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SOLVING OF SECOND ORDER NONLINEAR PDE PROBLEMS BY USING ARTIFICIAL CONTROLS WITH CONTROLLED ERROR

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Abstract. In this paper, we find the approximate solution of a second order nonlinear partial differential equation on a simple connected region in R^2. We transfer this problem to a new problem of second order nonlinear partial differential equation on a rectangle. Then, we transformed the later one to an equivalent optimization problem. Then we consider the optimization problem as a distributed parameter system with artificial controls. Finally, by using the theory of measure, we obtain the approximate solution of the original problem. In this paper also the global error in L_1 is controlled.

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1. Introduction

Numerical methods such as finite difference and finite element methods usually are used to find the approximate solution of boundary value problems. Nonlinear partial differential equations are difficult to solve by finite difference or by finite element method, because the system of algebraic equations which arises from discretization of these equations usually are nonlinear [14], [17]. In this paper we introduce a new technique for finding an approximate solution of a second order nonlinear partial differential equation.

Let A be a simple connected region. By the Riemann mapping theorem, there exists a conformal mapping which maps A onto a circle [13]. Also by using Schwarz-Christoffel theorem, we can map A the region A onto a rectangle [13].

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By this mapping may be the nonlinear partial differential equation convert to another one, which is defined on a rectangle. We will see that it is not important. So, we consider a nonlinear partial differential equation of the form

$$f_0\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}\right) = 0, \qquad (x, y) \in A^\circ \tag{1}$$

with the boundary condition

$$B_0(u(x, y)) = g(x, y), \qquad (x, y) \in \Gamma, \tag{2}$$

where $A = [x_a, x_b] \times [y_a, y_b]$ with interior A°, f_0 is a nonlinear function on R^8, B_0 is a linear partial differential operator on Γ of order at most one, $g : \Gamma \to I\!R$ is continuous functions. Our objective is to find an approximate solution for the problem (1) and (2) by using measure theory.

The solution of the classical optimal control problems by using measure theory has been introduced by Rubio [16]. Finding optimal control for diffusion and wave equations, using measure theory, has been considered by several authors, including [1], [4], [6], [7], [9], [10], [11], [12], [18]. The solution of systems of ordinary differential equations using measure theory has been considered in [2]. Also the solution of second order nonlinear PDE problems has been considered in [8] without controlling error function.

2. Transforming the problem into an optimization problem

Let $u(x, y)$ be the solution of the problem (1) and (2). Let us define new functions $v_1, \cdots, v_5$ on A° as follows:

$$\begin{aligned} v_1(x, y) &= \frac{\partial u(x, y)}{\partial x}, & v_2(x, y) &= \frac{\partial u(x, y)}{\partial y}, \\ v_3(x, y) &= \frac{\partial v_1(x, y)}{\partial x}, & v_4(x, y) &= \frac{\partial v_2(x, y)}{\partial y}, \\ v_5(x, y) &= \frac{\partial v_1(x, y)}{\partial y} = \frac{\partial v_2(x, y)}{\partial x}. \end{aligned} \tag{3}$$

We assume that v_1 and v_2 are $C^1(A)$ functions and v_3, v_4, and v_5 are continuous functions on A°. Now, we define a function $F : A \subset I\!R^2 \longrightarrow I\!R$ by

$$F(x, y) = |f_0(x, y, u(x, y), v_1(x, y), v_2(x, y), v_3(x, y), v_4(x, y), v_5(x, y))|.$$

