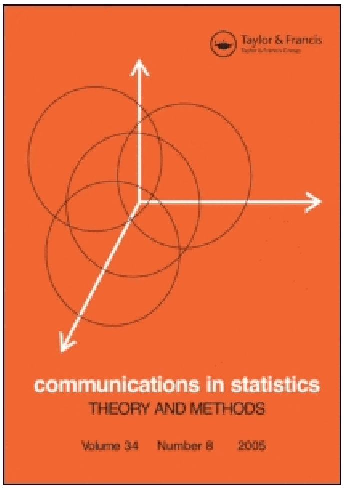

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Asymptotic Behaviors of the Lorenz Curve for Censored Data Under Strong Mixing

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In this article, we consider a nonparametric estimator of the Lorenz curve under censored dependent model. We show that this estimator is uniformly strongly consistent for the associated Lorenz curve. Also, a strong Gaussian approximation for the associated Lorenz process are established under appropriate assumptions. A law of the iterated logarithm for the Lorenz process is also derived.

Keywords Censored data; Kaplan–Meier estimator; Law of the iterated logarithm; Lorenz Curve; Quantile function; Strong consistency; Strong Gaussian approximation; Strong mixing.

Mathematics Subject Classification 62N01; 60F15.

1. Introduction and Preliminaries

Pietra (1915) and Gastwirth (1971) independently introduced the *Lorenz curve* corresponding to a non-negative random variable (rv) X with a distribution function (df) F, quantile function $Q(p)$, and finite mean $EX = \mu$ as:

$$L_F(t) := \frac{1}{\mu}\int_0^t Q(s)ds, \quad 0 \leq t \leq 1.$$

In econometrics, with X representing income, $L(t)$ gives the fraction of total income that the holders of the lowest tth fraction of income possesses. Most of the measures of income inequality are derived from the Lorenz curve. An important example is the Gini index associated with F defined by

$$G_F := \frac{\int_0^1 [u - L_F(u)]du}{\int_0^1 u\, du} = 1 - 2(CL)_F,$$

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where $(CL)_F = \int_0^1 L_F(u)du$ is the *cumulative Lorenz curve* corresponding to F. This is a ratio of the area between the Lorenz curve and the 45° line to the area under the 45° line. The numerator is usually called the *area of concentration*. Kendall and Stuart (1963) showed that this is equivalent to a ratio of a measure of dispersion to the mean. In general, these notions are useful for measuring concentration and inequality in distributions of resources, and in size distributions. For a list of applications in different areas, we refer the readers to Csörgő and Zitikis (1996a).

To estimate the Lorenz curve, one can use the *Lorenz statistic* $L_n(y)$ defined by

$$L_n(y) := \frac{1}{\mu_n}\int_0^y Q_n(u)du, \quad 0 \leq y \leq 1,$$

where μ_n is the sample mean and $Q_n(y)$ is the empirical quantile function constructed from i.i.d. sample taken from F.

Goldie (1977) proved the uniform consistency of L_n to L_F and derived the weak convergence of the *Lorenz process* $l_n(t) := \sqrt{n}[L_n(t) - L(t)]$, $0 \leq t \leq 1$ to a Gaussian process under suitable conditions. Csörgő et al. (1986) gave a unified treatment of strong and weak approximations of the Lorenz and other related processes. In particular, they established a strong invariance principle for the Lorenz process, by which Rao and Zhao (1995) derived one of their two versions of the law of the iterated logarithm (LIL) for the Lorenz process. Different versions of the LIL under weaker assumptions are also obtained by Csörgő and Zitikis (1996a, 1997). In Csörgő and Zitikis (1996b), confidence bands for the Lorenz curve that are based on weighted approximations of the Lorenz process are constructed. Csörgő et al. (1987), obtained weak approximations for Lorenz curves under random right censorship. Strong Gaussian approximations for the Lorenz process when data are subject to random right censorship and left truncation are established by Tse (2006), he is also derived a functional LIL for the Lorenz process.

