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Solving nonlinear integral equations in the Urysohn form by Newton–Kantorovich–quadrature method

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ABSTRACT

The Newton–Kantorovich method is a well-known method for solving nonlinear integral equations. This method attempts to solve a sequence of linear integral equations. In this article, we develop a new method, which is a combination of the Newton–Kantorovich and quadrature methods. The new method solves the nonlinear integral equations of the Urysohn form in a systematic procedure. Some numerical examples are provided, and the obtained numerical approximations are compared with the corresponding exact solutions.

1. Introduction and preliminaries

One kind of the nonlinear integral equation is the nonlinear integral equation in the Urysohn form. This kind of integral equation is defined in the following general form:

$$y(x) = f(x) + \int_{\Omega} K(x, t, y(t))\mathrm{d}t, \quad a \le x \le b. \tag{1.1}$$

Depending on $\Omega = (a, x)$ or $\Omega = (a, b)$, Eq. (1.1) is named a nonlinear Volterra integral equation or a nonlinear Fredholm integral equation, respectively.

To approximate the right-hand integral in (1.1), we use the usual quadrature methods similar to the ones used to approximate the linear integral equations that lead to the following nonlinear systems for Fredholm and Volterra equations, respectively. For further information on quadrature methods in this respect, see [1–11].

$$y(x_i) = f(x_i) + \sum_{j=0}^{n} w_j K(x_i, x_j, y(x_j)), \quad i = 0, 1, \ldots, n \tag{1.2}$$

$$\begin{cases} y(x_0) = f(x_0) \\ y(x_i) = f(x_i) + \displaystyle\sum_{j=0}^{i} w_{ij} K(x_i, x_j, y(x_j)), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.3}$$

where w_{ij}s and w_js are weights of the integration formula.

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We recall that in the Newton–Kantorovich method, we consider an initial solution to $y(x)$, say $y_0(x)$. Then by using the following iteration method, we solve the following sequence of linear integral equations instead of a nonlinear integral equation. For further information on the Newton–Kantorovich method, see [12–14].

$$\begin{cases} y_k(x) = y_{k-1}(x) + \phi_{k-1}(x) \\ \phi_{k-1}(x) = \varepsilon_{k-1}(x) + \displaystyle\int_\Omega K'_y(x, t, y_{k-1}(t))\phi_{k-1}(t)\mathrm{d}t \\ \varepsilon_{k-1}(x) = f(x) - y_{k-1}(x) + \displaystyle\int_\Omega K(x, t, y_{k-1}(t))\mathrm{d}t, \end{cases} \tag{1.4}$$

where $K'_y(x, t, y) = \frac{\partial}{\partial y} K(x, t, y)$.

In this article, we intend to combine these two methods to obtain a systematic and efficient method for solving nonlinear integral equations of the Urysohn form.

2. Solving nonlinear Fredholm integral equations

The general form of nonlinear Fredholm integral equations of the Urysohn form is as follows:

$$y(x) = f(x) + \int_a^b K(x, t, y(t))\mathrm{d}t, \quad a \le x \le b. \tag{2.1}$$

We apply the Newton–Kantorovich method, for solving this kind of equation in the following:

$$\begin{cases} y_k(x) = y_{k-1}(x) + \varphi_{k-1}(x) \\ \varphi_{k-1}(x) = \varepsilon_{k-1}(x) + \displaystyle\int_a^b K'_y(x, t, y_{k-1}(t))\varphi_{k-1}(t)\mathrm{d}t \\ \varepsilon_{k-1}(x) = f(x) - y_{k-1}(x) + \displaystyle\int_a^b K(x, t, y_{k-1}(t))\mathrm{d}t. \end{cases} \tag{2.2}$$

From (2.2), we have

$$\varphi_{k-1}(x) = f(x) - y_{k-1}(x) + \int_a^b K(x, t, y_{k-1}(t))\mathrm{d}t + \int_a^b K'_y(x, t, y_{k-1}(t))\varphi_{k-1}(t)\mathrm{d}t. \tag{2.3}$$