For given $\epsilon > 0$, which we call it admissible error, we want to find $u(\cdot, \cdot)$ such that $\int_A F(x, y)dxdy$ less than ϵ, and the condition (2) is satisfied too. Also, we compute the minimization of $\int_A F(x,y)dxdy$. On the other hand, we consider the following optimization problem:

$$\min \int_A F(x,y)dxdy$$

on K, where K is the set of all $(x,y,u,v_1,v_2,v_3,v_4,v_5)$ that satisfies the following conditions:

$$\begin{aligned}
&B_0(u(x,y)) = g(x,y), && && (x,y) \in \Gamma,\\
&\frac{\partial u(x,y)}{\partial x} = v_1(x,y), && \frac{\partial v_1(x,y)}{\partial x} = v_3(x,y), && (x,y) \in A^\circ,\\
&\frac{\partial u(x,y)}{\partial y} = v_2(x,y), && \frac{\partial v_2(x,y)}{\partial y} = v_4(x,y), && (x,y) \in A^\circ,\\
&\frac{\partial v_1(x,y)}{\partial y} = v_5(x,y), && && (x,y) \in A^\circ,\\
&\int_A F(x,y)dxdy \leq \epsilon.
\end{aligned} \tag{4}$$

Assume u, the solution of the problem (1) and (2), be bounded and $u \in U \equiv [u_a, u_b]$ and $v = (v_1, v_2, v_3, v_4, v_5) \in V \equiv V_1 \times V_2 \times V_3 \times V_4 \times V_5$, where $v_i \in V_i \equiv [v_{ia}, v_{ib}]$ and $\Omega = A \times U \times V \subseteq I\!\!R^8$. We call v a control function.

Definition 1. A trajectory for v is an absolutely continuous function $u(\cdot,\cdot)$ on A° such that $u(\cdot,\cdot)$ satisfies (3).

Definition 2. We call the pair $P = (u,v)$ as a trajectory-control pair, and we call it admissible if (u,v) satisfies the boundary condition of (4).

Note that in view of the above definitions, u is the solution of the problem (1) and (2).

The set of admissible pairs will be denoted by W. The set W is nonempty, since it is assumed that the problem (1) and (2) has a solution. We conclude that K is nonempty.

Consider the mapping $I : W \to R$ defined by

$$I(P) = \int_A F(x,y)dxdy. \tag{5}$$

If the minimum of the mapping I over the set W be zero, then we have $F(x,y)=0$ on A. Therefore,

$$f_0\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2},\frac{\partial^2 u}{\partial x\partial y}\right)=0$$

and

$$B_0(u(x,y))=g(x,y) \quad \text{on} \quad \Gamma,$$

that is, the exact solution for the problem (1) and (2) is obtained. If minimum of the mapping I is less than ϵ, then we obtain $F(x,y)\approx 0$, in the sense of L_1 topology, so we have approximate solution for the problem.

Let B be an open ball in $\mathbb{R}^3$ containing $A\times U$ and $C'(B)$ be the space of all real functions that are twice continuously differentiable on B such that they and their first and second partial derivatives are bounded on B. Now for all $\varphi\in C'(B)$ we define $\Phi:\Omega\subset\mathbb{R}^8\longrightarrow\mathbb{R}$ by

$$\Phi(x,y,u,v_1,v_2,v_3,v_4,v_5)=\varphi_{uu}u_xu_y+\varphi_{ux}u_y+\varphi_{uy}u_x+\varphi_u u_{xy}+\varphi_{xy}. \quad (6)$$

Let $u_{01}=u(x_a,y_a), u_{02}=u(x_a,y_b), u_{03}=u(x_b,y_a)$, and $u_{04}=u(x_b,y_b)$. Then, if $P=(u,v)$ is an admissible pair, we have

$$\int_A \Phi(x,y,u,v_1,v_2,v_3,v_4,v_5)dxdy=\triangle\varphi, \quad (7)$$

where $\triangle\varphi=\varphi(x_a,y_a,u_{01})-\varphi(x_a,y_b,u_{02})-\varphi(x_b,y_a,u_{03})+\varphi(x_b,y_b,u_{04})$. In particular, if $\varphi\in C'(B)$ is of the form $\varphi=\theta(x,y)$ then from (6) we have

$$\int_A \varphi_{xy}dxdy=\int_{x_a}^{x_b}\int_{y_a}^{y_b}\theta_{xy}dxdy=a_\theta, \quad (8)$$

