

Collective coordinate variable for soliton-potential system in sine-Gordon model

Kurosh Javidan^{a)}

Department of Physics, Ferdowsi University of Mashhad 91775-1436 Mashhad, Iran

(Received 1 August 2010; accepted 13 October 2010; published online 22 November 2010)

A collective coordinate variable for adding a space dependent potential to the sine-Gordon model is presented. Interaction of solitons with a delta function potential barrier and also a delta function potential well is investigated. A majority of the interactive characters are derived analytically. We find that the behavior of a solitonic solution is similar to a point particle which is moved under the influence of a complicated effective potential. The effective potential is a function of the initial field conditions and parameters of added potential. © 2010 American Institute of Physics. [doi:10.1063/1.3511337]

I. INTRODUCTION

Solitons are important models in most branches of science. They are stable against dispersive effects and propagate in a manner similar to classical pointlike particles.

Topological solitons play an important role in the nonperturbative aspects of quantum field theory. They are widely used as models of particles which are generated as nontrivial solutions of nonlinear field theories. Skyrmions are solitons which are used as a model of hadrons. Solitons also widely appear in models of condensed matter, such as Josephson junction, fluxon, spin chain dynamics, etc. Describing active regions of DNA is an example of a soliton application.¹⁻³

A sine-Gordon equation is a very famous soliton model with well-known topological solutions. This model widely appears in nonlinear problems. Some of the above-mentioned models are examples of sine-Gordon applications.

Recently, there has been an increasing interest in the scattering of solitons from defects or impurities, which generally come from medium properties. The motivations come from both the theoretical and the application aspects of physics. The effects of medium disorders and impurities can be added to the equation of motion as perturbative terms.^{4,5} These effects can also be generated by making some parameters of the equation of motion to a function of space or time.^{6,7} There still exists another interesting method which is mainly suitable for working with topological solitons.^{8,9} In this method, one can add such effects to the Lagrangian of the system by introducing a suitable nontrivial metric for the background spacetime without losing the topological boundary conditions.

The main investigation of the interaction of defective solitons uses numerical analysis and numerical simulations. Some analytical models presented are constructed using suitable collective coordinate systems. Analytical models for the sine-Gordon and ϕ^4 field theories are presented based on the method of adding the defects through the background spacetime metric.^{10,11} They predict most of the “soliton-potential” behaviors with very good precision. This method can be used for objects where the equation of motion results from a Lorentz invariant action, such as a sine-Gordon model, ϕ^4 theory, CP^N model, Skyrme model, Faddeev-Hopf equation, chiral quark-soliton model, Gross-Neveu model, nonlinear Klein-Gordon models, etc.

^{a)} Author to whom correspondence should be addressed. Electronic mail: javidan@um.ac.ir.

In this paper, a simple collective coordinate system is presented. This system is constructed based on the method of adding the defects by making some parameters of Lagrangian to be function of space. This method can be used for other nonlinear field theories where the equations of motion are not Lorentz invariant, also used for Lorentz invariant field theories.

A new collective coordinate system for solitons of the sine-Gordon field theory will be presented in Sec. II. Results of this model are discussed and also they are compared with the results of Ref. 10 in Sec. III. Some conclusion and remarks will be presented in Sec. IV.

II. COLLECTIVE COORDINATE VARIABLE

A sine-Gordon model in $(1 + 1)$ dimensions, $\mu = 0, 1$, is defined by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \lambda(x)(1 - \cos \phi), \quad (1)$$

where $\lambda(x) = \lambda_0 + V(x)$. $V(x)$ is a potential parameter and carries the effects of the external potential. Potential $V(x)$ is a localized function which is nonzero only in a certain region of space.

The equation of motion for the Lagrangian (1) is

$$\partial_\mu \partial^\mu \phi + \lambda(x) \sin(\phi) = 0. \quad (2)$$

This equation does not have an analytical solution with a general function for $V(x)$. If we take $V(x) = 0$, we have the usual sine-Gordon equation with the following one soliton solution:

$$\phi(x, X(t)) = 4 \tan^{-1} \left(\exp \left(\sqrt{\lambda_0} \frac{x - X(t)}{\sqrt{1 - \dot{X}^2}} \right) \right), \quad (3)$$

where $X(t) = x_0 - \dot{X}t$. x_0 and \dot{X} are the initial position of the soliton and its velocity. In the following calculations λ_0 has been set to $\lambda_0 = 1$.

