in the traditional perturbation technique is the over dependence on the existence of small parameter. This

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condition is over-strict and greatly affects the applications of the perturbation techniques because most of the nonlinear problems (especially those having strong nonlinearity) do not even contain the so-called small parameter; moreover, the determination of the small parameter is a complicated process and requires special techniques.

These facts have motivated to suggest alternative techniques such as the homotopy analysis method [11, 12], decomposition and the variational iteration method [8, 9, 10, 22]. In order to overcome these drawbacks, combining the standard homotopy method and perturbation, there are several different modified method have been made, which are called the *homotopy perturbation method* [20, 23, 24]. This technique has been used by Noor and Mohyud-Din [15-18] for solving boundary value problems of three various orders. Using the idea of Noor and Mohyud-Din [15-18], we develop a homotopy perturbation method for solving a system of integral equations associated with higher order boundary value problems. It is shown that this method provides the solution in a rapid convergent series. We show that this method is easy to implement and it is more efficient than the Adomian method. We remark that to apply the Adomian method, one has to evaluate the derivative of the so-called *Adomian polynomial*, which is itself a complicated problem. On the other hand, homotopy perturbation is very simple to apply [22], which is the main characteristic of this method. Several examples are given to illustrate the performance of the method.

Now suppose that a uniform magnetic field is also applied across the fluid [25] in the same direction as gravity. When instability sets in now as ordinary convection, it is modelled by a tenth-order boundary value problem; when instability sets in as over-stability, it is modelled by a $12^{th}$ order boundary value problem. For more details about the occurrences of high-order boundary value problems, see [4, 5, 7]. An eighth-order differential equation occurring in torsional vibration of uniform beams was investigated by Bishop, see [3]. A class of characteristic-value problems of high order (as high as 24) are known to arise in hydrodynamic and hydromagnetic stability [4, 5, 7]. The literature of numerical analysis contains little on the solution of the high-order boundary value problems [4, 7 and 26]. Research in this direction may be considered in its early stages. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are contained in a comprehensive survey in a book by Agarwal [1], though no numerical methods are contained therein for solving boundary value problems of higher order.

Recently, the boundary value problems of higher order have been investigated because of both their mathematical importance and their potential for applications in hydrodynamic and hydromagnetic stability. Baldwin [2] applied global phase-integral methods for solving BVPs of sixth order. However, numerical methods of solution were introduced implicitly by Chawla and Katti [6], although the authors focused their attention on fourth-order BVPs. Computational results have also been obtained by [7] for special nonlinear boundary value problems of order 2m by using finite-difference methods. In a later work [25], Octic splines solutions of linear eighth-order boundary value problems were implemented and the obtained results produced improvements over finite differences method. The spline function values at the midknots of the interpolation interval and the corresponding values of the even-order derivatives are related through consistency relations. Generally speaking, a considerable amount of interest [1-7, 25] was directed towards the use of finite differences methods and the spline solutions to handle boundary value problems of higher order. The present work is motivated by the desire to obtain numerical solutions to higher-order boundary value problems with a better accuracy level. In this paper, we use the homotopy perturbation method coupled with the integral equations to solve the higher order boundary value problems. Several examples are given to illustrate the performance and efficiency of the method. Our experience shows that the homotopy perturbation technique can be considered as an alternative to decomposition and variational iteration techniques.

## 2. Homotopy Perturbation Method

Consider the following system of the integral equations

$$F(t) = G(t) + \lambda \int_0^t K(t, s) F(s) ds, \tag{2.1}$$

where

$$F(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T,$$

$$G(t) = (g_1(t), g_2(t), \ldots, g_n(t))^T,$$

$$K(t, s) = [k_{ij}(t, s)], \quad i = 1, 2, 3, \ldots, n, \quad j = 1, 2, 3, \ldots, n. \tag{2.2}$$

To convey an idea of the homotopy perturbation method, we consider a general equation of the type

$$L(u) = 0, \tag{2.3}$$

where $L$ is an integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{2.4}$$

where $F(u)$ is a functional operator with known solutions $v_0$, which can be obtained easily. It is clear that

$$H(u, p) = 0, \tag{2.5}$$

from which we have $H(u, 0) = F(u)$ and $H(u, 1) = L(u)$.

This shows that $H(u, p)$, continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution $H(f, 1)$ where $f$ will be defined soon. The embedding parameter increases monotonically from zero to unit as the problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The embedding parameter can be considered as an expanding parameter .The homotopy perturbation method uses the homotopy parameter $p$ as an expanding parameter [10] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \dots. \tag{2.6}$$

If $p \to 1$, then (2.5) corresponds to (2.3) and becomes the approximate solution of the following form

$$f = \lim_{p \to l} u = \sum_{i=0}^{\infty} u_i. \tag{2.7}$$

It is well known that the series (2.7) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$ see [7], we assume that problem (2.1) has a unique solution. Consider the $i$-th equation of (2.1), we take

$$\begin{cases} f_1(t) = \sum_{i=0}^{\infty} p^i u_i, \\ f_2(t) = \sum_{i=0}^{\infty} p^i v_i, \\ f_3(t) = \sum_{i=0}^{\infty} p^i s_i, \\ \vdots \end{cases} \tag{2.8}$$

The comparisons of like powers of $p$ give solutions of the various orders.

## 3. Applications

In the examples that follow, the homotopy perturbation method will be tested by discussing three boundary value problems of 9th-order, 10th-order, and 12th-order, respectively. In the first example, boundary conditions at any-order derivative are used.