where a_θ is the value of the integral of $\theta_{xy}(x,y)$ on A. Now let $D(A^\circ)$ be the space of all infinitely differentiable real functions with compact support in A°. For any $\psi\in D(A^\circ)$ define the function Ψ as follows:

$$\begin{aligned}\Psi(x,y,u,v_1,v_2,v_3,v_4,v_5) &= B_0(u(x,y))(\psi_x(x,y)+\psi_y(x,y)) \\ &\quad +B_0((u_x(x,y)+u_y(x,y)))\psi(x,y),\end{aligned} \quad (9)$$

then we have [15]

$$\int_A \Psi(x,y,u,v_1,v_2,v_3,v_4,v_5)dxdy=\int_\Gamma B_0(u)\psi d\tau=0, \quad (10)$$

since ψ is a function with compact support in A° and $P=(u,v)$ is an admissible pair.

The relations $(7),(8)$ and (10) are the generalization of the properties of the admissible pairs in the formulation of the optimal control problems in distribution systems. Thus, we can obtain a new measure theoretical control problem.

For an admissible pair P, the mapping

$$\Lambda_P : G \mapsto \int_A G(x, y, u(x, y), v_1(x, y), v_2(x, y), v_3(x, y), v_4(x, y), v_5(x, y))dxdy \tag{11}$$

defines a positive linear functional on the space $C(\Omega)$ of continuous real-valued functions on Ω. The admissible pair $P \in W$ corresponds to a positive linear functional Λ_P on $C(\Omega)$ in one-to-one correspondence [16]. The equations (7), (8), and (10) can be written as follows:

$$\begin{cases} \Lambda_P(\Phi) = \triangle\varphi, & \varphi \in C'(B), \\ \Lambda_P(\theta) = a_\theta, & \theta \in C_1(\Omega), \\ \Lambda_P(\Psi) = 0, & \psi \in D(A^\circ), \end{cases} \tag{12}$$

where $C_1(\Omega)$ denotes the subspace of $C(\Omega)$ of those functions θ which depend on x and y only.

To enlarge the set W, and perhaps to overcome some of the difficulties associated with the formulation of the distributed optimal control problems, we shall develop a new framework by considering all positive linear functionals on $C(\Omega)$ satisfying (12).

By the Riesz representation theorem, it is convenient to identify each of such functions with a positive Radon measure on Ω [3]. Let $M^+(\Omega)$ be the set of all positive Radon measure on Ω. We can consider the set $Q \subset M^+(\Omega)$ such that for $\mu \in Q$ we have

$$\begin{cases} \mu(\Phi) = \triangle\varphi, & \varphi \in C'(B), \\ \mu(\theta) = a_\theta, & \theta \in C_1(\Omega), \\ \mu(\Psi) = 0, & \psi \in D(A^\circ). \end{cases} \tag{13}$$

Therefore, the new optimization problem consists of minimizing the linear functional $I : Q \longrightarrow R$, defined by

$$\mu \mapsto \mu(F) = \int_A F d\mu. \tag{14}$$

Hence, we have the following optimization problem:

$$\begin{cases} \min & \mu(F) \\ \text{s.t.}: & \mu \in Q. \end{cases} \tag{15}$$

Note that all the functions in (13) and (14) are linear in μ. Thus, the problem (15) is an infinite dimensional linear programming problem. Now, we consider the existence of an optimal measure in the set Q for the functional I. We define a topology on the set Q induced by the weak*-topology on $M^+(\Omega)$, then we have the following proposition [16]:

Proposition 1. *The measure-theoretical control problem, which is to find the minimum of the linear functional I over the set Q attains its minimum μ^* in Q.*

The proof of Proposition 1 is similar to theorem II.1 in [16].

3. Approximation

We shall approximate the infinite dimensional linear programming (2.13) by a finite dimensional one, then we shall obtain the approximate solution of the original problem (1.1) and (1.2) by using the optimal solution of the latter problem. In the following we state the stages of obtaining the approximation solution.

