Factorization law for two lower bounds of concurrence

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We study the dynamics of two lower bounds of concurrence in bipartite quantum systems when one party goes through an arbitrary channel. We show that these lower bounds obey the factorization law similar to that of [Konrad *et al.*, Nat. Phys. **4**, 99 (2008)]. We also discuss the application of this property in an example.

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I. INTRODUCTION

Entanglement, one of the important features of quantum systems, which does not exist classically, has been known as a key resource for some quantum computation and information processes. But the entanglement of a system changes due to its unavoidable interactions with environment. To study the entanglement changes, one needs to make use of an entanglement measure in order to specify the entanglement amount of a system. Unfortunately, most of the measures having been proposed for quantification of entanglement cannot be computed in general, and because of this, many lower and upper bounds, which can be computed easily, have been introduced for these entanglement measures. Using these bounds, one can estimate the amount of entanglement.

In Ref. [1], Konrad *et al.* have provided a factorization law for concurrence, which is one of the remarkable entanglement measures. They have shown that the concurrence of a two-qubit state, when one of its qubits goes through an arbitrary quantum channel, is equal to the product of its initial concurrence and concurrence of the maximally entangled state undergoing the effect of the same quantum channel. Then Li *et al.* [2] have shown that the generalization of the preceding factorization law to arbitrary dimensional bipartite states only leads to an upper bound for the concurrence of the system. If, besides this upper bound, we have a lower bound obeying a similar factorization law, then we can make better use of this useful dynamical property. So, it will be valuable to seek such entanglement lower bounds.

In Sec. II, we introduce the concurrence and some of its lower bounds. Next, in Sec. III, we briefly review the results of Refs. [1,2]. Then, in Secs. IV and V, we investigate the factorization property of the lower bounds introduced in Sec. II. In Sec. VI, as an application, we discuss an example. Finally, we give some conclusions in Sec. VII.

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II. CONCURRENCE AND SOME OF ITS LOWER BOUNDS

For a pure bipartite state $|\Psi\rangle$, $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, concurrence is defined as [3]

$$C(\Psi) = \sqrt{2\left[\langle\Psi|\Psi\rangle^2 - \operatorname{tr}\rho_r^2\right]}, \tag{1}$$

where ρ_r is the reduced density operator obtained by tracing over either subsystem A or B. Concurrence of $|\Psi\rangle$ can also be written in terms of the expectation value of an observable with respect to two identical copies of $|\Psi\rangle$ [3–5],

$$\begin{gathered} C(\Psi) = \sqrt{{}_{AB}\langle\Psi|_{A'B'}\langle\Psi|\mathcal{A}|\Psi\rangle_{AB}|\Psi\rangle_{A'B'}}, \\ \mathcal{A} = 4P_-^{AA'} \otimes P_-^{BB'}, \end{gathered} \tag{2}$$

where $P_-^{AA'}$ ($P_-^{BB'}$) is the projector onto the antisymmetric subspace of $\mathcal{H}_A \otimes \mathcal{H}_{A'}$ ($\mathcal{H}_B \otimes \mathcal{H}_{B'}$). A possible decomposition of $\mathcal{A}$ is

$$\begin{gathered} \mathcal{A} = \sum_{i<j,m<n} |\chi_{ij,mn}\rangle\langle\chi_{ij,mn}|, \\ |\chi_{ij,mn}\rangle = (|ij\rangle - |ji\rangle)_{AA'}(|mn\rangle - |nm\rangle)_{BB'}, \end{gathered} \tag{3}$$

where $|i\rangle$ and $|j\rangle$ ($|m\rangle$ and $|n\rangle$) are two different members of an orthonormal basis of the A (B) subsystem (instead of the index α in Ref. [3], we use the indices ij,mn because they seem most convenient for future usage).