The derivation of the collective action for the motion of the vortex centers starts with the elegant idea of Manton.¹² A collective action can be constructed by substituting the collective vortex ansatz for the field configuration with vortices at $X_i(t)$, $i = 1, \dots, N$, into the effective field theory action and reduce the action to a function of the collective coordinates, $L[X_i(t)] = \int \mathcal{L}(\psi(x, t, X_i(t))) dx$ (Ref. 13).

The center of the soliton can be considered a particle if we look at this as a collective coordinate variable. The collective coordinate can be related to the potential by using the suitable function $V(x)$ in Eq. (2). As a result, the model is able to give us an analytic description for the evolution of the soliton center during the soliton–potential interaction. By inserting the solution (3) in the Lagrangian (1) with adiabatic approximation,^{4,5} we have

$$\mathcal{L} = 2(\dot{X}^2 - 1) \operatorname{sech}^2(x - X(t)) - 2\lambda(x) \operatorname{sech}^2(x - X(t)), \quad (4)$$

where $X(t)$ remains as a collective coordinate if we integrate (4) over the variable x . If we take the potential $V(x) = \epsilon \delta(x)$, the effective action becomes

$$L = 4\dot{X}^2 - 8 - 2\epsilon \operatorname{sech}^2(x - X). \quad (5)$$

The equation of motion for the variable $X(t)$ is derived from (5)

$$8\dot{X} - 4\epsilon \operatorname{sech}^2(X) \tanh(X) = 0. \quad (6)$$

The above equation shows that the peak of the soliton moves under the influence of a complicated force which is a function of soliton position, soliton velocity, and characters of external potential $V(x)$. If $\epsilon > 0$, we have a barrier and $\epsilon < 0$ creates a potential well. Equation (6) has an exact solution as follows:

$$\dot{X}^2 = \frac{1}{2} \epsilon \operatorname{sech}^2(X_0) + \dot{X}_0^2 - \frac{1}{2} \epsilon \operatorname{sech}^2(X), \quad (7)$$

where X_0 and \dot{X}_0 are the initial position of the soliton and its initial velocity, respectively.

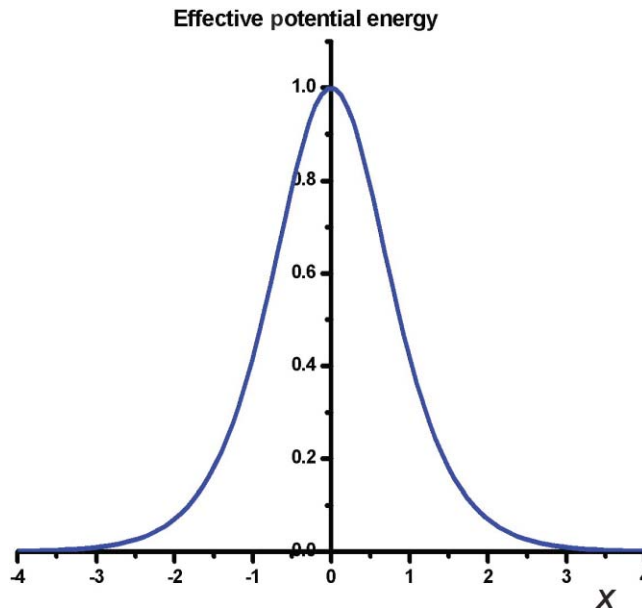


FIG. 1. Energy of a static soliton as a function of its distance from the potential $V(x) = \epsilon\delta(x)$ with $\epsilon = 0.5$.

Hamiltonian density can be obtained from the Lagrangian (4) as follows:

$$\mathcal{H} = 2(\dot{X}^2 + 1)\text{sech}^2(x - X(t)) + 2\lambda(x)\text{sech}^2(x - X(t)). \quad (8)$$

Collective energy density can be calculated with the integration of (4) over the variable x . Energy of the soliton in the presence of the potential $V(x) = \epsilon\delta(x)$ becomes

$$E = 4\dot{X}^2 + 8 + 2\epsilon\text{sech}^2 X. \quad (9)$$

It is the energy of a particle with the mass of $m = 8$ and velocity \dot{X} which is moved under the influence of external effective potential. Figure 1 presents the energy of a static soliton as a function of its position under the influence of the potential $V(x) = \epsilon\delta(x)$ with $\epsilon = 0.5$. Because of the extended nature of the soliton, the effective potential is not an exact delta function.