Now, we approximate the two integrals on the right-hand side of (2.3) by one of the numerical integration formulas such as repeated Simpson, repeated trapezoid or Gauss methods, so we get

$$\varphi_{k-1}(x) = f(x) - y_{k-1}(x) + \sum_{j=0}^n w_j K(x, x_j, y_{k-1}(x_j)) + \sum_{j=0}^n w_j K'_y(x, x_j, y_{k-1}(x_j))\varphi_{k-1}(x_j). \tag{2.4}$$

By substituting $x = x_i$ for $i = 0, 1, 2, \dots, n$ in (2.4), we obtain the following system:

$$\begin{aligned} \varphi_{k-1}(x_i) &= f(x_i) - y_{k-1}(x_i) + \sum_{j=0}^n w_j K(x_i, x_j, y_{k-1}(x_j)) + \sum_{j=0}^n w_j K'_y(x_i, x_j, y_{k-1}(x_j))\varphi_{k-1}(x_j), \\ & i = 0, 1, 2, \dots, n. \end{aligned} \tag{2.5}$$

Since $\varphi_{k-1}(x) = y_k(x) - y_{k-1}(x)$, we have

$$\begin{aligned} y_k(x_i) - y_{k-1}(x_i) &= f(x_i) - y_{k-1}(x_i) + \sum_{j=0}^n w_j K(x_i, x_j, y_{k-1}(x_j)) + \sum_{j=0}^n w_j K'_y(x_i, x_j, y_{k-1}(x_j)) \\ &\quad \times [y_k(x_j) - y_{k-1}(x_j)], \quad i = 0, 1, \dots, n. \end{aligned} \tag{2.6}$$

Therefore, we get

$$y_k(x_i) = f(x_i) + \sum_{j=0}^n w_j K(x_i, x_j, y_{k-1}(x_j)) + \sum_{j=0}^n w_j K'_y(x_i, x_j, y_{k-1}(x_j))[y_k(x_j) - y_{k-1}(x_j)], \quad i = 0, 1, \dots, n. \tag{2.7}$$

Now, we let

$$(F^{(k-1)})_{i+1} = f(x_i) + \sum_{j=0}^n w_j [K(x_i, x_j, y_{k-1}(x_j)) - K'_y(x_i, x_j, y_{k-1}(x_j))y_{k-1}(x_j)], \quad i = 0, 1, \dots, n, \tag{2.8}$$

$$(A^{(k-1)})_{i+1,j+1} = w_j K'_y(x_i, x_j, y_{k-1}(x_j)), \quad i, j = 0, 1, \dots, n, \tag{2.9}$$

$$(Y^{(k)})_{i+1} = y_k(x_i), \quad i = 0, 1, \dots, n \tag{2.10}$$

to obtain the following sequence of linear systems. Thus, by solving the following systems (2.11) we are able to solve this kind of equation:

$$(I - A^{(k-1)})Y^{(k)} = F^{(k-1)}, \quad k = 1, 2, 3, \ldots. \tag{2.11}$$

For this purpose, we first consider an initial solution y_0; so, $(Y^{(0)})_i = y_0(x_i)$ and by using (2.8) and (2.9) we can construct $F^{(0)}$ and $A^{(0)}$, then by solving the system $(I - A^{(0)})Y^{(1)} = F^{(0)}$, we obtain $Y^{(1)}$. By repeating this procedure and next using (2.11), we obtain the values $Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots$ and $Y^{(m)}$ for a selected $m \in N$. One can consider a constant value for m or may change it when n increases.

In the presented method depending upon n, we obtain an approximate solution to Eq. (2.1) and when we increase m, this solution tends to be the most accurate approximation with respect to n. In the following examples, one can see that we do not need to increase m significantly.