For mixed states, the concurrence is defined as follows [3]:

$$\begin{gathered} C(\rho) = \min_{\{p_k,\Psi_k\}} \sum_k p_k C(\Psi_k), \quad \rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|, \\ p_k \geqslant 0, \quad \sum_k p_k = 1, \end{gathered} \tag{4}$$

where the minimum is taken over all decompositions of ρ into pure states $|\Psi_k\rangle$. Like most of the other entanglement measures, $C(\rho)$ cannot be computed in general [i.e., in general, one cannot find the optimal decomposition of ρ minimizing Eq. (4)]. Any numerical effort to find the optimal decomposition is equivalent to find an upper bound for $C(\rho)$. So, some lower bounds have been introduced for $C(\rho)$ (e.g., Refs. [6,7]).

It has been shown that

$$C^{\rm ALB}_{ij,mn}(\rho) \equiv \min_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k |\langle \chi_{ij,mn}|\Psi_k\rangle|\Psi_k\rangle| \tag{5}$$

is a lower bound of concurrence (ALB is the abbreviation of the algebraic lower bound) [3,6,8]. $C^{\rm ALB}_{ij,mn}(\rho)$ can be computed analytically; $C^{\rm ALB}_{ij,mn}(\rho) = \max\{0, \mathscr{S}^{ij,mn}_1 - \sum_{l>1}\mathscr{S}^{ij,mn}_l\}$ [3]. $\mathscr{S}^{ij,mn}_l$ are the singular values of matrix $T^{ij,mn}$ in decreasing order. $T^{ij,mn}$'s entries are defined as $T^{ij,mn}_{rs} \equiv \sqrt{\lambda_r\lambda_s}\langle\chi_{ij,mn}|\Phi_r\rangle|\Phi_s\rangle$, where $|\Phi_r\rangle$ and λ_r are eigenvectors and eigenvalues of ρ, respectively.

Other lower bounds of concurrence are those introduced in Ref. [8]. In this reference, it has been shown that in terms of two identical copies of an arbitrary mixed state ρ_{AB}, we have

$$\begin{aligned} C^2(\rho_{AB}) &\geqslant C^{\rm MLB^2}_{(k)ij,mn}(\rho) \equiv {\rm tr}\,(\rho_{AB}\otimes\rho_{A'B'}V_{(k)ij,mn}),\\ k=1,2,\quad V_{(1)ij,mn} &= 4P^{AA'}_{-ij}\otimes\big(P^{BB'}_{-mn}-P^{BB'}_{+mn}\big),\\ V_{(2)ij,mn} &= 4\big(P^{AA'}_{-ij}-P^{AA'}_{+ij}\big)\otimes P^{BB'}_{-mn}. \end{aligned} \tag{6}$$

(MLB is the abbreviation of the measurable lower bound) where $2P^{AA'}_{-ij} = (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)$ and $2P^{AA'}_{+ij} = (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|) + 2|ii\rangle\langle ii| + 2|jj\rangle\langle jj|$ operate on $\mathcal{H}_A\otimes\mathcal{H}_{A'}$, whereas $2P^{BB'}_{-mn}=(|mn\rangle - |nm\rangle)(\langle mn| - \langle nm|)$ and $2P^{BB'}_{+mn} = (|mn\rangle + |nm\rangle)(\langle mn|+\langle nm|) + 2|mm\rangle\langle mm| + 2|nn\rangle\langle nn|$ operate on $\mathcal{H}_B\otimes\mathcal{H}_{B'}$ [$|i\rangle, |j\rangle, |m\rangle$, and $|n\rangle$ were introduced in Eq. (3)]. The previous expression means that measuring $V_{(k)ij,mn}$ on two identical copies of ρ (i.e., $\rho\otimes\rho$) gives us a measurable lower bound on $C^2(\rho)$.

In Ref. [9], another lower bound of concurrence was introduced. There, it was shown that

$$\begin{aligned} \tau(\rho) &\equiv \sum_{i<j,m<n} C^2_{ij,mn}(\rho) \leqslant C^2(\rho),\\ C_{ij,mn}(\rho) &= \min_{\{p_k,|\psi_k\rangle\}} \sum_k p_k|\langle\Psi_k|L_{A,ij}\otimes L_{B,mn}|\Psi^*_k\rangle|, \end{aligned} \tag{7}$$

where $L_{A,ij}$ and $L_{B,mn}$ are the generators of SO(d_A) and SO(d_B), respectively [$d_A(d_B)$ is the dimension of $\mathcal{H}_A(\mathcal{H}_B)$], and $|\Psi^*_k\rangle$ is the complex conjugate of $|\Psi_k\rangle$ in the computational basis. In this basis, $L_{A,ij}$ and $L_{B,mn}$ are [10]

$$\begin{aligned} L_{A,ij} &= |i\rangle_A\langle j| - |j\rangle_A\langle i|,\\ L_{B,mn} &= |m\rangle_B\langle n| - |n\rangle_B\langle m|. \end{aligned} \tag{8}$$