By substituting \dot{X} from (7) into (9), we show that the energy is a function of the initial conditions of the soliton X_0 and \dot{X}_0 only. Therefore, the energy of the system is conserved.

Some features of soliton–potential dynamics which can be investigated analytically using Eqs. (7) and (9), which are discussed in Sec. III.

III. SOLITON–POTENTIAL DYNAMICS

A soliton–barrier system is modeled with $\epsilon > 0$ in Eq. (7) or Eq. (9). Consider a soliton which is placed far away from the potential (namely at infinity), with initial velocity \dot{X}_0 . It moves toward the barrier and interacts with it. There exists two different kinds of trajectories for the soliton after the interaction with the barrier, depending on its initial velocity and which are separated by a critical velocity u_c . In low velocities, the soliton reflects back and reaches its initial place with final velocity $u_f \approx -\dot{X}_0$. A soliton with an initial velocity $\dot{X}_0 > u_c$ has enough energy for climbing the barrier and passing over the potential. At the velocities $\dot{X}_0 \approx u_c$ the soliton interacts with the potential slowly and spends more time near the barrier, but this situation is not a bound state. The presented model predicts all of the above features.

Energy of a soliton in the origin is calculated from (9) as $E(X = 0) = 4\dot{X}^2 + 8 + 2\epsilon$. The minimum value of the energy for a soliton in this situation is $E_{min}(X = 0) = 8 + 2\epsilon$. On the other hand, a soliton which comes from infinity with an initial velocity \dot{X}_0 has the energy of $E(X_0 = \infty) = 4\dot{X}_0^2 + 8$. It is clear that it can pass through the barrier if $E(X_0 = \infty) \geq E_{min}(X = 0)$.

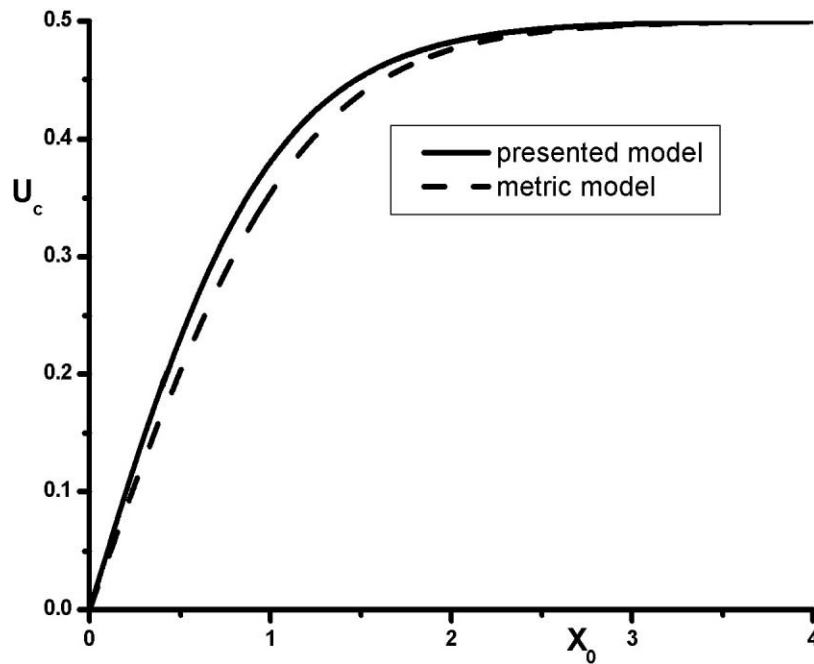


FIG. 2. Critical velocity as a function of initial position predicted in Ref. 10 (dashed line) and our presented model in Eq. (11) (solid line).