However, in most economic situations, the basic sequence of observations may not be independent. It is more realistic to assume some form of dependence among the data are observed. Csörgő and Yu (1999) obtained weak approximations for Lorenz curves and its inverse under the assumption of mixing dependence. Glivenko–Cantelli-type asymptotic behavior of the empirical generalized Lorenz curves based on random variables forming a stationary ergodic sequence with deterministic noise were considered by Davydov and Zitikis (2002). Davydov and Zitikis (2003) established a large sample asymptotic theory for the empirical generalized Lorenz curves when observations are stationary and either short-range or long-range dependent. Strong laws for the generalized absolute Lorenz curves when data are stationary and ergodic sequences established by Helmers and Zitikis (2005). Based on the generalized Lorenz curves, Davydov et al. (2007) proposed a statistical index for measuring the fluctuations of a stochastic process. They developed some of the asymptotic theory of the statistical index in the case where the stochastic process is a Gaussian process with stationary increments and a nicely behaved correlation function. The uniform strong convergence rate of the estimator under strong mixing hypothesis is obtained by Fakoor and Nakhaei Rad (2009). They also established a strong Gaussian approximation for the Lorenz process, by which they derived a functional LIL for the Lorenz process, under the assumption of strong mixing.

The purpose of this article is to provide some asymptotic results for Lorenz process $l_n(t)$, for the case in which data are assumed to be strong mixing subject to random right censorship.

Consider a sequence of strictly stationary rv's $X_1, X_2, \ldots, X_n$ with common unknown absolutely continuous df F and finite mean μ. The rv's are not assumed to be mutually independent (see Assumption **A1** for the kind of dependence stipulated). Let the rv X_i be censored on the right by the rv C_i, so that one observes only

$$Z_i = X_i \wedge C_i \quad \text{and} \quad \delta_i = I(X_i \leq C_i),$$

where $\wedge$ denotes minimum and $I(\cdot)$ is the indicator of the event specified in parentheses. In this random censorship model, we assume that the censoring rv's $C_1, \ldots, C_n$ are not mutually independent (see Assumption **A2** for the kind of dependence stipulated), having a common unknown continuous df G, and that they are independent of the X_i's. We assume that X_i and C_i are non negative. The actually observed Z_i's have a distribution function H satisfying

$$\overline{H}(t) = 1 - H(t) = (1 - F(t))(1 - G(t)).$$

Denote by

$$F_*(t) = P(Z \leq t, \delta = 1),$$

the sub-distribution function for the uncensored observations. Define

$$N_n(t) = \sum_{i=1}^{n} I(Z_i \leq t, \delta = 1) = \sum_{i=1}^{n} I(X_i \leq t \wedge C_i),$$

the number of uncensored observations less than or equal to t, and

$$Y_n(t) = \sum_{i=1}^{n} I(Z_i \geq t),$$

the number of censored or uncensored observations greater than or equal to t and also the empirical distribution functions of $\bar{H}(t)$ and $F_*(t)$ are, respectively, defined as

$$\overline{Y}_n(t) = n^{-1} Y_n(t), \quad \overline{N}_n(t) = n^{-1} N_n(t).$$

Then the Kaplan–Meier estimator for $1 - F(t)$, based on n pairs $\{(Z_i, \delta_i), 1 \leq i \leq n\}$ is given by

$$1 - \widehat{F}_n(t) = \prod_{s \leq t} \left(1 - \frac{dN_n(s)}{Y_n(s)}\right), \tag{1.1}$$

where $dN_n(t) = N_n(t) - N_n(t^-)$ and $N_n(t^-) = \lim_{\epsilon \to 0^+} N_n(t - \epsilon)$.