III. FACTORIZATION OF THE CONCURRENCE

According to the Schmidt decomposition, any pure bipartite state $|\Psi\rangle$, $|\Psi\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B$, can be expressed as

$$|\Psi\rangle = \sum_{i=1}^d \sqrt{\omega_i}|\alpha_i\beta_i\rangle,\quad 0\leqslant\sqrt{\omega_i}\leqslant 1,\quad \sum_{i=1}^d\omega_i = 1, \tag{9}$$

where $d = \min(d_A, d_B)$.

We can rewrite this $|\Psi\rangle$ as $|\Psi\rangle = (M\otimes I)|\phi^+\rangle$ where $|\phi^+\rangle = \sum_{i=1}^d \frac{1}{\sqrt{d}}|\alpha_i\beta_i\rangle$ is a maximally entangled state and $M = \sqrt{d}\sum_{i=1}^d\sqrt{\omega_i}|\alpha_i\rangle\langle\alpha_i|$.

Assume that the second part of this state goes through an arbitrary channel $\mathcal{S}$, then this state transforms to $\rho' = \frac{(\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|}{p'}$ where $p' = {\rm tr}\,[(\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|]$. Since M and $\mathcal{S}$ act on two different parts of $|\Psi\rangle$, ρ' can be written as $\rho' = \frac{(M\otimes\mathbf{I})\rho_S(M^\dagger\otimes\mathbf{I})}{p}$ where $\rho_S = \frac{(\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|}{p''}$, $p = {\rm tr}\,[(M\otimes\mathbf{I})\rho_S(M^\dagger\otimes\mathbf{I})]$, $p'' = {\rm tr}\,[(\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|]$, and $p' = pp''$.

By using these relations, for any two-qubit state $|\Psi\rangle$, Konrad *et al.* [1] have proved the following factorization law [11]:

$$C[(\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|] = C[(\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|]C(\Psi). \tag{10}$$

The right-hand side of the preceding equation is factorized into two independent parts. The first part is the concurrence of $|\phi^+\rangle$ after going through the channel $(\mathbf{1}\otimes\mathcal{S})$, which is independent of the initial state $|\Psi\rangle$, and the second part is the concurrence of the initial state $|\Psi\rangle$ (before going into the channel). So, if we know the concurrence of $|\phi^+\rangle$, after one of its qubits goes through a channel $\mathcal{S}$, we know, up to the factor $C(\Psi)$, the concurrence of any arbitrary state $|\Psi\rangle$ truly undergoing the same quantum channel.

For higher-dimensional bipartite systems, Li *et al.* [2] have shown that the previous equality changes to the following inequality:

$$C[(\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|] \leqslant \frac{d_B}{2}C[(\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|]C(\Psi). \tag{11}$$

For the $d_A\times 2$-dimensional states, we have the equality instead of the inequality in the foregoing relation. But, in general, the concurrence of $(\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|$ provides only an upper bound for $C[(\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|]$. We point out that, in relations (10) and (11), instead of $|\phi^+\rangle$, we can use any other maximally entangled state.

It is also interesting to investigate a similar relations for the lower bounds of concurrence. In Sec. IV, we study the factorization property of the lower bounds introduced in Sec. II.