Therefore, we have

$$u_c(X_0 = \infty) = \sqrt{\frac{\epsilon}{2}}. \quad (10)$$

Now consider a soliton which is placed at the initial position $X_0 < 0$ (which is not necessarily infinity) with initial velocity \dot{X}_0 . Equation (7) shows that the soliton reaches $-\infty(+\infty)$ with final speed $\dot{X}(\infty) = \sqrt{\dot{X}_0^2 + \frac{1}{2}\epsilon \operatorname{sech}^2(X_0)}$ if its initial velocity is less (more) than the critical velocity $u_c(X_0)$. The critical velocity in this situation is not $\sqrt{\frac{\epsilon}{2}}$. This soliton has the critical initial velocity if its initial energy becomes equal to the energy of the static soliton at the top of the barrier $X = 0$. In this situation the critical velocity is

$$u_c(X_0) = \sqrt{\frac{\epsilon}{2}(1 - \operatorname{sech}^2(X_0))}. \quad (11)$$

All presented models predict same critical velocity for a soliton at infinity. But critical velocity of a soliton at the initial position X_0 is model dependent. For example, the following expression has been calculated in Ref. 10:

$$u_c(X_0) = \sqrt{\frac{\epsilon}{2} \frac{\cosh^2 X_0 - 1}{\cosh^2 X_0 + \frac{3\epsilon}{4}}}. \quad (12)$$

The critical velocity of Ref. 10 (Eq. (12)) is compared with (11) in Fig. 2 with $\epsilon = 0.5$. The results of both models are the same at $X_0 = \infty$ and $X_0 = 0$. But the transient behavior of the two models between these limits is different. It is clear that we cannot fit one of the models on the other by choosing an effective potential.

There exists a return point if the soliton initial velocity is less than the critical velocity. This point can be found by comparing the soliton energy at its initial position and the point at which its

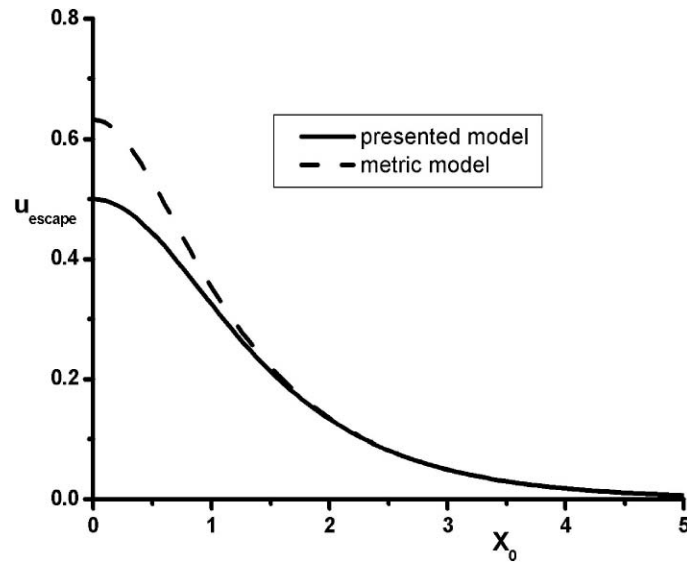


FIG. 3. Escape velocity as a function of initial position predicted in Ref. 10, Eq. (15) (dashed line) and our calculation in Eq. (14) (solid line).

velocity becomes zero. Therefore, we have

$$\frac{1}{\cosh^2 X_{stop}} = \frac{2}{\epsilon} \dot{X}_0^2 + \frac{1}{\cosh^2 X_0}. \quad (13)$$

Equation (13) clearly shows a linear relation between $\frac{1}{\cosh^2 X_{stop}}$ and $\frac{1}{\cosh^2 X_0}$. There is another linear relation between $\frac{1}{\cosh^2 X_{stop}}$ and $\frac{1}{\epsilon}$. Direct simulation and an analytic model of Ref. 10 also confirm these relations.