However, boundary conditions at evenorder derivatives are given in the last two examples.

**Example 1.** We first consider the linear ninth-order BVP

$$y^{(9)}(x) = -9e^x + y(x), \ \ 0 < x < 1,$$

subject to the boundary conditions

$$y^{(j)}(0) = (1 - j), \ \ j = 0, 1, 2, 3, 4,$$

$$y^{(j)}(1) = -je, \ \ j = 0, 1, 2, 3.$$

The exact solution for this problem is

$$y(x) = (1 - x)e^x.$$

We define

$$\frac{dy}{dx} = z_1(x), \ \ \frac{dz_1}{dx} = z_2(x), \ \ \frac{dz_2}{dx} = z_3(x), \ \ \frac{dz_3}{dx} = z_4(x), \ \ \frac{dz_4}{dx} = z_5(x),$$

$$\frac{dz_5}{dx} = z_6(x), \ \ \frac{dz_6}{dx} = z_7(x), \ \ \frac{dz_7}{dx} = z_8(x), \ \ \frac{dz_8}{dx} = -9e^x + y(x),$$

then we can rewrite the 9th-order BVP as the system of the following integral equations

$$\begin{cases} y(x) = 1 + \int_0^x z_1(t)dt, \\ z_1(x) = \int_0^x z_2(t)dt, \\ z_2(x) = -1 + \int_0^x z_3(t)dt, \\ z_3(x) = -2 + \int_0^x z_4(t)dt, \\ z_4(x) = -3 + \int_0^x z_5(t)dt, \\ z_5(x) = A + \int_0^x z_6(t)dt, \\ z_6(x) = B + \int_0^x z_7(t)dt, \\ z_7(x) = C + \int_0^x z_8(t)dt, \\ z_8(x) = D + \int_0^x (-9e^t + y(t))dt. \end{cases}$$

Using the homotopy perturbation method, we obtain

$$\begin{cases} (y_0 + py_1 + p^2y_2 + \ldots) = 1 + p\int_0^x (z_{10} + pz_{11} + p^2z_{12} + \ldots)dt, \\ (z_{10} + pz_{11} + p^2z_{12} + \ldots) = p\int_0^x (z_{20} + pz_{21} + p^2z_{22} + \ldots)dt, \\ (z_{20} + pz_{21} + p^2z_{22} + \ldots) = -1 + p\int_0^x (z_{30} + pz_{31} + p^2z_{32} + \ldots)dt, \\ (z_{30} + pz_{31} + p^2z_{32} + \ldots) = -2 + p\int_0^x (z_{40} + pz_{41} + p^2z_{42} + \ldots)dt, \\ (z_{40} + pz_{41} + p^2z_{42} + \ldots) = -3 + p\int_0^x (z_{50} + pz_{51} + p^2z_{52} + \ldots)dt, \\ (z_{50} + pz_{51} + p^2z_{52} + \ldots) = A + p\int_0^x (z_{60} + pz_{61} + p^2z_{62} + \ldots)dt, \\ (z_{60} + pz_{61} + p^2z_{62} + \ldots) = B + p\int_0^x (z_{70} + pz_{71} + p^2z_{72} + \ldots)dt, \\ (z_{70} + pz_{71} + p^2z_{72} + \ldots) = C + p\int_0^x (z_{80} + pz_{81} + p^2z_{82} + \ldots)dt, \\ (z_{80} + pz_{81} + p^2z_{82} + \ldots) = D + p\int_0^x (-9e^t + (y_0 + py_1 + p^2y_2 + \ldots))dt. \end{cases}$$

Comparing the coefficient of like powers of $p$, we get

$$p^{(0)}:\begin{cases} y_0 = 1 \\ z_{10} = 0 \\ z_{20} = -1 \\ z_{30} = -2 \\ z_{40} = -3, \\ z_{50} = A \\ z_{60} = B \\ z_{70} = C \\ z_{80} = D \end{cases} \qquad p^{(1)}:\begin{cases} y_1 = 0 \\ z_{11} = -x \\ z_{21} = -2x \\ z_{31} = -3x \\ z_{41} = Ax \\ z_{51} = Bx \\ z_{61} = Cx \\ z_{71} = Dx \\ z_{81} = -9e^x + 9 + x \end{cases},$$

$$p^{(2)}:\begin{cases} y_2 = \dfrac{-x^2}{2} \\ z_{12} = -x^2 \\ z_{22} = \dfrac{-3x^2}{2} \\ z_{32} = \dfrac{Ax^2}{2} \\ z_{42} = \dfrac{Bx^2}{2} \\ z_{52} = \dfrac{Cx^2}{2} \\ z_{62} = \dfrac{Dx^2}{2} \\ z_{72} = -9e^x + 9 + 9x + \dfrac{x^2}{2} \\ z_{82} = 0 \end{cases}, \qquad p^{(3)}:\begin{cases} y_3 = \dfrac{-x^3}{3} \\ z_{13} = \dfrac{-x^3}{2} \\ z_{23} = \dfrac{Ax^3}{6} \\ z_{33} = \dfrac{Bx^3}{6} \\ z_{43} = \dfrac{Cx^3}{6} \\ z_{53} = \dfrac{Bx^3}{6} \\ z_{63} = -9e^x + 9\left(\displaystyle\sum_{n=0}^{2}\frac{x^n}{n!}\right) + \dfrac{x^3}{3!} \\ z_{73} = 0 \\ z_{83} = \dfrac{-x^3}{6} \end{cases},$$