IV. FACTORIZATION OF THE LOWER BOUNDS OF CONCURRENCE

Let us at first consider the lower bound introduced in expression (6). From this relation, we have

$$\begin{aligned} p^2C^{\rm MLB^2}_{(1)ij,mn}(\rho') &= p^2\,{\rm tr}\,(\rho'_{AB}\otimes\rho'_{A'B'}V_{(1)ij,mn})\\ &= {\rm tr}\,[(M_A\otimes\mathbf{I}_B)\rho_{SAB}(M^\dagger_A\otimes\mathbf{I}_B)\\ &\quad\otimes(M_{A'}\otimes\mathbf{I}_{B'})\rho_{SA'B'}(M^\dagger_{A'}\otimes\mathbf{I}_{B'})V_{(1)ij,mn}]\\ &= {\rm tr}\,[(M_A\otimes\mathbf{I}_B\otimes M_{A'}\otimes\mathbf{I}_{B'})(\rho_{SAB}\otimes\rho_{SA'B'})\\ &\quad\times(M^\dagger_A\otimes\mathbf{I}_B\otimes M^\dagger_{A'}\otimes\mathbf{I}_{B'})V_{(1)ij,mn}]\\ &= {\rm tr}\,[(\rho_{SAB}\otimes\rho_{SA'B'})(M^\dagger_A\otimes\mathbf{I}_B\otimes M^\dagger_{A'}\otimes\mathbf{I}_{B'})\\ &\quad\times V_{(1)ij,mn}(M_A\otimes\mathbf{I}_B\otimes M_{A'}\otimes\mathbf{I}_{B'})]\\ &= d^2\omega_i\omega_j\,{\rm tr}\,[\rho_{SAB}\otimes\rho_{SA'B'}V_{(1)ij,mn}]. \end{aligned} \tag{12}$$

In order to obtain the last equality, we have used $(M^\dagger_A\otimes M^\dagger_{A'})P^{AA'}_{-ij}(M_A\otimes M_{A'}) = d^2\omega_i\omega_j P^{AA'}_{-ij}$ where $P^{AA'}_{-ij}$ is written in the Schmidt basis (i.e., we choose $|i\rangle = |\alpha_i\rangle$ and $|j\rangle = |\alpha_j\rangle$ in construction of $P^{AA'}_{-ij}$). Also, writing $P^{BB'}_{-mn}$ and $P^{BB'}_{+mn}$ in the

Schmidt basis, we have $C^{\mathrm{MLB}^2}_{(1)ij,mn}(|\Psi\rangle) = 4\omega_i\omega_j\delta_{im}\delta_{jn}$. Using this relation, Eq. (12) can be written in the form:

$$C^{\mathrm{MLB}^2}_{(1)ij,mn}\big((\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|\big) = \frac{d^2}{4}C^{\mathrm{MLB}^2}_{(1)ij,mn}\big((\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|\big)C^{\mathrm{MLB}^2}_{(1)ij,ij}(|\Psi\rangle). \quad (13)$$

The preceding equation (which is our main result) is similar to Eq. (10), so $C^{\mathrm{MLB}^2}_{(1)ij,mn}(\rho)$ has the same factorization property as concurrence [i.e., knowing the effect of $(\mathbf{1}\otimes\mathcal{S})$ on the $C^{\mathrm{MLB}^2}_{(1)ij,mn}(\rho)$ when the initial state is $|\Phi^+\rangle$, we know this effect for any other initial state $|\Psi\rangle$, up to a factor $C^{\mathrm{MLB}^2}_{(1)ij,ij}(|\Psi\rangle)$].

For the $C^{\mathrm{MLB}^2}_{(2)ij,mn}(\rho)$, we obtain exactly the same result as earlier if instead of the second part, the first part of the state $|\Psi\rangle$ goes through the channel $\mathcal{S}$.

Now, we discuss the factorization property of $C^{\mathrm{ALB}}_{ij,mn}(\rho')$. We use a similar method as Ref. [2], namely, at first, we restrict ourselves to those cases where $\rho_{\mathcal{S}}$ is a pure state (i.e., $\rho_{\mathcal{S}} = |\psi\rangle\langle\psi|$). In this case, ρ' is also a pure state [i.e., $\rho' \equiv |\psi'\rangle\langle\psi'| = \frac{(M\otimes\mathbf{I})|\psi\rangle\langle\psi|(M^\dagger\otimes\mathbf{I})}{p}$]. From Eq. (5), we have