A potential well can be constructed with $\epsilon < 0$. Consider a particle moving toward a frictionless potential well from infinity. It falls in the well with an increasing velocity and reaches the bottom of the well with its maximum speed. After that, it will climb the well with decreasing velocity and finally pass through the well. Its final velocity after the interaction is equal to its initial speed. A soliton–well system is constructed by changing ϵ into $-\epsilon$ in the above equations. But the dynamics of a soliton–well system is very different from what we have seen in a soliton–barrier interaction. There is no critical velocity for a soliton–well system, but we can define an escape velocity. A soliton at initial position X_0 reaches infinity with a zero final velocity if its initial velocity is

$$\dot{X}_{escape} = \sqrt{\frac{\epsilon}{2} \operatorname{sech}^2(X_0)}. \quad (14)$$

The predicted value for this Ref. 10 is

$$\dot{X}_{escape} = \sqrt{\frac{\epsilon}{2} \frac{1}{\cosh^2 X_0 - \frac{3\epsilon}{4}}}. \quad (15)$$

Figure 3 presents \dot{X}_{escape} as a function of the initial position of a soliton using (14) and (15). The predicted function \dot{X}_{escape} at the center of the potential in the two models is different. But one can show that we can fit this value by defining an effective potential parameter $\epsilon_{effective}$; however, the transient behavior between the origin and infinity does not fit. Figure 3 also shows that the predicted escape velocity for infinity in the two models is zero.

Consider a potential well with the depth of ϵ and a soliton at the initial position X_0 which moves toward the well with initial velocity \dot{X}_0 smaller than the function \dot{X}_{escape} . The soliton interacts with the potential well and reaches a maximum distance X_{max} from the center of the potential with a zero velocity and then comes back toward the well for another interaction. The soliton oscillates around

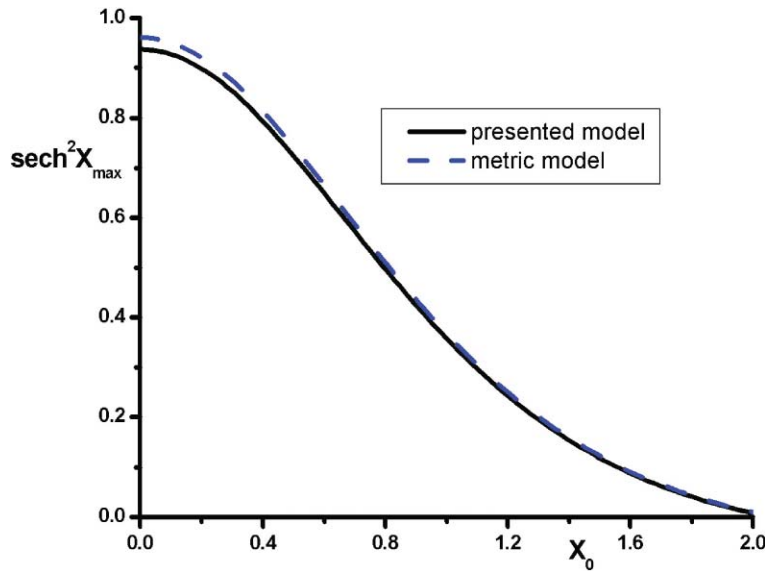


FIG. 4. $sech^2 X_{max}$ as a function of initial position predicted in Ref. 10, Eq. (17) (dashed line) and our calculation in Eq. (16) (solid line).

the well with amplitude X_{max} ,

$$X_{max} = sech^{-1} \sqrt{sech^2 X_0 - \frac{2}{\epsilon} \dot{X}_0^2}. \tag{16}$$

Maximum distance has been calculated with another model in Ref. 10 as follows:

$$X_{max} = sech^{-1} \sqrt{sech^2 X_0 - \frac{2}{\epsilon} \dot{X}_0^2 \left(1 - \frac{3\epsilon}{4} sech^2 X_0\right)}. \tag{17}$$

Figure 4 presents $sech^2 X_{max}$ as a function of X_0 using (16) and (17) with $\epsilon = 0.5$ and $\dot{X}_0 = 0.5$. They are very similar to each other and it is possible to fit one of them on the other in the origin by defining an effective potential parameter $\epsilon_{effective}$.