$$p^{(4)}:\begin{cases} y_4 = \dfrac{-x^4}{8} \\ z_{14} = \dfrac{Ax^4}{24} \\ z_{24} = \dfrac{Bx^4}{24} \\ z_{34} = \dfrac{Cx^4}{24} \\ z_{44} = \dfrac{Dx^4}{24} \\ z_{54} = -9e^x + 9\left(\displaystyle\sum_{n=0}^{3}\frac{x^n}{n!}\right) + \dfrac{x^4}{4!} \\ z_{64} = 0 \\ z_{74} = \dfrac{-x^4}{24} \\ z_{84} = \dfrac{-x^4}{12} \end{cases}, \quad p^{(5)}:\begin{cases} y_5 = \dfrac{Ax^5}{5!} \\ z_{15} = \dfrac{Bx^5}{5!} \\ z_{25} = \dfrac{Cx^5}{5!} \\ z_{35} = \dfrac{Dx^5}{5!} \\ z_{45} = -9e^x + 9\left(\displaystyle\sum_{n=0}^{4}\frac{x^n}{n!}\right) + \dfrac{x^5}{5!}, \\ z_{55} = 0 \\ z_{65} = \dfrac{-x^5}{5!} \\ z_{75} = \dfrac{-x^5}{60} \\ z_{85} = \dfrac{-x^5}{40} \end{cases}$$

$$p^{(6)}:\begin{cases} y_6 = \dfrac{Bx^6}{6!} \\ z_{16} = \dfrac{Cx^6}{6!} \\ z_{26} = \dfrac{Dx^6}{6!} \\ z_{36} = -9e^x + 9\left(\displaystyle\sum_{n=0}^{5}\frac{x^n}{n!}\right) + \dfrac{x^6}{6!} \\ z_{46} = 0 \\ z_{56} = \dfrac{-x^6}{6!} \\ z_{66} = \dfrac{-x^6}{360} \\ z_{76} = \dfrac{-x^6}{240} \\ z_{86} = \dfrac{Ax^6}{6!} \end{cases}, \quad p^{(7)}:\begin{cases} y_7 = \dfrac{Cx^7}{7!} \\ z_{17} = \dfrac{Dx^7}{7!} \\ z_{27} = -9e^x + 9\left(\displaystyle\sum_{n=0}^{6}\frac{x^n}{n!}\right) + \dfrac{x^7}{7!} \\ z_{37} = 0 \\ z_{47} = \dfrac{-x^7}{7!} \\ z_{57} = \dfrac{-x^7}{2520} \\ z_{67} = \dfrac{-x^7}{1680} \\ z_{77} = \dfrac{Ax^7}{7!} \\ z_{87} = \dfrac{Bx^7}{7!} \end{cases},$$

$$p^{(8)}:\begin{cases} y_8 = \dfrac{Dx^8}{8!} \\ z_{18} = -9e^x + 9\left(\displaystyle\sum_{n=0}^{7}\frac{x^n}{n!}\right) + \dfrac{x^8}{8!} \\ z_{28} = 0 \\ z_{38} = \dfrac{-x^8}{8!} \\ z_{48} = \dfrac{-x^8}{20160} \\ z_{58} = \dfrac{-x^8}{13440} \\ z_{68} = \dfrac{Ax^8}{8!} \\ z_{78} = \dfrac{Bx^8}{8!} \\ z_{88} = \dfrac{Cx^8}{8!}, \end{cases} \qquad p^{(9)}:\begin{cases} y_9 = -9e^x + 9\left(\displaystyle\sum_{n=0}^{8}\frac{x^n}{n!}\right) + \dfrac{x^9}{9!} \\ z_{19} = 0 \\ z_{29} = \dfrac{-x^9}{9!} \\ z_{39} = \dfrac{-x^9}{181440} \\ z_{49} = \dfrac{-x^9}{120960} \\ z_{59} = \dfrac{Ax^9}{9!} \\ z_{69} = \dfrac{Bx^9}{9!} \\ z_{79} = \dfrac{Cx^9}{9!} \\ z_{89} = \dfrac{Dx^9}{9!}, \end{cases}$$

$$p^{(10)}:\begin{cases} y_{10} = 0 \\ z_{1,10} = \dfrac{-x^{10}}{10!} \\ z_{2,10} = \dfrac{-x^{10}}{1814400} \\ z_{3,10} = \dfrac{-x^{10}}{1209600} \\ z_{4,10} = \dfrac{Ax^{10}}{10!} \\ z_{5,10} = \dfrac{Bx^{10}}{10!} \\ z_{6,10} = \dfrac{Cx^{10}}{10!} \\ z_{7,10} = \dfrac{Dx^{10}}{10!} \\ z_{8,10} = -9e^x + 9\left(\displaystyle\sum_{n=0}^{9}\frac{x^n}{n!}\right) + \dfrac{x^{10}}{10!} \end{cases}, \quad p^{(11)}:\begin{cases} y_{11} = \dfrac{-x^{11}}{11!} \\ z_{1,11} = \dfrac{-x^{11}}{19958400} \\ z_{2,11} = \dfrac{-x^{11}}{13305600} \\ z_{3,11} = \dfrac{Ax^{11}}{11!} \\ z_{4,11} = \dfrac{Bx^{11}}{11!} \\ z_{5,11} = \dfrac{Cx^{11}}{11!} \\ z_{6,11} = \dfrac{Dx^{11}}{11!} \\ z_{7,11} = -9e^x + 9\left(\displaystyle\sum_{n=0}^{10}\frac{x^n}{n!}\right) + \dfrac{x^{11}}{11!} \\ z_{8,11} = 0 \end{cases},$$