$$\begin{aligned} pC^{\mathrm{ALB}}_{ij,mn}(|\psi'\rangle) &= p|\langle\chi_{ij,mn}|\psi'\rangle|\psi'\rangle| \\ &= |\langle\chi_{ij,mn}|M\otimes\mathbf{I}\otimes M\otimes\mathbf{I}|\psi\rangle|\psi\rangle| \\ &= d\sqrt{\omega_i\omega_j}C^{\mathrm{ALB}}_{ij,mn}(\rho_{\mathcal{S}}), \end{aligned} \quad (14)$$

where we used $(M^\dagger\otimes\mathbf{I}\otimes M^\dagger\otimes\mathbf{I})|\chi_{ij,mn}\rangle\langle\chi_{ij,mn}|(M\otimes\mathbf{I}\otimes M\otimes\mathbf{I}) = d^2\omega_i\omega_j|\chi_{ij,mn}\rangle\langle\chi_{ij,mn}|$ and $|\chi_{ij,mn}\rangle$ is written in the Schmidt basis. Using $C^{\mathrm{ALB}^2}_{ij,mn}(|\Psi\rangle) = 4\omega_i\omega_j\delta_{im}\delta_{jn}$, we obtain

$$pC^{\mathrm{ALB}}_{ij,mn}(|\psi'\rangle) = \frac{d}{2}C^{\mathrm{ALB}}_{ij,ij}(|\Psi\rangle)C^{\mathrm{ALB}}_{ij,mn}(\rho_{\mathcal{S}}). \quad (15)$$

Next, we consider the general case where $\rho_{\mathcal{S}}$ is a mixed state. Corresponding to any pure state decomposition of $\rho_{\mathcal{S}}$ as $\rho_{\mathcal{S}} = \sum_k p_k|\psi_k\rangle\langle\psi_k|$, there exists a pure state decomposition for ρ' in terms of pure states $|\psi'_k\rangle = \frac{(M\otimes\mathbf{I})|\psi_k\rangle}{\sqrt{pq_k}}$, $q_k = \mathrm{tr}\,[\frac{(M\otimes\mathbf{I})|\psi_k\rangle\langle\psi_k|(M^\dagger\otimes\mathbf{I})}{p}]$ such that $\rho' = \sum_k p_k q_k|\psi'_k\rangle\langle\psi'_k|$. Thus, by using the same arguments as before, for any $|\psi'_k\rangle$, we have $pq_k|\langle\chi_{ij,mn}|\psi'_k\rangle|\psi'_k\rangle| = d\sqrt{\omega_i\omega_j}|\langle\chi_{ij,mn}|\psi_k\rangle|\psi_k\rangle|$. Now, assume $\rho_{\mathcal{S}} = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ is the optimal pure state decomposition, which gives $C^{\mathrm{ALB}}_{ij,mn}(\rho_{\mathcal{S}})$ [i.e., $C^{\mathrm{ALB}}_{ij,mn}(\rho_{\mathcal{S}}) = \sum_k p_k|\langle\chi_{ij,mn}|\psi_k\rangle|\psi_k\rangle|$ so $p\sum_k p_k q_k|\langle\chi_{ij,mn}|\psi'_k\rangle|\psi'_k\rangle| = d\sqrt{\omega_i\omega_j}C^{\mathrm{ALB}}_{ij,mn}(\rho_{\mathcal{S}})$]. But $\sum_k p_k|\psi'_k\rangle\langle\psi'_k|$ is *not* necessarily the optimal pure state decomposition of ρ' such that $C^{\mathrm{ALB}}_{ij,mn}(\rho') = \sum_k p_k|\psi'_k\rangle\langle\psi'_k|$. Therefore, in general,

$$C^{\mathrm{ALB}}_{ij,mn}\big((\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|\big) \leqslant \frac{d}{2}C^{\mathrm{ALB}}_{ij,ij}(|\Psi\rangle)C^{\mathrm{ALB}}_{ij,mn}\big((\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|\big). \quad (16)$$