The quantile function Q and its empirical counterpart Q_n are defined by

$$Q(p) = \inf\{x \in R; F(x) \geq p\} \quad \text{and} \quad Q_n(p) = \inf\{x \in R; \widehat{F}_n(x) \geq p\} \tag{1.2}$$

where $\widehat{F}_n(x)$ is the KM estimator defined in (1.1). Suppose that $0 < p_0 \leq p_1 < 1$. We defined the Lorenz curve corresponding to rv X as:

$$L_F(t) := \frac{1}{\mu}\int_{p_0}^{t} Q(s)ds, \quad p_0 \leq t \leq p_1,$$

where $\mu = \int_{p_0}^{p_1} Q(s)ds$. Therefore the natural estimator for the Lorenz curve $L_F(t)$ is

$$L_n(t) := \frac{1}{\mu_n}\int_{p_0}^{t} Q_n(s)ds, \quad p_0 \leq t \leq p_1,$$

where $\mu_n = \int_{p_0}^{p_1} Q_n(s)ds$.

The main aims of this article are to derive strong uniform consistency of the Lorenz statistic and a strong Gaussian approximation for Lorenz process, for the case in which data are assumed to be dependent subject to random right censorship. As a result of our strong Gaussian approximation, we obtain a functional LIL for the Lorenz process.

In this article, we consider the strong mixing dependence, which amounts to a form of asymptotic independence between the past and the future as shown by its definition.

Definition 1.1. Let $\{X_i, i \geq 1\}$ denote a sequence of random variables. Given a positive integer n, set

$$\alpha(n) = \sup_{k\geq 1}\{|P(A\cap B) - P(A)P(B)|;\, A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}^{\infty}\}, \tag{1.3}$$

where $\mathcal{F}_i^k$ denote the σ-field of events generated by $\{X_j; i \leq j \leq k\}$. The sequence is said to be strong mixing (α-mixing) if the mixing coefficient $\alpha(n) \to 0$ as $n \to \infty$.

Among various mixing conditions used in the literature, α-mixing is reasonably weak and has many practical applications (see, e.g., Doukhan, 1994, or Cai, 1998, 2001 for more details). In particular, Masry and Tjostheim (1995) proved that, both ARCH processes and nonlinear additive AR models with exogenous variables, which are particularly popular in finance and econometrics, are stationary and α-mixing.

Now we introduce some assumptions that are used to state our results gathered below for easy reference.

- **A1.** $\{X_i\}_{i\geq 1}$ is a sequence of stationary α-mixing random variables with continuous df F and mixing coefficient $\alpha_1(n)$.
- **A2.** $\{C_i\}_{i\geq 1}$ is a sequence of stationary α-mixing random variables with continuous df G and mixing coefficient $\alpha_2(n)$. Moreover, the censoring times are independent of $\{X_i\}_{i\geq 1}$.
- **A3.** $\alpha(n) = O(e^{-(\log n)^{1+\zeta}})$ for some $\zeta > 0$, with $\alpha(n) = \max\{\alpha_1(n), \alpha_2(n)\}$ (For the interpretation of this assumption, see Remark 2.1 in Ould-Saïd and Sadki (2005).

In the next section, we present our main results.

2. Asymptotic Behaviors of Lorenz Curve

2.1. *Strong Uniform Consistency*

Theorem 2.1 below proves the uniform strong consistency with rate of the estimator L_n.

Theorem 2.1. *Under Assumptions* **A1–A3**, *assuming that* $F' = f$ *is continuous and strictly positive on* $[Q(p_0) - \delta, Q(p_1) + \delta]$, *for some* $\delta > 0$. *Then,*

$$\sup_{p_0 \le t \le p_1} |L_n(t) - L_F(t)| = O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad a.s. \tag{2.1}$$

Proof. An elementary computation shows that,

$$L_n(t) - L_F(t) = \frac{1}{\mu_n}\int_{p_0}^{t}[Q_n(s) - Q(s)]ds - \frac{\mu_n - \mu}{\mu_n}L_F(t). \tag{2.2}$$

It is easy to see that

$$\mu_n - \mu = \int_{p_0}^{p_1}[Q_n(s) - Q(s)]ds. \tag{2.3}$$

Now, by using (2.2), (2.3), and Lemma 3.2 of Ould-Saïd and Sadki (2005), we obtain the result.

