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Variational Iteration Method for Solving Nonlinear Differential-difference Equations (NDDEs)

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Abstract: In this paper, the variational iteration method is applied to solve nonlinear differential-difference equations, such as the discretized nonlinear Schrödinger equation and the Toda lattice equation, which they need only one- iteration for each and the obtained solutions are of remarkable accuracy. Comparisons are made between the results of the proposed method and exact solutions. The results show that the variational iteration method is an attractive method in solving the NDDEs.

Key words: Variational iteration method; Nonlinear differential-difference equation; Approximate solution

INTRODUCTION

Nonlinear differential-difference equations are encountered in various fields of physics such as particle vibrations in lattices, currents flow in electrical networks, and pulses in biological chains. The solution of these DDEs can provide numerical simulations of nonlinear partial differential equations, queuing problems, and discretizations in solid state and quantum physics. Unlike difference equations which are fully discretized, differential-difference equations are semi-discretized, with some (or all) of their spatial variables discretized, while time variable is usually continuous. Since the work of Fermi et al. in the 1950 s (Fermi, E., 1965), there was quite a number of research works developed during the last decades on DDEs (Levi, D., R.I. Yamilov, 1997; Yamilov, R.I., 1994; Cherdantsev, IYu, 1997,1995; Svinolupov, S.I., R.I. Yamilov, 1997; Yang, H.X., X.X. Xu, H.Y. Ding, 2005).

The variational iteration method (VIM) is powerful in investigating the approximate or analytical solutions of the nonlinear differential equations. This method is proposed by the Chinese mathematician (Ji-Huan, He, 1997; Ji-Huan He, 1998; Ji-Huan He, 1999; Ji-Huan He, 2000) as a modification of a general Lagrange multiplier method (M. Inokuti, *et al*., 1978). It has been shown that this procedure is a powerful tool for solving various kinds of problems.

The aim of this article is to directly extend the VIM to solve NDDEs, such as the discretized nonlinear Schrödinger equation (Dai, C.Q., J.F. Zhang, 2006):

$$i \cdot \frac{du_n}{dt} = (u_{n+1} + u_{n-1} - 2u_n) - |u_n|^2 (u_{n+1} + u_{n-1}), \tag{1}$$

and the Toda lattice equation (Suris, Y.U.B., 1997):

$$\begin{aligned} \frac{du_n}{dt} &= u_n(v_n - v_{n-1}), \\ \frac{dv_n}{dt} &= v_n(u_{n+1} - u_n), \end{aligned} \tag{2}$$

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where the subscript n in Eqs. (1) and (2) represents the nth lattice.

2. Variational Iteration Method:

To illustrate the basic idea of VIM, we consider the following general nonlinear system:

$$Lu(x) + Nu(x) = g(x) \quad , \tag{3}$$

where L is a linear operator, N is a nonlinear operator and g (x) is an inhomogeneous forcing term. According to the variational iteration method (Yang, H.X., X.X. Xu, H.Y. Ding, 2005; Ji-Huan, He, 1997; Ji-Huan He, 1998; Ji-Huan He, 1999), we can construct a correction functional for the system as follows:

$$u_{m+1}(x) = u_m(x) + \int_0^x \lambda(s)\{Lu_m(s) + N\tilde{u}_m(s) - g(s)\}\, ds, \tag{4}$$

where λ is a Lagrange multiplier, which can be identified optimally via the variational theory (M. Inokuti, *et al*., 1978; He, J.H., 1997; Finlayson, B.A., 1972), the subscripts m denotes the mth approximation, and $\tilde{u}_m$ is considered as a restricted variation. i.e. $\delta\tilde{u}_m$ =0.

3. Applications of the Variational Iteration Method:

In this section, we will use the method demonstrated in section 2 to find the approximate solutions for the two above mentioned NDDEs.

Example 1: First, we consider Eq. (1) with the initial condition:

$$u_n(0) = \tanh(k)\cdot e^{ipn}\cdot\tanh(kn), \tag{5}$$

and the exact solution (Dai, C.Q., J.F. Zhang, 2006):

$$u = \tanh(k) * \exp(i[pn + (2 - 2\cos(p)\cdot \sec h(k))t])\cdot\tanh(kn + 2\sin(p)\tanh(k)\cdot t). \tag{6}$$

According to (4), we can construct the following correction functional:

$$\begin{aligned} u_{m+1}(t,n) = u_m(t,n) + \int_0^t \lambda(s)\{i\frac{du_m(s,n)}{ds} - (\tilde{u}_m(s,n+1) + \tilde{u}_m(s,n-1) - 2\tilde{u}_m(s,n)) \\ + \left|\tilde{u}_m(s,n)\right|^2 (\tilde{u}_m(s,n+1) + \tilde{u}_m(s,n-1))\}\, ds, \end{aligned} \tag{7}$$

To find the optimal value of λ we have

$$\begin{aligned} \delta u_{m+1}(t,n) = u_m(t,n) + \delta\int_0^t \lambda(s)\{i\frac{du_m(s,n)}{ds} - (\tilde{u}_m(s,n+1) + \tilde{u}_m(s,n-1) - 2\tilde{u}_m(s,n)) \\ + \left|\tilde{u}_m(s,n)\right|^2 (\tilde{u}_m(s,n+1) + \tilde{u}_m(s,n-1))\}\, ds, \end{aligned}$$