$\vdots$

Adding up the above relations from $p^{(0)}$ to $p^{(11)}$ and so on, the solution is given as

$$y(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} + \frac{Ax^5}{5!} + \frac{Bx^6}{6!} + \frac{Cx^7}{7!} A$$

$$+ \frac{Dx^8}{8!} - \frac{8x^9}{9!} - \frac{9x^{10}}{10!} - \frac{10x^{11}}{11!} - \frac{11x^{12}}{12!} + \dots,$$

where $A, B, C$ and $D$ are as yet undetermined. Imposing the boundary conditions at $x = 1$, leads to the following system

$$\begin{bmatrix} \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{8!} \\ \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} \\ \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} \\ \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -\frac{2849503}{68428800} \\ -e + \frac{9072821}{3628800} \\ -2e + \frac{5445427}{1209600} \\ -3e + \frac{259883}{51840} \end{bmatrix}.$$

Solving this system gives

$$A = -3.999992,$$

$$B = -5.00017,$$

$$C = -5.9985,$$

$$D = -7.005.$$

Consequently, the series solution is given by

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - 0.03333326667x^5 - 0.006944680556x^6$$

$$- 0.001190178571x^7 - 0.000173735119x^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10}$$

$$- \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \dots.$$

Table 1 below shows the numerical results for Example 1 as well as the results of the analytical solution.

Examining the errors obtained by using the proposed HPM shows the high accuracy of the method.

**Table 1.** Numerical results for Example 1.

| $x$ | Analytical Solution | Numerical Solution | Errors* |
|---|---|---|---|
| 0.0 | 1.00000000 | 1.0000000000 | 0.000000 |
| 0.1 | 0.99465383 | 0.9946538264 | 3.6E−9 |
| 0.2 | 0.97712221 | 0.9771222066 | 3.4E−9 |
| 0.3 | 0.94490117 | 0.9449011654 | 4.6E−9 |
| 0.4 | 0.89509482 | 0.8950948186 | 1.4E−9 |
| 0.5 | 0.82436064 | 0.8243606355 | 4.5E−9 |
| 0.6 | 0.72884752 | 0.7288475206 | –6E−6 |
| 0.7 | 0.60412581 | 0.6041258131 | –3.1E−9 |
| 0.8 | 0.44510819 | 0.4451081876 | –2.4E−9 |
| 0.9 | 0.24596031 | 0.2459603145 | –4.5E−9 |
| 1.0 | 0.00000000 | 0.0000000000 | 0.000000 |

$*$Error = analytical solution – numerical solution.

**Example 2.** Now, we consider the nonlinear tenth-order BVP

$$y^{(10)}(x) = e^{-x} y^2(x), \;\; 0 < x < 1,$$

subject to the boundary conditions

$$y^{(2i)}(0) = 1, \;\; y^{(2i)}(1) = e, \;\; i = 0, 1, 2, 3, 4.$$

The exact solution for this problem is $y(x) = e^x$.

Using the following transformations, we obtain

$$\frac{dy}{dx} = z_1(x), \;\; \frac{dz_1}{dx} = z_2(x), \;\; \frac{dz_2}{dx} = z_3(x), \;\; \frac{dz_3}{dx} = z_4(x), \;\; \frac{dz_4}{dx} = z_5(x),$$

$$\frac{dz_5}{dx} = z_6(x), \;\; \frac{dz_6}{dx} = z_7(x), \;\; \frac{dz_7}{dx} = z_8(x), \;\; \frac{dz_8}{dx} = -9e^x + y(x),$$

with considering the boundary conditions, we can rewrite the 10th-order BVP as the system of the following integral equations

$$\begin{cases} y(x) = 1 + \int_0^x z_1(t)dt, \\ z_1(x) = A + \int_0^x z_2(t)dt, \\ z_2(x) = 1 + \int_0^x z_3(t)dt, \\ z_3(x) = B + \int_0^x z_4(t)dt, \\ z_4(x) = 1 + \int_0^x z_5(t)dt, \\ z_5(x) = C + \int_0^x z_6(t)dt, \\ z_6(x) = 1 + \int_0^x z_7(t)dt, \\ z_7(x) = D + \int_0^x z_8(t)dt, \\ z_8(x) = 1 + \int_0^x z_9(t)dt, \\ z_9(x) = E + \int_0^x e^{-t}\, y^2(t)dt. \end{cases}$$