3. Solving nonlinear Volterra integral equations

The general form of the nonlinear Volterra integral equations of the Urysohn form is as follows:

$$y(x) = f(x) + \int_a^x K(x, t, y(t))\mathrm{d}t, \quad a \le x \le b. \tag{3.1}$$

Similar to that of the Fredholm integral equations, the method of Newton–Kantorovich, which has been described to solve this kind of equation as follows:

$$\begin{cases} y_k(x) = y_{k-1}(x) + \varphi_{k-1}(x) \\ \varphi_{k-1}(x) = \varepsilon_{k-1}(x) + \displaystyle\int_a^x K'_y(x, t, y_{k-1}(t))\varphi_{k-1}(t)\mathrm{d}t \\ \varepsilon_{k-1}(x) = f(x) - y_{k-1}(x) + \displaystyle\int_a^x K(x, t, y_{k-1}(t))\mathrm{d}t. \end{cases} \tag{3.2}$$

Now, if we follow the similar procedure, which was explained in the previous section, we would have

$$\varphi_{k-1}(x) = f(x) - y_{k-1}(x) + \int_a^x K(x, t, y_{k-1}(t))\mathrm{d}t + \int_a^x K'_y(x, t, y_{k-1}(t))\varphi_{k-1}(t)\mathrm{d}t. \tag{3.3}$$

Consequently, by approximating the integrals on the right-hand side of (3.3), we obtain the following system:

$$\begin{cases} \varphi_{k-1}(x_0) = f(x_0) - y_{k-1}(x_0) \\ \varphi_{k-1}(x_i) = f(x_i) - y_{k-1}(x_i) + \displaystyle\sum_{j=0}^{i} w_{ij}K(x_i, x_j, y_{k-1}(x_j)) + \sum_{j=0}^{i} w_{ij}K'_y(x_i, x_j, y_{k-1}(x_j))\varphi_{k-1}(x_j), \quad i = 1, 2, \ldots, n. \end{cases} \tag{3.4}$$

By interchanging $\varphi_{k-1}(x)$ with $y_k(x) - y_{k-1}(x)$, similar to the Fredholm integral equations' case, we obtain

$$\begin{cases} y_k(x_0) = f(x_0) \\ y_k(x_i) = f(x_i) + \displaystyle\sum_{j=0}^{i} w_{ij}K(x_i, x_j, y_{k-1}(x_j)) + \sum_{j=0}^{i} w_{ij}K'_y(x_i, x_j, y_{k-1}(x_j))[y_k(x_j) - y_{k-1}(x_j)], \quad i = 1, 2, \ldots, n. \end{cases} \tag{3.5}$$

Now we consider

$$(F^{(k-1)})_{i+1} = \begin{cases} f(x_0) \quad i = 0 \\ f(x_i) + \displaystyle\sum_{j=0}^{i} w_{ij}K(x_i, x_j, y_{k-1}(x_j)) - \sum_{j=0}^{i} w_{ij}K'_y(x_i, x_j, y_{k-1}(x_j))y_{k-1}(x_j) \quad i = 1, 2, \ldots, n \end{cases} \tag{3.6}$$

$$(A^{(k-1)})_{i+1,j+1} = \begin{cases} w_{ij}K'_y(x_i, x_j, y_{k-1}(x_j)) \quad i = 1, 2, \ldots, n,\ j = 0, 1, \ldots, i \\ 0 \quad \text{o.w.} \end{cases} \tag{3.7}$$

$$(Y^{(k)})_{i+1} = y_k(x_i), \quad i = 0, 1, \ldots, n. \tag{3.8}$$

Similar to the Fredholm integral equations, by considering an initial solution $y_0(x)$ and constructing the $Y^{(0)}$, we can solve these equations by using the following repetition sequence:

$$(I - A^{(k-1)})Y^{(k)} = F^{(k-1)}, \quad k = 1, 2, 3, \ldots. \tag{3.9}$$

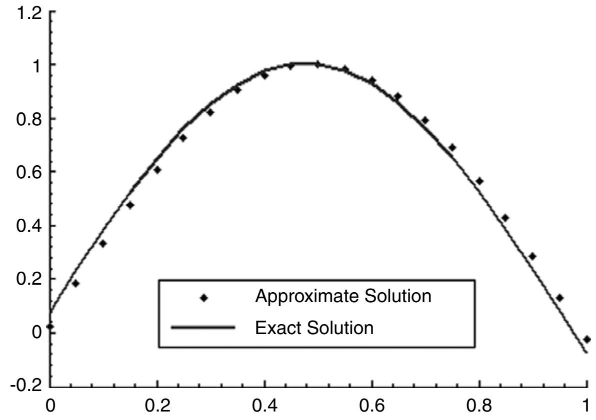

Fig. 1. Solutions to the integral equation (4.1) for $m = 4$.