We consider the minimization of I over the subset of $M^+(\Omega)$ defined by requiring that only a finite number of the constraint in (13) be satisfied. This is achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate space, and then selecting a finite number of them. Let the functions $\psi \in D(A^\circ)$ be in the form

$$\begin{aligned} &\sin(2\pi\ell(x-x_a)/hx)\sin(2\pi\ell(y-y_a)/hy),\\ &\sin(2\pi\ell(x-x_a)/hx)(1-\cos(2\pi\ell(y-y_a)/hy)),\\ &(1-\cos(2\pi\ell(x-x_a)/hx))\sin(2\pi\ell(y-y_a)/hy),\\ &(1-\cos(2\pi\ell(x-x_a)/hx))(1-\cos(2\pi\ell(y-y_a)/hy)), \end{aligned} \tag{16}$$

where $hx = x_b - x_a$, $hy = y_b - y_a$, and $\ell = 1, 2, \cdots$. Then, we have the following theorem:

Theorem 1. *Consider the following linear programming:*

$$\begin{cases} \min & \mu(F) \\ s.t.: & \mu(\Phi_i) = \triangle\varphi_i, \quad i = 1, 2, \cdots, M_1, \\ & \mu(\Psi_j) = 0, \qquad j = 1, 2, \cdots, M_2, \end{cases} \tag{17}$$

where $\Psi_j, j = 1, 2, \cdots, M_2$, are obtained from the functions ψ which are of the form (16). As M_1 and M_2 tend to infinity, the solution of (17) tends to the solution of (15).

Proof. Let $Q(M_1, M_2)$ be the set of measures in $M^+(\Omega)$ satisfying in the constraints of problem (17), and η and $\gamma(M_1, M_2)$ be the optimal values of problems (15) and (17) respectively. As M_1 and M_2 tend to infinity, $\gamma(M_1, M_2)$ converges to ξ, and also $\xi \leq \eta$, [16]. Let

$$P = \bigcap_{M_1=1}^{\infty} \bigcap_{M_2=1}^{\infty} Q(M_1, M_2).$$

Then $P \supset Q$, [16]. So it is sufficient to show that $P \subset Q$. Let $S_1 = \text{span}\{\varphi_i, i = 1, 2, \cdots\}$ and $S_2 = \text{span}\{\psi_j, j = 1, 2, \cdots\}$. If $\mu \in P$ then $\mu(\Phi) = \triangle\varphi$ for all $\varphi \in S_1$. Now, for all $\varphi \in C'(B)$ there exists a sequence $\{\varphi^k\}$ in S_1 such that as k tends to infinity, the following sequences:

$$\sup_B \left|\varphi_{uu}(x,y,u) - \varphi^k_{uu}(x,y,u)\right|, \ \sup_B \left|\varphi_{ux}(x,y,u) - \varphi^k_{ux}(x,y,u)\right|,$$
$$\sup_B \left|\varphi_{uy}(x,y,u) - \varphi^k_{uy}(x,y,u)\right|, \ \sup_B \left|\varphi_{u}(x,y,u) - \varphi^k_{u}(x,y,u)\right|,$$
$$\sup_B \left|\varphi_{xy}(x,y,u) - \varphi^k_{xy}(x,y,u)\right|$$

tend to zero. Thus

$$\begin{aligned}
|\mu(\Phi) - \triangle\varphi| &= \left|\mu(\Phi) - \triangle\varphi - \mu(\Phi^k) + \triangle\varphi^k\right| \\
&= \Bigg|\int_\Omega \big\{\left[\varphi_{uu} - \varphi^k_{uu}\right] u_x\ u_y + \left[\varphi_{ux} - \varphi^k_{ux}\right] u_y + \left[\varphi_{uy} - \varphi^k_{uy}\right] u_x \\
&\quad + [\varphi_u - \varphi^k_u] u_{xy} + [\varphi_{xy} - \varphi^k_{xy}]\big\}\, d\mu - (\triangle\varphi - \triangle\varphi^k)\Bigg| \\
&\le K_1 \sup_B |\varphi_{uu} - \varphi^k_{uu}| + K_2 \sup_B |\varphi_{ux} - \varphi^k_{ux}| \\
&\quad + K_3 \sup_B \left|\varphi_{uy} - \varphi^k_{uy}\right| + K_4 \sup_B \left|\varphi_u - \varphi^k_u\right| \\
&\quad + K_5 \sup_B |\varphi_{xy} - \varphi^k_{xy}|.
\end{aligned}$$