Using the HPM, we have get the following system

$$\begin{cases}(y_0 + py_1 + p^2y_2 + \ldots) = 1 + p\int_0^x (z_{10} + pz_{11} + p^2z_{12} + \ldots)dt,\\ (z_{10} + pz_{11} + p^2z_{12} + \ldots) = A + p\int_0^x (z_{20} + pz_{21} + p^2z_{22} + \ldots)dt,\\ (z_{20} + pz_{21} + p^2z_{22} + \ldots) = 1 + p\int_0^x (z_{30} + pz_{31} + p^2z_{32} + \ldots)dt,\\ (z_{30} + pz_{31} + p^2z_{32} + \ldots) = B + p\int_0^x (z_{40} + pz_{41} + p^2z_{42} + \ldots)dt,\\ (z_{40} + pz_{41} + p^2z_{42} + \ldots) = 1 + p\int_0^x (z_{50} + pz_{51} + p^2z_{52} + \ldots)dt,\\ (z_{50} + pz_{51} + p^2z_{52} + \ldots) = C + p\int_0^x (z_{60} + pz_{61} + p^2z_{62} + \ldots)dt,\\ (z_{60} + pz_{61} + p^2z_{62} + \ldots) = 1 + p\int_0^x (z_{70} + pz_{71} + p^2z_{72} + \ldots)dt,\\ (z_{70} + pz_{71} + p^2z_{72} + \ldots) = D + p\int_0^x (z_{80} + pz_{81} + p^2z_{82} + \ldots)dt,\\ (z_{80} + pz_{81} + p^2z_{82} + \ldots) = 1 + p\int_0^x (z_{90} + pz_{91} + p^2z_{92} + \ldots)dt,\\ (z_{90} + pz_{91} + p^2z_{92} + \ldots) = E + p\int_0^x e^{-t}(y_0 + py_1 + p^2y_2 + \ldots)^2 dt.\end{cases}$$

Comparing the coefficients of the similar powers of $p$, yields to

$$p^{(0)}:\begin{cases} y_0 = 1 \\ z_{10} = A \\ z_{20} = 1 \\ z_{30} = B \\ z_{40} = 1 \\ z_{50} = C \\ z_{60} = 1 \\ z_{70} = D \\ z_{80} = 1 \\ z_{90} = E \end{cases}, \quad p^{(1)}:\begin{cases} y_1 = Ax \\ z_{11} = x \\ z_{21} = Bx \\ z_{31} = x \\ z_{41} = Cx \\ z_{51} = x \\ z_{61} = Dx \\ z_{71} = x \\ z_{81} = Ex \\ z_{91} = 1 - e^{-x} \end{cases}, \quad p^{(2)}:\begin{cases} y_2 = \frac{x^2}{2} \\ z_{12} = \frac{Bx^2}{2} \\ z_{22} = \frac{x^2}{2} \\ z_{32} = \frac{Cx^2}{2} \\ z_{42} = \frac{x^2}{2} \\ z_{52} = \frac{Dx^2}{2} \\ z_{62} = \frac{x^2}{2} \\ z_{72} = \frac{Ex^2}{2} \\ z_{82} = e^{-x} + x - 1 \\ z_{92} = 2A(1 - xe^{-x} - e^{-x}), \end{cases}$$

$$p^{(3)}:\begin{cases} y_3 = \frac{Bx^3}{3!} \\ z_{13} = \frac{x^3}{3!} \\ z_{23} = \frac{Cx^3}{3!} \\ z_{33} = \frac{x^3}{3!} \\ z_{43} = \frac{Dx^3}{3!} \\ z_{53} = \frac{x^3}{3!} \\ z_{63} = \frac{Ex^3}{3!} \\ z_{73} = \frac{x^2}{2!} - x - e^{-x} + 1 \\ z_{83} = 2A(x + xe^{-x} + 2e^{-x} - 2) \\ z_{93} = (A^2 + 1)(-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + 2), \end{cases}$$

$$p^{(4)}:\begin{cases} y_4 = \dfrac{x^4}{4!} \\ z_{14} = \dfrac{Cx^4}{4!} \\ z_{24} = \dfrac{x^4}{4!} \\ z_{34} = \dfrac{Dx^4}{4!} \\ z_{44} = \dfrac{x^4}{4!} \\ z_{54} = \dfrac{Ex^4}{4!} \\ z_{64} = \dfrac{x^3}{3!} - \dfrac{x^2}{2!} + x + e^{-x} - 1 \\ z_{74} = 2A\left(\dfrac{x^2}{2!} - 2x - 3e^{-x} - xe^{-x} + 3\right) \\ z_{84} = (A^2+1)(x^2e^{-x} + 4xe^{-x} + 6e^{-x} + 2x - 6) \\ z_{94} = \left(\dfrac{B}{3} + A\right)(-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + 6), \end{cases}$$

$$p^{(5)}:\begin{cases} y_5 = \dfrac{Cx^5}{5!} \\ z_{15} = \dfrac{x^5}{5!} \\ z_{25} = \dfrac{Dx^5}{5!} \\ z_{35} = \dfrac{x^5}{5!} \\ z_{45} = \dfrac{Ex^5}{5!} \\ z_{55} = \dfrac{x^4}{4!} - \dfrac{x^3}{3!} + \dfrac{x^2}{2!} - x - e^{-x} + 1 \\ z_{65} = 2A\left(\dfrac{x^3}{3!} - x^2 + 3x - 4 + 4e^{-x} + xe^{-x}\right) \\ z_{75} = (A^2+1)(x^2 - 6x - x^2e^{-x} - 6xe^{-x} - 12e^{-x} + 12) \\ z_{85} = \left(\dfrac{B}{3} + A\right)(x^3e^{-x} + 6x^2e^{-x} + 18xe^{-x} + 24e^{-x} + 6x - 24) \\ z_{95} = \dfrac{1}{3}(1 + AB)(-x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x} + 24), \end{cases}$$

$$\vdots$$

Combining all the terms from $p^{(0)}$ to $p^{(12)}$ and so on, the solution is given as

$$y(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7$$
$$+ \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \left(\frac{1}{19958400}A - \frac{1}{39916800}\right)x^{11}$$
$$+ \left(-\frac{1}{119750400}A + \frac{1}{159667200}\right)x^{12} + \dots.$$