Table 1
Solutions to the integral equation (4.1) for $m = 1$.

Nodes	Exact solutions	Approximate solutions	Error
0	0.07542668890494	0	−0.07542668890494
0.05	0.23093252624133	0.15643446504023	−0.07449806120110
0.1	0.38075203836055	0.30901699437495	−0.07173504398561
0.15	0.52119617165177	0.45399049973955	−0.06720567191223
0.2	0.64880672544600	0.58778525229247	−0.06102147315353
0.25	0.76044150439368	0.70710678118655	−0.05333472320713
0.3	0.85335168974252	0.80901699437495	−0.04433469536757
0.35	0.92524952437802	0.89100652418837	−0.03424300018965
0.4	0.97436464499621	0.95105651629515	−0.02330812870106
0.45	0.99948767432374	0.98768834059514	−0.01179933372860
0.5	1	1	0
0.55	0.97588900686654	0.98768834059514	0.01179933372860
0.6	0.92774838759410	0.95105651629515	0.02330812870106
0.65	0.85676352399872	0.89100652418837	0.03424300018965
0.7	0.76468229900737	0.80901699437495	0.04433469536757
0.75	0.65377205797942	0.70710678118655	0.05333472320713
0.8	0.52676377913895	0.58778525229247	0.06102147315353
0.85	0.38678482782732	0.45399049973955	0.06720567191223
0.9	0.23728195038934	0.30901699437495	0.07173504398561
0.95	0.08193640383913	0.15643446504023	0.07449806120110
1	−0.07542668890494	0	0.07542668890494

4. Numerical examples

In this section, we intend to show the efficiency of the Newton–Kantorovich–quadrature method for solving nonlinear integral equations of the Urysohn form by illustrating some examples. For calculating the results in each table, we use MATLAB v7.2.

Example 1. Consider the following nonlinear Fredholm integral equation:

$$y(x) = \sin \pi x + \frac{1}{5}\int_0^1 \cos \pi x \sin \pi t y^3(t)\mathrm{d}t, \quad 0 \le x \le 1. \tag{4.1}$$

We solved this equation by combining the Newton–Kantorovich method and repeating the Simpson quadrature method, which we explained before. For approximating the right-hand integrals, we repeat the Simpson method and divide the integration interval into 20 equal subintervals. The results have been obtained and given in Tables 1–3, and Fig. 1. The exact solution to this integral equation is $y(x) = \sin \pi x + \frac{1}{3}\left(20 - \sqrt{391}\right) \cos \pi x$.

Example 2. In this example, we intend to solve a nonlinear Volterra integral equation by choosing $y_0(x) = 0$ and dividing the integration interval into 20 equal parts, but to get the result that is more accurate, we use the block-by-block method quadrature formula. For further information on block-by-block methods, see [13,2,3,8].

Table 2
Solutions to the integral equation (4.1) for $m = 3$.