In the above inequality, the expression on the right tends to zero as k tends to infinity. Thus, it follows that $\mu(\Phi) = \triangle\varphi$ for all $\varphi \in C'(B)$.

Now, we show that, $\mu(\Psi) = 0$ for all $\psi \in D(A^\circ)$. If $\mu \in P$ then $\mu(\Psi) = 0$ for all $\psi \in S_2$. Let $\psi \in D(A^\circ)$. For all $\psi \in D(A^\circ)$, the double Fourier series for ψ, ψ_x, and ψ_y converge uniformly on any subdomain of A. Thus, any function Ψ can be approximated uniformly on Ω by a sequence of functions in S_2. In a similar manner we can prove that $\mu(\Psi) = 0$ for all $\psi \in D(A^\circ)$. Hence, we have proved that $P = Q$ and so $\xi = \eta$. □

The dimension of $M^+(\Omega)$ is not finite. By using the following theorem, we approximate μ^* which is optimal measure of linear programming (17), [16].

Theorem 2. *The measure μ^* in the set $Q(M_1, M_2)$ which minimizes $\mu(F)$ has the form*

$$\mu^* = \sum_{k=1}^{M_1+M_2} \alpha^*_k \delta(Z^*_k),$$

where $Z^*_k \in \Omega$ and δ is unitary atomic measure and $\alpha^*_k \ge 0,\ k = 1, 2, \cdots, M_1 + M_2$.

Of course, the support of these atomic measures, i.e., Z_k^* are unknown. These supports can, however, be approximated by introducing a dense set in Ω. The following theorem has been proved in [16]:

Theorem 3. *Let ω be a countable dense subset of Ω. Then, for each $\xi > 0$, a measure $\nu \in M^+(\Omega)$ can be found such that*

$$|(\mu^* - \nu)F| < \xi, \quad |(\mu^* - \nu)\Phi_i| < \xi, \quad i = 1, 2, \cdots, M_1,$$

$$|(\mu^* - \nu)\Psi_j| < \xi, \quad j = 1, 2, \cdots, M_2$$

and the measure ν has the form

$$\nu = \sum_{k=1}^{M_1+M_2} \alpha_k^* \delta(Z_k),$$

where $Z_k \in \omega$ and $\alpha_k^* \geq 0,\ k = 1, 2, \cdots, M_1+M_2$ are the same as in the previous theorem.

The set Ω will be covered with a grid, by dividing each of the intervals to which x, y, u, v_1, v_2, v_3, v_4, and v_5 belong, into a number of equal subintervals. Let Ω be divided to N cells Ω_j, $j = 1, 2, ..., N$; we choose points $Z_j = (x_j, y_j, u_j, v_{1_j}, v_{2_j}, v_{3_j}, v_{4_j}, v_{5_j}) \in \Omega_j$, and let $\sigma = \{Z_j : j = 1, 2, \cdots, N\}$. The set σ is an approximate dense subset of the set Ω. Thus, by using the above theorem we can approximate the solution of the infinite dimensional linear programming problem (15) by the following problem:

$$\min \sum_{j=1}^{N} \alpha_j f_0(Z_j) \tag{18}$$

such that:

$$\begin{cases} \sum_{j=1}^{N} \alpha_j \Phi_k(Z_j) = \triangle\varphi_k, & k = 1, 2, \cdots, M_1, \\ \sum_{j=1}^{N} \alpha_j \Psi_k(Z_j) = 0, & k = 1, 2, \cdots, M_2, \\ \sum_{j=1}^{N} \alpha_j f_k(x_j, y_j) = a_k, & k = 1, 2, \cdots, M_3 \times M_3', \\ \sum_{j=1}^{N} \alpha_j f_0(Z_j) \leq \epsilon, & \\ \alpha_j \geq 0, & j = 1, 2, \cdots, N, \end{cases} \tag{19}$$

where x_j and y_j are the first two components of Z_j and $a_k = \int_A f_k(x, y)dxdy$. The functions $\{\varphi_k;\ k = 1, 2, \cdots, M_1\}$ are monomials in the components of x, y

and u only, and $\{\psi_k;\ k=1,2,\cdots,M_2\}$ is obtained from functions ψ of the form (16). The functions $\{f_k; k=1,2,\cdots,M_3\times M_3'\}$ are defined by

$$f_k(x,y)=\begin{cases}1, & (x,y)\in A_{kk'}\\ 0, & \text{otherwise,}\end{cases}$$

where

$$A_{kk'}=(x_a+(k-1)d_1, x_a+kd_1)\times(y_a+(k'-1)d_2, y_a+k'd_2), d_1=\frac{x_b-x_a}{M_3},$$

and

$$d_2=\frac{y_b-y_a}{M_3'}.$$

The above linear programming problem contains M constraints where $M=M_1+M_2+M_3\times M_3'$, and N variables.

We now construct $u(\cdot,\cdot)$, the approximate solution to the original problem (1) and (2), by the optimal solution $\{\alpha_j : j=1,2,...,N\}$ of the linear programming problem (18) and (19). We construct the functions v_i and then by using the condition (2) we can obtain $u(x,y)$ for $(x,y)\in A$.

Let $[x_a,x_b]$ and $[y_a,y_b]$ be divided into r_1 and r_2 equal subintervals respectively. Let t be define such that $N=r_1\times r_2\times t$. We can correspond to each $1\le m\le N$, a triple (i,j,k) as

$$m=(i-1)r_2t+(j-1)t+k, \qquad \begin{array}{l} i=1,2,\cdots,r_1\\ j=1,2,\cdots,r_2\\ k=1,2,\cdots,t\end{array}$$

and define $\lambda(i,j,k)=m$. Then, we define the piecewise-constant functions as follows [5]:

$$v_r(x,y)=(v_r)_m,\ (x,y)\in B_{ijk},\quad r=1,2,\cdots,5, \tag{20}$$

where $(v_r)_m$ are the $(r+3)$th component of Z_m and

$$B_{ijk}=\left[x_{i-1}+\sum_{k'<k} r_{ik'}, x_{i-1}+\sum_{k'\le k} r_{ik'}\right)\times\left[y_{j-1}+\sum_{k'<k} s_{jk'}, y_{j-1}+\sum_{k'\le k} s_{ik'}\right) \tag{21}$$

and

$$\begin{array}{l} r_{ik}=\alpha_{\lambda(i,1,k)}+\alpha_{\lambda(i,2,k)}+\cdots+\alpha_{\lambda(i,r_2,k)},\ \ i=1,2,\cdots,r_1,\\ s_{jk}=\alpha_{\lambda(1,j,k)}+\alpha_{\lambda(2,j,k)}+\cdots+\alpha_{\lambda(r_1,j,k)},\ \ j=1,2,\cdots,r_2.\end{array} \tag{22}$$

By using control functions and the boundary condition (1.2) we can obtain the solution of original problem as follows:

$$u(x,y)=(v_1)_m x+(v_2)_m y+c,\quad (x,y)\in B_{ijk}, \tag{23}$$

where c is a constant and is determind by the boundary condition (2).