Trajectory of a soliton during the interaction with the potential $X(t)$ follows from (7) as

$$t = \int_{X(t=0)}^{X(t)} \frac{dx}{\sqrt{\dot{X}_0^2 + \frac{\epsilon}{2} sech^2 X_0 - \frac{\epsilon}{2} sech^2 X}} \tag{18}$$

for soliton–barrier ($\epsilon > 0$) and soliton–well ($\epsilon < 0$) systems. The metric model prediction following complicated equation¹⁰

$$t = \int_{X(t=0)}^{X(t)} \frac{dx}{\sqrt{\left(\dot{X}_0^2 + \frac{2}{3}\right) \frac{\cosh^2 X \left(\frac{3\epsilon}{4} + \cosh^2 X_0\right)}{\cosh^2 X_0 \left(\frac{3\epsilon}{4} + \cosh^2 X\right)} - \frac{2}{3}}}. \tag{19}$$

Figure 5 presents a soliton trajectory with respect to time for a soliton with initial velocity $\dot{X}_0 = 0.1$ and initial position $X = -3$ in a potential well of $\epsilon = -0.3$, plotted using (18) and (19). Trajectory of the soliton also simulated in numerical integrating of complete Eq. (2). Figure 5 clearly shows a very good agreement between the presented model, the metric model of Ref. 10, and direct simulation of equation of motion in low velocities. One can find that proposed model is a little better than the metric model of Ref. 10.

If the soliton initial velocity is lower than the escape velocity the soliton oscillates around the well. The period of oscillation can be calculated numerically using Eq. (18). Some simulations have been done using (18) and also (19) and the results have been compared. The results are different

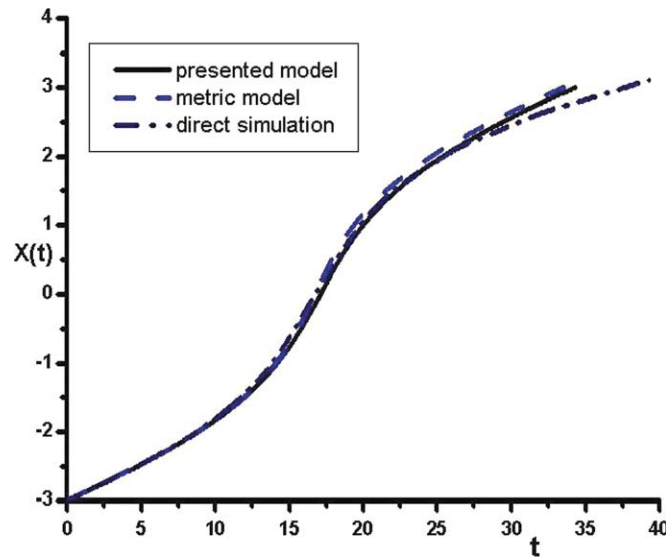


FIG. 5. Soliton trajectory as a function of time for a soliton with initial velocity $\dot{X}_0 = 0.1$ and $X_0 = -3$ during the interaction with potential well $\epsilon = -0.3$ plotted with presented model Eq. (18) (solid line), metric model of Ref. 10 Eq. (19) (dashed line) and direct simulation of Eq. (2) (dash-dotted line).

as shown in Fig. 6. One can adjust period of oscillation by using an effective potential parameter $\epsilon_{effective}$. But general behavior of oscillation in two models is not adjustable at all. It is clear because the presented model comes from a different method which added the potential to the system. The comparison of the two models shows that the results are acceptable with a very good approximation.