2.2. *Strong Gaussian Approximation*

We first introduce the following Gaussian process, which plays an important role to present our strong approximation.

Let $g_j(s) = I(Z_j \le s) - H(s), \quad j \ge 0,$

$$\Gamma(s, s') = Cov(g_1(s), g_1(s')) + \sum_{j=2}^{\infty}[Cov(g_1(s), g_j(s')) + Cov(g_1(s'), g_j(s))]. \tag{2.4}$$

Define, for $t \ge 0$ two-parameter mean zero Gaussian process,

$$B(t, n) = \int_0^t \frac{K(x, n)/\sqrt{n}}{(\bar{H}(x))^2}dF_*(x),$$

where $\{K(s, t), s, t \ge 0\}$ is a Kiefer process in Theorem 3 of Dhompongsa (1984) with covariance function

$$\Gamma^*(t, t', s, s') = \min(t, t')\Gamma(s, s'),$$

and $\Gamma(s, s')$ given by (2.4).

We now restate below a strong approximation by Fakoor and Nakhaei Rad (2010) for the normed quantile process $\rho_n(u) := \sqrt{n}f(Q(u))[Q(u) - Q_n(u)]$ by a two-parameter Gaussian process at the rate $O((\log n)^{-\lambda})$, for some $\lambda > 0$.

Theorem 2.2 (Fakoor and Nakhaei Rad, 2010). *Let* $0 < p_0 \leq p_1 < 1$. *Under Assumptions* **A1–A3**, *assume that* F *is Lipschitz continuous and that* F *is twice continuously differentiable on* $[Q(p_0) - \delta, Q(p_1) + \delta]$, *for some* $\delta > 0$, *such that* f *is bounded away from zero. Then there exists a two-parameter mean zero Gaussian process* $B(t, u)$ *for* $t, u \geq 0$, *such that,*

$$\sup_{p_0 \leq u \leq p_1} |\rho_n(u) - (1-u)B(Q(u), n)| = O((\log n)^{-\lambda}) \quad a.s.,$$

with $\lambda > 0$.

We will give strong Gaussian approximation of the Lorenz process over restricted interval $[p_0, p_1]$ for fixed $0 < p_0 \leq p_1 < 1$.

In the full model, Langberg et al. (1980) defined the total time on test transform curve corresponding to a continuous distribution F on $[0, \infty)$, $H_F^{-1}(u)$, for $u \in [0, 1]$ as

$$H_F^{-1}(u) = \int_0^u (1-y)dQ(y) = (1-u)Q(u) + \int_0^u Q(y)dy, \quad Q(0) = 0.$$

Obviously, $H_F^{-1}(u) \leq H_F^{-1}(1) := \lim_{u \uparrow 1} H_F^{-1}(u) = \mu$. For the our model, we modify the definition of $H_F^{-1}(u)$ as

$$H_F^{-1}(u) = (p_1 - u)Q(u) + \int_{p_0}^u Q(y)dy, \quad u \in [p_0, p_1]. \tag{2.5}$$

As $p_0 \downarrow 0$ and $p_1 \uparrow 1$, $H_F^{-1}(p_1) \to \int_0^1 Q(y)dy = \mu$. We can regard $H_F^{-1}(p_1)$ as a surrogate for the finite mean μ. A natural estimator for $H_F^{-1}(u)$ is

$$H_n^{-1}(u) = (p_1 - u)Q_n(u) + \int_{p_0}^u Q_n(y)dy, \quad u \in [p_0, p_1].$$

In the next theorem, we construct a two-parameter mean zero Gaussian process that strongly uniformly approximate the empirical process $l_n(t)$.