It remains to determine the approximations of the constants $A$, $B$, $C$, $D$ and $E$. Imposing the boundary conditions at $x = 1$, leads to the following system

$$\begin{bmatrix} \frac{23950081}{23950080} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} \\ \frac{1}{226800} & 1 & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} \\ \frac{1}{3360} & 0 & 1 & \frac{1}{3!} & \frac{1}{5!} \\ \frac{1}{90} & 0 & 0 & 1 & \frac{1}{3!} \\ \frac{1}{3!} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} e - \frac{246379361}{159667200} \\ e - \frac{5599523}{3628800} \\ e - \frac{20737}{13340} \\ e - \frac{123}{80} \\ e - \frac{35}{24} \end{bmatrix}.$$

Solving this algebraic system gives

$$A = 1.00001436,$$
$$B = 0.999858964,$$
$$C = 1.001365775,$$
$$D = 0.987457318,$$
$$E = 1.093279434.$$

Putting the results in the series solution, we obtain

$$y(x) = 1 + 1.00001436x + \frac{1}{2}x^2 + 0.1666431607x^3 + \frac{1}{4!}x^4$$
$$+ 0.008344714791x^5 + \frac{1}{6!}x^6 + 0.000195924071x^7 + \frac{1}{8!}x^8$$
$$+ 3.013 \times 10^{-6}x^9 + \frac{1}{10!}x^{10} + 2.51 \times 10^{-8}x^{11} - 2.087 \times 10^{-9}x^{12} + \dots.$$

Table 2 below shows the exact values of the solution, numerical solutions, and the errors obtained by using the homotopy perturbation method. Table 2 also provides some numerical evidence which suggests that the performance of the HPM is promising.

**Table 2.** Numerical results for Example 2.

| $x$ | Analytical Solution | Numerical Solution | Errors* |
|---|---|---|---|
| 0.0 | 1.000000000 | 1.000000000 | 0.00000 |
| 0.1 | 1.105170918 | 1.10517233 | –1.41E–6 |
| 0.2 | 1.221402758 | 1.221405446 | –2.69E–6 |
| 0.3 | 1.349858808 | 1.349862509 | –3.70E–6 |
| 0.4 | 1.491824698 | 1.49182905 | –4.35E–6 |
| 0.5 | 1.648721271 | 1.648725849 | –4.58E–6 |
| 0.6 | 1.822118800 | 1.822123158 | –4.36E–6 |
| 0.7 | 2.013752707 | 2.013756415 | –3.71E–6 |
| 0.8 | 2.225540928 | 2.225543623 | –2.69E–6 |
| 0.9 | 2.459603111 | 2.459604528 | –1.42E–6 |
| 1.0 | 2.718281828 | 2.7182830 | 2.00E–9 |

$*$Error = analytical solution – numerical solution.

We close our analysis by discussing a 12th-order boundary value problem, where the 12th-order is a function of a lower-order derivative.

**Example 3.** Assume, we consider the following nonlinear 12th-order BVP

$$y^{(12)}(x) = 2e^x y^2(x) + y^{(3)}(x), \quad 0 < x < 1,$$

subject to the boundary conditions

$$y^{(2i)}(0) = 1, \quad y^{(2i)}(1) = e^{-1}, \quad i = 0, 1, 2, 3, 4, 5,$$

with the exact solution, $y(x) = e^{-x}$.

Proceeding as before, we set

$$\frac{dy}{dx} = z_1(x), \frac{dz_1}{dx} = z_2(x), \frac{dz_2}{dx} = z_3(x), \frac{dz_3}{dx} = z_4(x), \frac{dz_4}{dx} = z_5(x), \frac{dz_5}{dx} = z_6(x),$$

$$\frac{dz_6}{dx} = z_7(x), \frac{dz_7}{dx} = z_8(x), \frac{dz_8}{dx} = z_9(x), \frac{dz_9}{dx} = z_{10}(x), \frac{dz_{10}}{dx} = z_{11}(x),$$

$$\frac{dz_{11}}{dx} = 2e^x y^2(x) + y^{(3)}(x).$$

By considering the boundary conditions, we can rewrite the 12th-order BVP as the following system of integral equations

$$\begin{cases} y(x) = 1 + \int_0^x z_1(t)dt, \\ z_1(x) = A + \int_0^x z_2(t)dt, \\ z_2(x) = 1 + \int_0^x z_3(t)dt, \\ z_3(x) = B + \int_0^x z_4(t)dt, \\ z_4(x) = 1 + \int_0^x z_5(t)dt, \\ z_5(x) = C + \int_0^x z_6(t)dt, \\ z_6(x) = 1 + \int_0^x z_7(t)dt, \\ z_7(x) = D + \int_0^x z_8(t)dt, \\ z_8(x) = 1 + \int_0^x z_9(t)dt, \\ z_9(x) = E + \int_0^x z_{10}(t)dt, \\ z_{10}(x) = 1 + \int_0^x z_{11}(t)dt, \\ z_{11}(x) = F + \int_0^x (2e^t y^2(t) + y^{(3)}(t))dt. \end{cases}$$