In the cases where M^{-1} exists [i.e., when in Eq. (9) for all ω_i we have $\omega_i \neq 0$], as for the $d_A\times 2$-dimensional systems (the case of the separable initial states is not of interest), corresponding to any pure state decomposition for ρ', there is a pure state decomposition for $\rho_{\mathcal{S}}$ and vice versa, namely, for any $|\psi'_k\rangle$ in the expression $\rho' = \sum_k p_k|\psi'_k\rangle\langle\psi'_k|$, we have $|\psi_k\rangle = \sqrt{p}(M^{-1}\otimes\mathbf{I})|\psi'_k\rangle$ such that $\rho_{\mathcal{S}} = \sum_k p_k|\psi_k\rangle\langle\psi_k|$. So, if the $\rho_{\mathcal{S}} = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ is the optimal decomposition for $C^{\mathrm{ALB}}_{ij,mn}(\rho_{\mathcal{S}})$, then $\sum_k p_k|\psi'_k\rangle\langle\psi'_k|$ is the optimal pure state decomposition of ρ' for $C^{\mathrm{ALB}}_{ij,mn}(\rho')$. Therefore, in Eq. (16), we have an equality instead of the inequality.

V. FACTORIZATION OF THE LOWER BOUND OF SQUARED CONCURRENCE (τ)

In Ref. [12], Liu and Fan have shown that τ [Eq. (7)], for a $d\times d$ bipartite quantum state, obeys the relation

$$\tau\big((\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|\big) \leqslant \frac{d^2}{4}\tau\big((\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|\big)C^2(\Psi). \quad (17)$$

The preceding relation is the factorization law for τ similar to Eq. (11), which is for the concurrence itself.

Now, we show that $C^{\mathrm{ALB}}_{ij,mn}(\rho)$ is closely related to τ; for an arbitrary $|\Psi\rangle$, according to the definition of $|\chi_{ij,mn}\rangle$ in Eq. (3), it can be seen that $|\langle\Psi|L_{A,ij}\otimes L_{B,mn}|\Psi^*\rangle| = |\langle\chi_{ij,mn}|\Psi\rangle|\Psi\rangle|$. So, from Eq. (5), we have

$$\begin{aligned} C^{\mathrm{ALB}}_{ij,mn}(\rho) &= \min_{\{p_k,|\Psi_k\rangle\}}\sum_k p_k|\langle\chi_{ij,mn}|\Psi_k\rangle|\Psi_k\rangle| \\ &= \min_{\{p_k,|\Psi_k\rangle\}}\sum_k p_k|\langle\Psi_k|L_{A,ij}\otimes L_{B,mn}|\Psi^*_k\rangle|. \end{aligned} \quad (18)$$

From Eq. (7) and the previous equation, we deduced that

$$C_{ij,mn}(\rho) = C^{\mathrm{ALB}}_{ij,mn}(\rho) \quad (19)$$

and so

$$\tau(\rho) = \sum_{i<j,m<n} C^{\mathrm{ALB}^2}_{ij,mn}(\rho). \quad (20)$$

Therefore, from Eqs. (16) and (19), we deduce that Eq. (12) of Ref. [12], that is,

$$C^2_{ij,mn}\big((\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|\big) = \frac{d^2}{4}\left(\sum_{l>k=0}^{d-1}C_{ij,kl}(|\Psi\rangle)\right)C_{kl,mn}\big((\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|\big)^2, \quad (21)$$

and so Eq. (15) of the same reference, that is,

$$\tau\big((\mathbf{1}\otimes\mathcal{S})|\Psi\rangle\langle\Psi|\big) \geqslant \frac{2d\eta}{d-1}\frac{d^2}{4}\tau\big((\mathbf{1}\otimes\mathcal{S})|\phi^+\rangle\langle\phi^+|C^2(|\Psi\rangle)\big), \quad (22)$$

where $\eta = \min_{\{p,r\}}\omega_p\omega_r$ for any pair $p<r$ satisfying $\omega_p\omega_r \neq 0$ does not hold in general.