Nodes	Exact solutions	Approximate solutions	Error
0	0.07542668890494	0.02558828156747	−0.04983840733747
0.05	0.23093252624133	0.18170771240029	−0.04922481384105
0.1	0.38075203836055	0.33335289630048	−0.04739914206007
0.15	0.52119617165177	0.47678982555893	−0.04440634609284
0.2	0.64880672544600	0.60848660693741	−0.04032011850859
0.25	0.76044150439368	0.72520042860182	−0.03524107579186
0.3	0.85335168974252	0.82405740891181	−0.02929428083071
0.35	0.92524952437802	0.90262336092466	−0.02262616345336
0.4	0.97436464499621	0.95896373015635	−0.01540091483986
0.45	0.99948767432374	0.99169122973344	−0.00779644459029
0.5	1	1	0
0.55	0.97588900686654	0.98368545145683	0.00779644459029
0.6	0.92774838759410	0.94314930243395	0.01540091483986
0.65	0.85676352399872	0.87938968745208	0.02262616345336
0.7	0.76468229900737	0.79397657983808	0.02929428083071
0.75	0.65377205797942	0.68901313377128	0.03524107579186
0.8	0.52676377913895	0.56708389764754	0.04032011850859
0.85	0.38678482782732	0.43119117392016	0.04440634609284
0.9	0.23728195038934	0.28468109244941	0.04739914206007
0.95	0.08193640383913	0.13116121768018	0.04922481384105
1	−0.07542668890494	−0.02558828156747	0.04983840733747

Table 3
Solutions to the integral equation (4.1) for $m = 4$.

Nodes	Exact solutions	Approximate solutions	Error
0	0.07542668890494	0.02559478612236	−0.04983190278257
0.05	0.23093252624133	0.18171413687332	−0.04921838936802
0.1	0.38075203836055	0.33335908249980	−0.04739295586075
0.15	0.52119617165177	0.47679562115978	−0.04440055049199
0.2	0.64880672544600	0.60849186923286	−0.04031485621314
0.25	0.76044150439368	0.72520502801669	−0.03523647637699
0.3	0.85335168974252	0.82406123219325	−0.02929045754927
0.35	0.92524952437802	0.90262631393079	−0.02262321044723
0.4	0.97436464499621	0.95896574017436	−0.01539890482186
0.45	0.99948767432374	0.99169224727001	−0.00779542705373
0.5	1	1	0
0.55	0.97588900686654	0.98368443392027	0.00779542705373
0.6	0.92774838759410	0.94314729241595	0.01539890482185
0.65	0.85676352399872	0.87938673444595	0.02262321044723
0.7	0.76468229900737	0.79397275655664	0.02929045754927
0.75	0.65377205797942	0.68900853435640	0.03523647637699
0.8	0.52676377913895	0.56707863535209	0.04031485621314
0.85	0.38678482782732	0.43118537831931	0.04440055049199
0.9	0.23728195038934	0.28467490625009	0.04739295586075
0.95	0.08193640383913	0.13115479320715	0.04921838936802
1	−0.07542668890494	−0.02559478612236	0.04983190278257

We note that if we use the block-by-block quadrature method, the resulting system for solving the nonlinear Volterra integral equations is slightly different from (3.5). Its resulting system will be as follows: (By applying the similar way, which was explained in Section 3, one can obtain this system.)

$$\begin{cases} y_k(x_0) = f(x_0) \\ y_k(x_{2i-1}) = f(x_{2i-1}) + \sum\limits_{j=0}^{2i} w_{2i-1,j} K(x_{2i-1}, x_j, y_{k-1}(x_j)) + \sum\limits_{j=0}^{2i} w_{2i-1,j} K'_y(x_{2i-1}, x_j, y_{k-1}(x_j))[y_k(x_j) - y_{k-1}(x_j)], \\ \quad i = 1, 2, \ldots, n \\ y_k(x_{2i}) = f(x_{2i}) + \sum\limits_{j=0}^{2i} w_{2i,j} K(x_{2i}, x_j, y_{k-1}(x_j)) + \sum\limits_{j=0}^{2i} w_{2i,j} K'_y(x_{2i}, x_j, y_{k-1}(x_j))[y_k(x_j) - y_{k-1}(x_j)], \\ \quad i = 1, 2, \ldots, n, \end{cases} \tag{4.2}$$

where w_{ij}s are weights for the block-by-block method.