Or

$$\delta u_{m+1}(t,n) = u_m(t,n) + \delta\int_0^t \lambda(s)(i\frac{du_m(s,n)}{ds})\, ds,$$

witch yields

$$\lambda'(s) = 0,$$
$$1 + i\lambda(t) = 0,$$

Thus, we have

$$\lambda = i, \tag{8}$$

and we obtain the following iteration formula:

$$u_{m+1}(t,n) = u_m(t,n) + \int_0^t \{-\frac{du_m(s,n)}{ds} - i(u_m(s,n+1) + u_m(s,n-1) - 2u_m(s,n)) + i\left|u_m(s,n)\right|^2 (u_m(s,n+1) + u_m(s,n-1))\}\, ds. \tag{9}$$

Choosing $u_0(t,n) = u(0,n)$, for simplicity, as the initial approximation, we obtain from (9) an improved approximation

$$\begin{aligned} u_1 = {} & \tanh(k)e^{ipn}\tanh(kn) + i\,(-\tanh(k)e^{ip(n+1)}\tanh(k(n+1)) \\ & -\tanh(k)e^{ip(n-1)}\tanh(k(n-1)) + 2\tanh(k)e^{ipn}\tanh(kn) \\ & +\tanh(k)e^{ipn}\tanh(kn)\cdot\overline{\tanh(k)e^{ipn}\tanh(kn)}\cdot(\tanh(k)e^{ip(n+1)}\tanh(k(n+1)) \\ & +\tanh(k)e^{ip(n-1)}\tanh(k(n-1))))t. \end{aligned} \tag{10}$$

In Tables 1 and 2, we compare the one-iteration VIM solution (10) with the exact solution (6).

Table. 1: Comparison between the exact solution and the first order approximate when K = 0.1, P = 0.5 and t = 0.5.

n	| Exact solution |	$\|u_1\|$	Absolute error
-25	0.0982010639	0.0990501911	0.00092697
-15	0.0893155105	0.0901334506	0.0009263
-5	0.0422312592	0.0427417531	0.00081647
0	0.004758853	0.004762474	0.0006032
5	0.0497200827	0.0501686269	0.00054232
15	0.0910386415	0.0918400593	0.00083057
25	0.0984547238	0.0993013305	0.00091231

Table. 2: Comparison between the exact solution and the first order approximate when, K = 0.1, P = 0.5 and t = 0.5.

n	| Exact solution |	$\|u_1\|$	Absolute error
-25	0.0978948457	0.1053162249	0.007433811
-15	0.0872706681	0.0945465349	0.007287058
-5	0.0341123408	0.0392541622	0.006298122
0	0.014190354	0.0142874216	0.005384179
5	0.0565293256	0.0600915154	0.005595815
15	0.0924858639	0.0993514973	0.007015895
25	0.0986647435	0.1060233411	0.007392019

Example 2: Now, we consider Eq. (2) with initial conditions

$$\begin{aligned} u_n(0) &= -\coth(d)c + c\tanh(dn), \\ v_n(0) &= -\coth(d)c - c\tanh(dn), \end{aligned} \tag{11}$$

and the exact solution

$$u = -\coth(d)c + c\tanh(dn+ct),$$
$$v = -\coth(d)c - c\tanh(dn+ct), \tag{12}$$

Using the variational iteration method, the correction functionals can be written in the following form

$$u_{m+1}(t,n) = u_m(t,n) + \int_0^t \lambda_1(s)\{\frac{du_m(s,n)}{ds} - \tilde{u}_m(s,n)(v_m(s,n) - v_m(s,n-1))\}\,ds, \tag{13}$$

$$v_{m+1}(t,n) = v_m(t,n) + \int_0^t \lambda_2(s)\{\frac{dv_m(s,n)}{ds} - \tilde{v}_m(s,n)(u_m(s,n+1) - u_m(s,n-1))\}\,ds, \tag{14}$$

Identification of the Lagrange multipliers results in

$$\lambda_1 = \lambda_2 = -1, \tag{15}$$

Now, we assume that an initial approximation has the following form:

$$u_0 = -\coth(d)c + c\tanh(dn),$$
$$v_0 = -\coth(d)c - c\tanh(dn). \tag{16}$$

Therefore, we obtain the following first order approximate:

$$u_1 = (-\coth(d)c + c\tanh(dn))(-c\tanh(dn) + c\tanh(d(n-1)))t$$
$$\quad -\coth(d)c + c\tanh(dn),$$
$$v_1 = (-\coth(d)c - c\tanh(dn))(c\tanh(d(n+1)) - c\tanh(d(n))t$$
$$\quad -\coth(d)c - c\tanh(dn), \tag{17}$$

Tables 3 and 4 are shown the comparison between the exact solution and the first-order approximate solution. We observe that higher accuracy is obtained without any difficulty.

Table. 3: Comparison between the exact solution and the first order approximate when $c = d = 0.1$, and $t = 0.5$.

n	$\mid u - u_1 \mid$	$\mid v - v_1 \mid$
-25	0.000007	0.000007
-15	0.000042	0.000042
-5	0.0000896	0.0000896
0	0.000004	0.000004
5	0.0000919	0.0000919
15	0.0000398	0.0000398
25	0.000006	0.000006

Table. 4: Comparison between the exact solution and the first order approximate when $c = d = 0.1$, and $t = 1.5$.

n	$\mid u - u_1 \mid$	$\mid v - v_1 \mid$
-25	0.0000652	0.0000652
-15	0.0003989	0.0003989
-5	0.00077745	0.00077745
0	0.0001115	0.0001115
5	0.00084144	0.00084144
15	0.00033966	0.00033966
25	0.0000537	0.0000537

3. Conclusion:

In this paper, we have used the variational iteration method for finding the solution of nonlinear differential-difference equations. The method is applied in a direct way without using linearization transformation. The numerical results in the Tables 1-4 show that the present method provides highly accurate numerical solutions for solving this type of NDDEs.

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