Theorem 2.3. *Let* $0 < p_0 \leq p_1 < 1$. *Under Assumptions* **A1–A3**, *assume that* F *is Lipschitz continuous and that* F *is twice continuously differentiable on* $[Q(p_0) - \delta, Q(p_1) + \delta]$, *for some* $\delta > 0$ *such that* f *is bounded away from zero, then there exists a two-parameter mean zero Gaussian process* $B(t, u)$ *for* $t, u \geq 0$, *such that, almost surely,*

$$\sup_{p_0 \leq u \leq p_1} \left| l_n(u) - \frac{1}{H_F^{-1}(p_1)} \left(\int_{p_0}^u \frac{(p_1 - y)B(Q(y), n)}{f(Q(y))} dy - L_F(u) \int_{p_0}^{p_1} \frac{(p_1 - y)B(Q(y), n)}{f(Q(y))} dy \right) \right| = O((\log n)^{-\lambda}), \tag{2.6}$$

with $\lambda > 0$.

Proof. See the Appendix.

2.3. *Functional LIL*

The next theorem gives a functional LIL for the Lorenz process. We work on the probability space of Theorem 2.3. Let $D[a, b]$ be the space of functions on $[a, b]$ that are right continuous and have left limits and B is the unit ball in the reproduce kernel Hilbert space $H(\Gamma^*)$.

Theorem 2.4. *Suppose that conditions of Theorem* 2.3 *are satisfied. On a rich enough probability space,* $l_n(.)/\sqrt{2\log\log n}$ *is almost surly relatively compact in* $D[p_0, p_1]$ *with respect to the supremum norm and its set of limit points is*

$$G = \left\{ g_h : g_h(u) = \frac{1}{H_F^{-1}(p_1)} \left(\int_{p_0}^{u} \frac{h(y)}{f(Q(y))} dy - L_F(u) \int_{p_0}^{p_1} \frac{h(y)}{f(Q(y))} dy \right), \right.$$
$$\left. p_0 \le u \le p_1, h \in \mathscr{H} \right\},$$

where

$$\mathscr{H} = \left\{ h : [p_0, p_1] \to \mathbf{R}, h(u) = \int_0^{Q(u)} \frac{g(x)}{(\overline{H}(x))^2} dF_*(x) : g \in B \right\}.$$

Proof. Theorem 2.4 follows at once from (2.6) and Theorem A in Berkes and Philipp (1977).

Appendix

In establishing Theorem 2.3, we were aided by some ideas found in Tse (2006), but first we start with the following lemmas which are necessary for achieving the establishment of the our results.

Lemma A.1. *Suppose the conditions of Theorem* 2.2 *are satisfied. We have*:

$$\lim_{n\to\infty} \sup_{p_0 \le u \le p_1} |H_n^{-1}(u) - H_F^{-1}(u)| = O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad a.s.$$

Proof. By Lemma 3.2 of Ould-Saïd and Sadki (2005), we have:

$$\begin{aligned} &\sup_{p_0 \le u \le p_1} |H_n^{-1}(u) - H_F^{-1}(u)| \\ &\le \sup_{p_0 \le u \le p_1} [(p_1 - u)|Q_n(u) - Q(u)|] + \sup_{p_0 \le u \le p_1} \int_{p_0}^{u} |Q_n(y) - Q(y)| dy \\ &= O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad a.s. \end{aligned}$$

□

Next, define the normed total time on test empirical process $t_n(u)$ by

$$t_n(u) = \sqrt{n}[H_n^{-1}(u) - H_F^{-1}(u)], \quad u \in [p_0, p_1].$$

Lemma A.2 characterize the asymptotic limit of $t_n(u)$.

Lemma A.2. *Suppose the conditions of Theorem* 2.2 *are satisfied. Then there exists a two-parameter mean zero Gaussian process* $B(t, u)$ *for* $t, u \geq 0$, *such that,*

$$\sup_{p_0 \leq u \leq p_1} \left| t_n(u) - \left(\int_{p_0}^{u} \frac{(p_1 - y)B(Q(y), n)}{f(Q(y))} dy + \frac{(p_1 - u)^2 B(Q(u), n)}{f(Q(u))} \right) \right|$$
$$= O((\log n)^{-\lambda}) \quad a.s.$$

Proof. Proof of this lemma can be done using similar augment of Lemma 3.2 in Tse (2006), we therefore omit the proof.