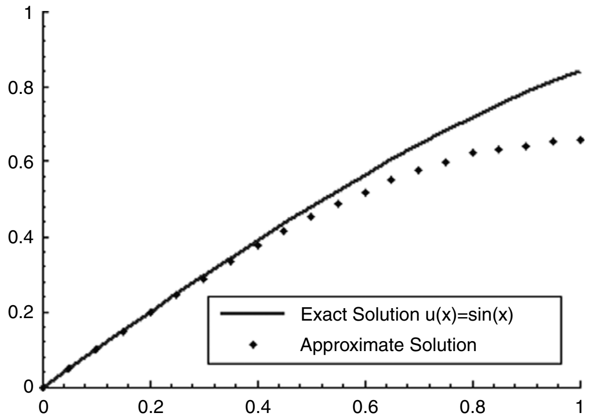

Fig. 2. Solutions to the integral equation (4.6) for $m = 4$.

Table 4
Solutions to the integral equation (4.6) for $m = 1$.

Nodes	Exact solutions	Approximate solutions	Error
0	0	0	0
0.05	0.04997916927068	0.04993752343239	−0.00004164583829
0.1	0.09983341664683	0.09950074934559	−0.00033266730123
0.15	0.14943813247360	0.14831818413893	−0.00111994833467
0.2	0.19866933079506	0.19602391637222	−0.00264541442284
0.25	0.24740395925452	0.24226034390557	−0.00514361534895
0.3	0.29552020666134	0.28668082501010	−0.00883938165124
0.35	0.34289780745545	0.32895222926487	−0.01394557819058
0.4	0.38941834230865	0.36875736503353	−0.02066097727512
0.45	0.43496553411123	0.40579726151810	−0.02916827259313
0.5	0.47942553860420	0.43979328480618	−0.03963225379803
0.55	0.52268722893066	0.47048906894602	−0.05219815998464
0.6	0.56464247339504	0.49765224488684	−0.06699022850819
0.65	0.60518640573604	0.52107595209034	−0.08411045364570
0.7	0.64421768723769	0.54058011973481	−0.10363756750289
0.75	0.68163876002333	0.55601250667435	−0.12562625334899
0.8	0.71735609089952	0.56724949165990	−0.15010659923962
0.85	0.75128040514029	0.57419660775341	−0.17708379738688
0.9	0.78332690962748	0.57678881734703	−0.20653809228045
0.95	0.81341550478937	0.57499052671123	−0.23842497807815
1	0.84147098480790	0.56879534151432	−0.27267564329358

So, we consider

$$(F^{(k-1)})_{i+1} = \begin{cases} f(x_0) & i = 0 \\ f(x_i) + \sum_{j=0}^{i+1} w_{ij} K(x_i, x_j, y_{k-1}(x_j)) - \sum_{j=0}^{i+1} w_{ij} K'_y(x_i, x_j, y_{k-1}(x_j)) y_{k-1}(x_j) & i = 1, 3, \ldots, 2n-1 \\ f(x_i) + \sum_{j=0}^{i} w_{ij} K(x_i, x_j, y_{k-1}(x_j)) - \sum_{j=0}^{i} w_{ij} K'_y(x_i, x_j, y_{k-1}(x_j)) y_{k-1}(x_j) & i = 2, 4, \ldots, 2n, \end{cases} \tag{4.3}$$

$$(A^{(k-1)})_{i+1,j+1} = \begin{cases} w_{ij} K'_y(x_i, x_j, y_{k-1}(x_j)) & \begin{array}{l} \text{if } i = 2, 4, \ldots, 2n,\ j = 0, 1, \ldots, i \\ \text{or } i = 1, 3, \ldots, 2n-1,\ j = 0, 1, \ldots, i+1 \end{array} \\ 0 \quad \text{o.w.} \end{cases} \tag{4.4}$$

$$(Y^{(k)})_{i+1} = y_k(x_i), \quad i = 0, 1, \ldots, 2n. \tag{4.5}$$

Moreover, by operating the same procedure as we explained in Section 3, we can solve the nonlinear Volterra integral equations.

Now, consider the following nonlinear Volterra integral equation. We are going to solve this equation by combining the Newton–Kantorovich and block-by-block methods. The results are listed in Tables 4–6, and Fig. 2. The exact solution to the following equation is $y(x) = \sin x$.

$$y(x) = \sin x - \frac{x}{2} + \frac{1}{4}\sin 2x + \int_0^x y^2(t)\,dt, \quad 0 \le x \le 1. \tag{4.6}$$

Table 5
Solutions to the integral equation (4.6) for $m = 3$.