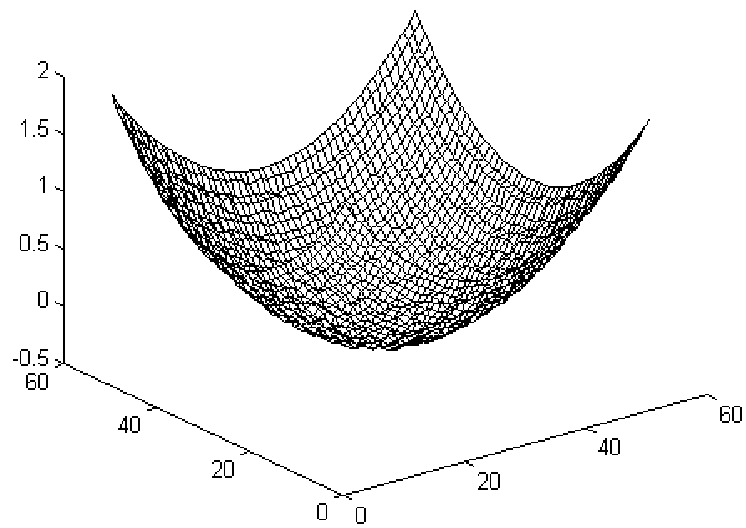

Figure 1. approximate solution

4. Example

Consider the boundary value problem

$$\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-8u=4(x-1)(y-1)\ \text{ in }\ A^\circ=(0,2)\times(0,2),$$

$$u(x,y)=\begin{cases}1+(y-1)^2 & \text{if}\quad x=0\\ (x-1)^2+1 & \text{if}\quad y=0\\ 1+(y-1)^2 & \text{if}\quad x=2\\ (x-1)^2+1 & \text{if}\quad y=2\end{cases}\quad \text{on}\ \ \Gamma.$$

This problem has the exact solution $u(x,y)=(x-1)^2+(y-1)^2$. Now, we obtain an approximate solution of the above nonlinear partial differential equation. Let $A=[0,2]\times[0,2], U=[0,2], V=[-2,2]\times[-2,2]\times[-2,2]\times[-2,2]\times[-2,2]$ and $\epsilon=0.025$. Divide A into 6×6 subsets (in other hand $M_3=M_3'=6$), U into 4 subsets and V into $4^2\times 3^3$ subsets. So, $N=62208$.

By choosing Z_j in Ω_j, the linear programming (18) and (19) contains 62208 variables. We solve the above linear programming problem using a programme in Matlab software, by choosing different value for M_1 and M_2. We can obtain an optimal solution for problem by $M_1=4$ and $M_2=8$ with the optimal

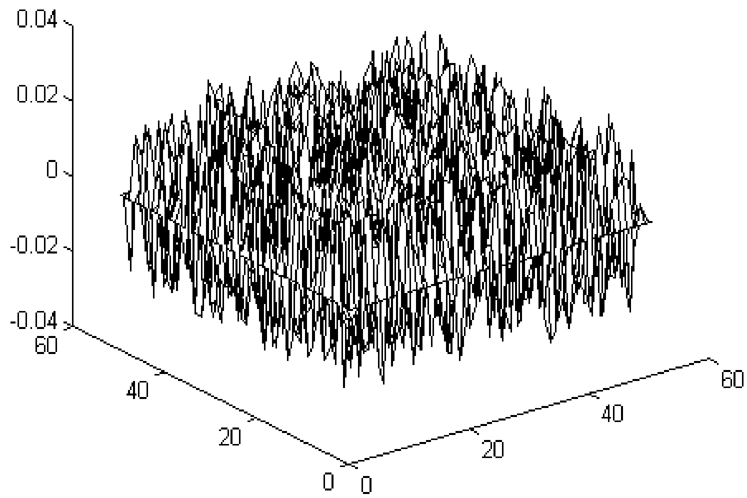

Figure 2. error function

value of objective function as $I^* = 0.0236$. By using the result of this finite dimensional linear programming, we obtain an approximate piecewise constant control function, and then we can obtain an approximate solution $u(x, y)$. The graphs of approximate solution, exact solution, and error function are shown in Figures 1 and 2 respectively.

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