Next, we define the scaled total time on test transform, its statistic and associated empirical process corresponding to F.

$$W_F(u) := \frac{H_F^{-1}(u)}{H_F^{-1}(p_1)}, \quad W_n(u) := \frac{H_n^{-1}(u)}{H_n^{-1}(p_1)} \tag{2.7}$$

and

$$w_n(u) := \sqrt{n}[W_n(u) - W_F(u)]$$

for $u \in [p_0, p_1]$.

The following lemmas give the strong uniform consistency of $W_n(u)$ and strong Gaussian approximation of the scaled total time on test empirical process respectively.

Lemma A.3. *Suppose that conditions of Theorem* 2.2 *are satisfied. We have*:

$$\sup_{p_0 \leq u \leq p_1} |W_n(u) - W_F(u)| = O\left(\sqrt{\frac{\log \log n}{n}} \right) \quad a.s.$$

Proof. By triangular inequality and Lemma A.1, the left-hand side is bounded by

$$\sup_{p_0 \leq u \leq p_1} \left| \frac{H_n^{-1}(u)}{H_n^{-1}(p_1)} - \frac{H_n^{-1}(u)}{H_F^{-1}(p_1)} \right| + \sup_{p_0 \leq u \leq p_1} \left| \frac{H_n^{-1}(u)}{H_F^{-1}(p_1)} - \frac{H_F^{-1}(u)}{H_F^{-1}(p_1)} \right|$$
$$\leq \sup_{p_0 \leq u \leq p_1} \left| H_n^{-1}(u) \frac{H_F^{-1}(p_1) - H_n^{-1}(p_1)}{H_n^{-1}(p_1) H_F^{-1}(p_1)} \right| + \sup_{p_0 \leq u \leq p_1} \left| \frac{1}{H_F^{-1}(p_1)} [H_F^{-1}(u) - H_n^{-1}(u)] \right|$$
$$= O\left(\sqrt{\frac{\log \log n}{n}} \right) \quad a.s.$$

Lemma A.4. *Suppose that conditions of Theorem* 2.2 *are satisfied. Then there exists a two-parameter mean zero Gaussian process* $B(t, u)$ *for* $t, u \geq 0$, *such that,*

$$\sup_{p_0 \leq u \leq p_1} \left| w_n(u) - \frac{1}{H_F^{-1}(p_1)} \left(\int_{p_0}^{u} \frac{(p_1 - y)B(Q(y), n)}{f(Q(y))} dy + \frac{(p_1 - u)^2 B(Q(u), n)}{f(Q(u))} \right) \right.$$
$$\left. + \frac{H_F^{-1}(u)}{\left(H_F^{-1}(p_1)\right)^2} \int_{p_0}^{p_1} \frac{(p_1 - y)B(Q(y), n)}{f(Q(y))} dy \right| = O((\log n)^{-\lambda}) \quad a.s.$$

for some $\lambda > 0$.

Proof. Proof can be done along the lines of Lemma 3.5 of Tse (2006), we therefore omit the proof.

Proof of Theorem 2.2. By definition of the Lorenz curve corresponding to F in the our model and by using (2.5) and (2.7) we have:

$$W_F(y) = \frac{(p_1 - y)Q(y)}{\int_{p_0}^{p_1} Q(u)du} + L_F(y). \tag{2.8}$$

We have also

$$W_n(y) = \frac{(p_1 - y)Q_n(y)}{\int_{p_0}^{p_1} Q_n(u)du} + L_n(y), \quad y \in [p_0, p_1]. \tag{2.9}$$

Substituting (2.8) and (2.9) in Lemma A.4, we obtain the result.

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