Nodes	Exact solutions	Approximate solutions	Error
0	0	0	0
0.05	0.04997916927068	0.04993752343239	−0.00004164583829
0.1	0.09983341664683	0.09950074934559	−0.00033266730123
0.15	0.14943813247360	0.14864782752524	−0.00079030494836
0.2	0.19866933079506	0.19713420136978	−0.00153512942529
0.25	0.24740395925452	0.24348891515832	−0.00391504409620
0.3	0.29552020666134	0.28886963085576	−0.00665057580558
0.35	0.34289780745545	0.33402432780083	−0.00887347965462
0.4	0.38941834230865	0.37551688104730	−0.01390146126135
0.45	0.43496553411123	0.41536435885283	−0.01960117525840
0.5	0.47942553860420	0.45520325261229	−0.02422228599192
0.55	0.52268722893066	0.48739205117169	−0.03529517775897
0.6	0.56464247339504	0.51766002578542	−0.04698244760962
0.65	0.60518640573604	0.55050912526944	−0.05467728046660
0.7	0.64421768723769	0.57526361325793	−0.06895407397976
0.75	0.68163876002333	0.59709408901407	−0.08454467100927
0.8	0.71735609089952	0.62144057454843	−0.09591551635110
0.85	0.75128040514029	0.63181372699168	−0.11946667814861
0.9	0.78332690962748	0.63952099663276	−0.14380591299472
0.95	0.81341550478937	0.65368545142539	−0.15973005336399
1	0.84147098480790	0.65608333976798	−0.18538764503992

Table 6
Solutions to the integral equation (4.6) for $m = 4$.

Nodes	Exact solutions	Approximate solutions	Error
0	0	0	0
0.05	0.04997916927068	0.04993752343239	−0.00004164583829
0.1	0.09983341664683	0.09950074934559	−0.00033266730123
0.15	0.14943813247360	0.14864782752524	−0.00079030494836
0.2	0.19866933079506	0.19713420136978	−0.00153512942529
0.25	0.24740395925452	0.24348891515832	−0.00391504409620
0.3	0.29552020666134	0.28886963085576	−0.00665057580558
0.35	0.34289780745545	0.33402432780083	−0.00887347965462
0.4	0.38941834230865	0.37551688104730	−0.01390146126135
0.45	0.43496553411123	0.41536435885283	−0.01960117525840
0.5	0.47942553860420	0.45520325261229	−0.02422228599192
0.55	0.52268722893066	0.48739205117169	−0.03529517775897
0.6	0.56464247339504	0.51766002578542	−0.04698244760962
0.65	0.60518640573604	0.55050912526944	−0.05467728046660
0.7	0.64421768723769	0.57526361325793	−0.06895407397976
0.75	0.68163876002333	0.59709408901407	−0.08454467100927
0.8	0.71735609089952	0.62144057454843	−0.09591551635110
0.85	0.75128040514029	0.63181372699168	−0.11946667814861
0.9	0.78332690962748	0.63952099663276	−0.14380591299472
0.95	0.81341550478937	0.65368545142539	−0.15973005336399
1	0.84147098480790	0.65608333976798	−0.18538764503992

5. Conclusion

As we explained above, solving the nonlinear integral equations leads to a nonlinear system of equations, which is the same as (1.2) and (1.3), which may not be solvable easily, but in the Newton–Kantorovich–quadrature method, the solutions of nonlinear integral equations lead to a sequence of linear systems of equations that are solvable by different methods.

For solving the nonlinear integral equations by this method, which we explained in Sections 2 and 3, we need to increase values of m and n, but we do not need to increase m significantly. For this purpose, if one compares the two latter tables in each of our examples, it can be easily seen that the results of these tables are very close; even in Example 2 the results in their tables are identical. This shows that for solving a nonlinear integral equation, we only need to increase the value of n largely and do not need to increase m at the same rate as n increases.

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