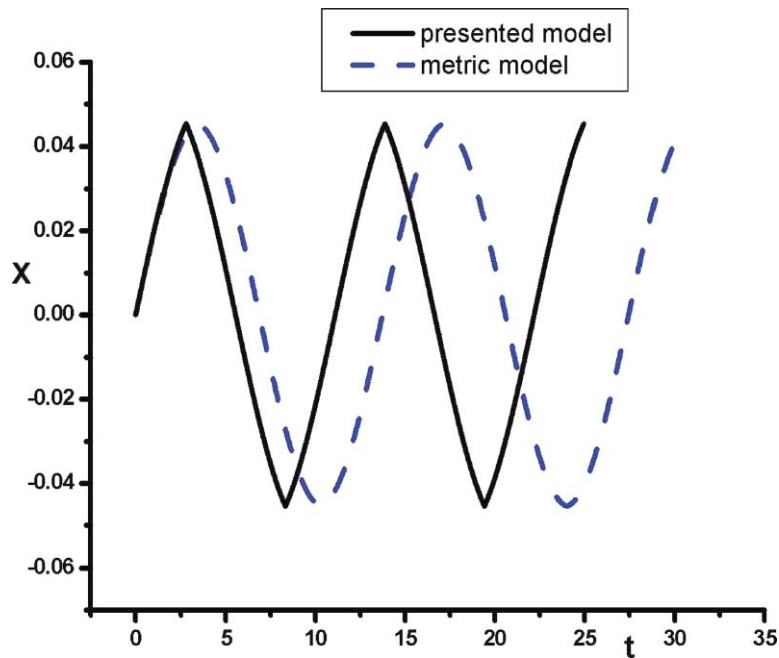


FIG. 6. Soliton trajectory as a function of time for a soliton with initial velocity $\dot{X}_0 = 0.02$ and $X_0 = 0$ during the interaction with potential well $\epsilon = -0.3$ simulated with presented model (solid line) and metric model of Ref. 10 (dashed line).

TABLE I. Comparison between results of presented collective coordinate and model of Ref. 10.

Model	Presented model	Model of Ref. 10
Usage	Actions in the form of $L = T - V$	Only Lorentz invariant actions
Soliton mass	Constant ($M = 2$)	Function of $X(M = 2 + \frac{3\epsilon}{2 \cosh^2 X})$
Kinetic energy	Smaller kinetic energy	Greater kinetic energy
Escape velocity	Smaller escape velocity	Greater escape velocity

Differences between results of proposed collective coordinate approach with the results of Ref. 10 have been summarized in Table I. Note that the difference between predicted critical velocity (and also escape velocity) in two models is small (order of ϵ^2).

IV. CONCLUSION AND REMARKS

An analytical model for scattering of sine-Gordon solitons from delta function potential barriers and also potential wells has been presented. Several features of soliton–potential characters were calculated using this model. A critical velocity for the soliton during the interaction with a potential barrier as a function of its initial conditions and the potential characters has been found. The model predicts specific relations between some functions of initial conditions and other functions of the final state of the soliton after the interaction. An escape velocity has been derived for the soliton–well system. The oscillation period of a soliton in a potential well has also been investigated using this model.

Calculated characters have been compared with the results of another analytical model. All of the simulations for soliton–barrier and soliton–well systems show the validity of the presented analytic model. Therefore, we can conclude that the presented collective coordinate method explains many of the features of the sin-Gordon soliton behavior during the interaction with a potential. But this model (similar to analytical model¹⁰) is not able to explain the fine structure of the islands of reflecting in a soliton–well system. This phenomenon is a very interesting features of soliton–potential systems. We expect to find an acceptable explanation for this behavior using a better model with a suitable collective coordinate method. On the other hand, using this method for investigation of other nonlinear models in potentials is an interesting subject.

¹ V. Vasumathi and M. Daniel, *Phys. Lett. A*, **373**, 76 (2008).

² Marek Jaworski, *Phys. Rev. B* **71**, 214515 (2005).

³ S. A. Zvyagin, A. K. Kolezhuk, J. Krzystek, and R. Feyerherm, *Phys. Rev. Lett.* **93**, 027201 (2004).

⁴ Y. S. Kivshar, Z. Fei, and L. Vazquez, *Phys. Rev. Lett* **67**, 1177 (1991).

⁵ Z. Fei, Y. S. Kivshar, and L. Vazquez, *Phys. Rev. A* **46**, 5214 (1992).

⁶ H. Al-Alawi Jassem and W. J. Zakrzewski, *J. Phys. A* **40**, 11319 (2007).

⁷ B. Piette and W. J. Zakrzewski, *J. Phys. A: Math. Theor.* **40**, 329 (2007).

⁸ K. Javidan and M. Sarbishaei, *Indian J. Physics B* **75**, 413 (2001).

⁹ K. Javidan *J. Phys. A: Math. Gen.* **39**, 10565 (2006).

¹⁰ K. Javidan, *Phys. Rev. E* **78**, 046607 (2008).

¹¹ A. Ghahraman and K. Javidan, e-print arXiv:0809.0210.

¹² N. S. Manton, *Phys. Lett. B* **110**, 54 (1982).

¹³ F. Lund, *Phys. Lett. A* **159**, 245 (1991).