Using the HPM, we have

$$\begin{cases}
(y_0 + py_1 + p^2y_2 + \dots) = 1 + p\int_0^x (z_{10} + pz_{11} + p^2 z_{12} + \dots)dt, \\
(z_{10} + pz_{11} + p^2z_{12} + \dots) = A + p\int_0^x (z_{20} + pz_{21} + p^2 z_{22} + \dots)dt, \\
(z_{20} + pz_{21} + p^2z_{22} + \dots) = 1 + p\int_0^x (z_{30} + pz_{31} + p^2 z_{32} + \dots)dt, \\
(z_{30} + pz_{31} + p^2z_{32} + \dots) = B + p\int_0^x (z_{40} + pz_{41} + p^2 z_{42} + \dots)dt, \\
(z_{40} + pz_{41} + p^2z_{42} + \dots) = 1 + p\int_0^x (z_{50} + pz_{51} + p^2 z_{52} + \dots)dt, \\
(z_{50} + pz_{51} + p^2z_{52} + \dots) = C + p\int_0^x (z_{60} + pz_{61} + p^2 z_{62} + \dots)dt, \\
(z_{60} + pz_{61} + p^2z_{62} + \dots) = 1 + p\int_0^x (z_{70} + pz_{71} + p^2 z_{72} + \dots)dt, \\
(z_{70} + pz_{71} + p^2z_{72} + \dots) = D + p\int_0^x (z_{80} + pz_{81} + p^2 z_{82} + \dots)dt, \\
(z_{80} + pz_{81} + p^2z_{82} + \dots) = 1 + p\int_0^x (z_{90} + pz_{91} + p^2 z_{92} + \dots)dt, \\
(z_{90} + pz_{91} + p^2z_{92} + \dots) = E + p\int_0^x (z_{10,0} + pz_{10,1} + p^2 z_{10,2} + \dots)dt, \\
(z_{10,0} + pz_{10,1} + p^2z_{10,2} + \dots) = 1 + p\int_0^x (z_{11,0} + pz_{11,1} + p^2 z_{11,2} + \dots)dt, \\
(z_{11,0} + pz_{11,1} + p^2z_{11,2} + \dots) = F + p\int_0^x (2e^{-t}(y_0 + py_1 + p^2 y_2 + \dots)^2 \\
\qquad + (z_{30} + pz_{31} + p^2 z_{32} + \dots))dt.
\end{cases}$$

Comparing the coefficients of the like powers of $p$, we get

$$p^{(0)}: \begin{cases}
y_0 = 1 \\
z_{10} = A \\
z_{20} = 1 \\
z_{30} = B \\
z_{40} = 1 \\
z_{50} = C \\
z_{60} = 1 \\
z_{70} = D \\
z_{80} = 1 \\
z_{90} = E \\
z_{10,0} = 1 \\
z_{11,0} = F
\end{cases}, \qquad
p^{(1)}: \begin{cases}
y_1 = Ax \\
z_{11} = x \\
z_{21} = Bx \\
z_{31} = x \\
z_{41} = Cx \\
z_{51} = x \\
z_{61} = Dx \\
z_{71} = x \\
z_{81} = Ex \\
z_{91} = x \\
z_{10,1} = Fx \\
z_{11,1} = 2e^x - 2 + Bx
\end{cases},$$

$$p^{(2)}:\begin{cases} y_2 = \dfrac{x^2}{2} \\ z_{12} = \dfrac{Bx^2}{2} \\ z_{22} = \dfrac{x^2}{2} \\ z_{32} = \dfrac{Cx^2}{2} \\ z_{42} = \dfrac{x^2}{2} \\ z_{52} = \dfrac{Dx^2}{2} \\ z_{62} = \dfrac{x^2}{2} \\ z_{72} = \dfrac{Ex^2}{2} \\ z_{82} = \dfrac{x^2}{2} \\ z_{92} = \dfrac{Fx^2}{2} \\ z_{10,2} = 2e^x - 2x - 2 + \dfrac{Bx^2}{2} \\ z_{11,2} = 4A(xe^x - e^x + 1) + \dfrac{x^2}{2} \end{cases},$$

$$
p^{(3)}:\begin{cases}
y_3 = \dfrac{Bx^3}{3!}\\
z_{13} = \dfrac{x^3}{3!}\\
z_{23} = \dfrac{Cx^3}{3!}\\
z_{33} = \dfrac{x^3}{3!}\\
z_{43} = \dfrac{Dx^3}{3!}\\
z_{53} = \dfrac{x^3}{3!}\\
z_{63} = \dfrac{Ex^3}{3!}\\
z_{73} = \dfrac{x^3}{3!}\\
z_{83} = \dfrac{Fx^3}{3!}\\
z_{93} = \dfrac{Bx^3}{3!} - x^2 - 2x + 2e^x - 2\\
z_{10,3} = 4A(xe^x - 2e^x + 2 + x) + \dfrac{x^3}{3!}\\
z_{11,3} = 2(A^2 + 1)(x^2e^x - 2xe^x + 2e^x - 2) + \dfrac{Cx^3}{3!}
\end{cases},
$$

$$p^{(4)} : \begin{cases} y_4 = \dfrac{x^4}{4!} \\ z_{14} = \dfrac{Cx^4}{4!} \\ z_{24} = \dfrac{x^4}{4!} \\ z_{34} = \dfrac{Dx^4}{4!} \\ z_{44} = \dfrac{x^4}{4!} \\ z_{54} = \dfrac{Ex^4}{4!} \\ z_{64} = \dfrac{x^4}{4!} \\ z_{74} = \dfrac{Fx^4}{4!} \\ z_{84} = \dfrac{Bx^4}{4!} - \dfrac{x^3}{3} - x^2 - 2x + 2e^x - 2 \\ z_{94} = 4A\left(xe^x - 3e^x + 3 + x + \dfrac{x^2}{2}\right) + \dfrac{x^4}{4!} \\ z_{10,4} = 2(A^2 + 1)(x^2e^x - 4xe^x + 6e^x - 2x - 6) + \dfrac{Cx^4}{4!} \\ z_{11,4} = 2\left(A + \dfrac{B}{3}\right)(x^3e^x - 3x^2e^x + 6xe^x - 6e^x + 6) + \dfrac{x^4}{4!} \end{cases},$$