VI. EXAMPLE

Consider a two-qutrit system where one of its qutrits interacts with an environment. The time evolution of this system is given by the following master equation:

$$\dot{\rho} = \mathcal{L}\rho, \quad \mathcal{L} = 1_A\otimes\mathcal{L}_B, \quad (23)$$

where $\mathcal{L}_B$, for a one-qutrit ρ_B, is

$$\mathcal{L}_B = \frac{\Gamma}{2}(2\gamma\rho_B\gamma^\dagger - \rho_B\gamma^\dagger\gamma - \gamma^\dagger\gamma\rho_B).$$

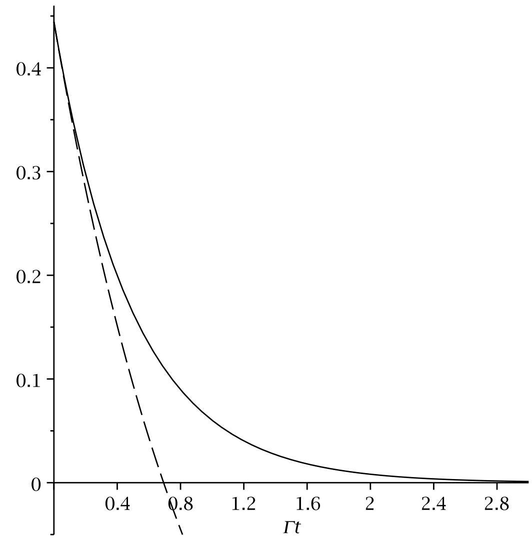

FIG. 1. Time evolution of the $C^{\mathrm{MLB}^2}_{(1)12,12}$ when the initial state of the system is $|\phi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ for the cases (a) spontaneous decay(dashed line) (b) decoherence (solid line).

Γ is the decay constant, and γ is a coupling operator characterizing the dynamics of system.

$$\text{For } \gamma = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

Eq. (23) represents the spontaneous decay of the system. and for

$$\gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Eq. (23) represents the system's decoherence [13].

In order to evaluate the entanglement dynamics of this system, we use the $C^{\mathrm{MLB}^2}_{(1)ij,mn}(\rho)$ (which is a lower bound of squared concurrence). Figure 1 shows the time evolution of $C^{\mathrm{MLB}^2}_{(1)ij,mn}(\rho)$ for the case $i = 1$, $j = 2$, $m = 1$, and $n = 2$, when the initial state of the system is $|\phi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ (for other values of i, j, m, and n, $C^{\mathrm{MLB}^2}_{(1)ij,mn}(\rho)$ does not give a better estimate for entanglement). From this figure and using Eq. (13), we can deduce the behavior of $C^{\mathrm{MLB}^2}_{(1)12,12}(\rho)$ for any initial states of the form $|\psi\rangle = a|00\rangle + b|11\rangle + c|22\rangle$. For any such initial state, the ability of the $C^{\mathrm{MLB}}_{(1)12,12}$ in detecting the entanglement of $\rho' = (\mathbf{1} \otimes \mathcal{S})|\Psi\rangle\langle\Psi|$ is determined by the ability of $C^{\mathrm{MLB}}_{(1)12,12}$ in detecting the entanglement of $\rho_{\mathcal{S}} = (\mathbf{1} \otimes \mathcal{S})|\phi^+\rangle\langle\phi^+|$, which is shown in Fig. 1. Also, the amount of the lower bound $C^{\mathrm{MLB}}_{(1)12,12}(\rho')$ is, up to a factor, equal to $C^{\mathrm{MLB}}_{(1)12,12}(\rho_{\mathcal{S}})$.

VII. CONCLUSIONS

We have studied the dynamics of two lower bounds of bipartite concurrence introduced in Eqs. (5) and (6), when one party goes through an arbitrary channel. In Eq. (13), we have shown that, for arbitrary bipartite quantum states, $C^{\mathrm{MLB}}_{(1)ij,mn}(\rho)$ obeys the factorization law similar to that of Eq. (10) for the concurrence. In an example, we have discussed the application of this factorization law in determining the behavior of the $C^{\mathrm{MLB}}_{(1)ij,mn}(\rho)$ in estimating the entanglement of the system. Also, we have shown that the $C^{\mathrm{ALB}}_{ij,mn}(\rho)$ obeys a similar factorization law for concurrence as Eq. (11).

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