$$
p^{(5)}:\begin{cases}
y_5 = \dfrac{Cx^5}{5!} \\
z_{15} = \dfrac{x^5}{5!} \\
z_{25} = \dfrac{Dx^5}{5!} \\
z_{35} = \dfrac{x^5}{5!} \\
z_{45} = \dfrac{Ex^5}{5!} \\
z_{55} = \dfrac{x^5}{5!} \\
z_{65} = \dfrac{Fx^5}{5!} \\
z_{75} = \dfrac{Bx^5}{5!} - \dfrac{x^4}{12} - \dfrac{x^3}{3} - x^2 - 2x + 2e^x - 2 \\
z_{85} = 4A\left(xe^x - 4e^x + 3x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + 4\right) + \dfrac{x^5}{5!} \\
z_{95} = 2(A^2 + 1)(x^2e^x - 6xe^x + 12e^x - x^2 - 6x - 12) + \dfrac{Cx^5}{5!} \\
z_{10,5} = 2\left(A + \dfrac{B}{3}\right)(x^3e^x - 6x^2e^x + 18xe^x - 24e^x + 6x + 24)\dfrac{x^5}{5!} \\
z_{11,5} = \dfrac{2}{3}(1 + AB)(x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x - 24) + \dfrac{Dx^5}{5!}
\end{cases},
$$

$$\vdots$$

Combining all the terms from $p^{(0)}$ to $p^{(12)}$ and so on, the solution is given as

$$
y(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7
$$

$$
+ \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}Fx^{11} + \left(\frac{2}{12!} + \frac{1}{12!}\right)x^{12} + \dots .
$$

It remains to determine approximations of the constants $A$, $B$, $C$, $D$ and $E$. Imposing the boundary conditions at $x = 1$, leads to the following system

$$\begin{bmatrix} 1 & \frac{79833601}{479001600} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} \\ 0 & \frac{3628801}{3628800} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} \\ 0 & \frac{1}{8!} & 1 & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} \\ 0 & \frac{1}{6!} & 0 & 1 & \frac{1}{3!} & \frac{1}{5!} \\ 0 & \frac{1}{4!} & 0 & 0 & 1 & \frac{1}{3!} \\ 0 & \frac{1}{2!} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} e^{-1} - \frac{369569047}{239500800} \\ e^{-1} - \frac{1399883}{907200} \\ e^{-1} - \frac{31109}{20160} \\ e^{-1} - \frac{139}{90} \\ e^{-1} - \frac{19}{12} \\ e^{-1} - 2 \end{bmatrix}.$$

It follows that

$$A = -0.9999983604,$$

$$B = -1.000016174,$$

$$C = -0.9998407313,$$

$$D = -1.001558298,$$

$$E = -0.981011393,$$

$$F = -1.132112472.$$

so that the series solution is therefore given by

$$y(x) = 1 - 0.9999983604x + \frac{1}{2!}x^2 - 0.1666693624x^3 + \frac{1}{4!}x^4$$

$$- 0.008332006094x^5 + \frac{1}{6!}x^6 - 0.0001987218845x^7 + \frac{1}{8!}x^8$$

$$- 2.715 \times 10^{-6}x^9 + \frac{1}{10!}x^{10} - 2.836 \times 10^{-8}x^{11} + 2.087 \times 10^{-9}x^{12} + \ldots.$$

In Table 3, we show the results of the numerical solution and the analytical solution and the errors obtained by using the approximant of above relation.

**Table 3.** Numerical results for Example 3.

| $x$ | Analytical Solution | Numerical Solution | Errors* |
|---|---|---|---|
| 0.0 | 1.000000000 | 1.000000000 | 0.00000 |
| 0.1 | 0.904837418 | 0.904837579 | –1.61E−7 |
| 0.2 | 0.818730753 | 0.818731060 | –3.07E−7 |
| 0.3 | 0.740818221 | 0.740818643 | –4.22E−7 |
| 0.4 | 0.670320046 | 0.670320543 | –4.97E−7 |
| 0.5 | 0.606530659 | 0.606531182 | –5.22E−7 |
| 0.6 | 0.548811636 | 0.548812133 | –4.97E−7 |
| 0.7 | 0.496585304 | 0.496585726 | –4.22E−7 |
| 0.8 | 0.449328964 | 0.44932971 | –3.07E−7 |
| 0.9 | 0.406569659 | 0.406569821 | –1.61E−7 |
| 1.0 | 0.367879441 | 0.367879441 | 2.00E−10 |

*Error = analytical solution – numerical solution.

## 4. Conclusion

In this paper, we have shown that the homotopy perturbation method can be used successfully for finding the solution of linear and nonlinear boundary value problems of some higher-order by reformulating it as a system of integral equations. It may be concluded that this technique is a very powerful and efficient in finding highly accurate numerical solutions for a large class of integral and differential equations. Impose of the fact the numerical results obtained through using this method are the same as the modified Adomian method (MAM) [26], the HPM is easier in practice because the MAM requires computing Adomain polynomials which is usually a tedious process.

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