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VARIATIONAL ITERATION METHOD FOR SOLVING SEVENTH ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper, the variational iteration method is applied to solve boundary value problems for seventh order integro-differential equations. The obtained numerical results show that only one iteration is needed to apply, and the obtained solutions are of remarkable accuracy. By giving two examples and comparing with the exact solution, the efficiency of the method will be shown.

1. Introduction

In recent years, many different methods were proposed to solve boundary value problems (BVPs), such as homotopy perturbation method (HPM) [5, 9], variational iteration method (VIM) [6, 11] and modified decomposition method (MDM) [8]. Recently, Sweilam [10] implemented the VIM to solve fourth order integro-differential equations. In this paper, we apply the variational iteration method proposed by Ji-Huan He [1-4] to find approximate solutions for boundary value problems of seventh order integro-differential equations.

To illustrate the basic idea of VIM, we consider the following general nonlinear system

$$Lu + Nu = g(x), \tag{1}$$

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where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous forcing term. According to the variational iteration method [1-4], we can construct a correction functional for the system, as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\{Lu_n(s) + N\tilde{u}_n(s) - g(s)\}ds, \quad (2)$$

where λ is a Lagrange multiplier, which can be identified optimally via the variational theory [7], the subscripts n denotes the n-th approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

We consider the general boundary value problem of the following type, to solve by using VIM

$$y^{(vii)}(x) = g(x) + \int_0^x f(t, y(t), y'(t), y''(t), \ldots, y^{(vii)})dt, \quad (3)$$

with suitable boundary conditions.

2. Applications

According to VIM, the correction functional for (3) can be constructed as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\{y_n^{(vii)}(s) - g(s) - \int_0^s \tilde{f}(t, y_n(t), y_n'(t), \ldots, y_n^{(vii)}(t))dt\}ds, \quad (4)$$

where λ is general Lagrange multiplier, $\tilde{y}_n$ denotes restricted variation i.e., $\delta\tilde{y}_n = 0$. Making the above correction functional stationary, we obtain the following stationary conditions

$$1 + \lambda^{(vi)}(x) = 0, \quad \lambda^{(v)}(x) = 0, \quad \lambda^{(iv)}(x) = 0, \quad \lambda'''(x) = 0,$$
$$\lambda''(x) = 0, \quad \lambda'(x) = 0, \quad \lambda(x) = 0, \quad \lambda^{(vii)}(s) = 0.$$

The Lagrange multiplier, therefore, can be obtaining in the following form

$$\lambda(s) = -\frac{(s-x)^6}{6!}. \quad (5)$$

Therefore equation (4) can be rewritten as

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(s-x)^6}{6!}\{y_n^{(vii)}(s) - g(s)$$
$$- \int_0^s f(t, y_n(t), y_n'(t), \ldots, y_n^{(vii)}(t))dt\}ds. \quad (6)$$

Now, to demonstrate the accuracy of the variational iteration method we consider two following examples with known exact solutions.

2.1. Linear integro-differential equation

First, we consider the following integro-differential equation

$$y^{(vii)}(x) = 2 - 8e^x + \int_0^x y(t)dt, \quad 0 \le x \le 1 \tag{7}$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = -2, \\ &y(1) = 0, \quad y'(1) = -e, \quad y''(1) = -2e. \end{aligned} \tag{8}$$

The exact solution of (7) is $y(x) = (1 - x)e^x$. According to (6), we have the following iteration formulation

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(s-x)^6}{6!}\left\{y_n^{(vii)}(s) - 2 + 8e^s - \int_0^s y_n(t)dt\right\}ds. \tag{9}$$

Now, starting with the initial solution

$$y_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6, \tag{10}$$

where a_0, a_1, a_2, a_3, a_4, a_5 and a_6 which are unknown constants to be further determined.

By the iteration formula (9), we obtain the following first-order approximation

$$\begin{aligned} y_1(x) &= y_0(x) - \int_0^x \frac{(s-x)^6}{6!}\left\{y_0^{(vii)}(s) - 2 + 8e^s - \int_0^s y_0(t)dt\right\}ds \\ &= (8 + a_0) + (8 + a_1)x + (a_2 + 4)x^2 + \left(\frac{4}{3} + a_3\right)x^3 + \left(\frac{1}{3} + a_4\right)x^4 \\ &\quad + \left(\frac{1}{15} + a_5\right)x^5 + \left(\frac{1}{90} + a_6\right)x^6 + \frac{1}{2520}x^7 + \frac{a_0}{40320}x^8 + \frac{a_1}{362880}x^9 \\ &\quad + \frac{a_2}{1814400}x^{10} + \frac{a_3}{6652800}x^{11} + \frac{a_4}{19958400}x^{12} + \frac{a_5}{51891840}x^{13} \\ &\quad + \frac{a_6}{121080960}x^{14} - 8e^x. \end{aligned} \tag{11}$$

Incorporating the boundary conditions, equation (8), into $y_1(x)$, we get

$$y_1(0) = a_0 = 1, \quad y_1'(0) = a_1 = 0, \quad y_1''(0) = 2a_2 = -1, \quad y_1'''(0) = 6a_3 = -2.$$

$$y(1)=\frac{874638257}{39916800}+\frac{19958401}{19958400}a_4+\frac{51891841}{51891840}a_5+\frac{121080961}{121080960}a_6-8e=0,$$

$$y'(1)=\frac{5968259}{3024400}+\frac{6652801}{1663200}a_4+\frac{19958401}{3991680}a_5+\frac{51891841}{8648640}a_6-7e=0,$$

$$y''(1)=\frac{6780301}{362880}+\frac{1814401}{151200}a_4+\frac{6652801}{332640}a_5+\frac{19958401}{665280}a_6-6e=0,$$

Solving the above system simultaneously, we obtain the unknowns as follows:

$$c_0=1,\ a_1=0,\ a_2=\frac{-1}{2},\ a_3=\frac{-1}{3},$$

$$a_4=-\frac{3001266767266598497485 4029}{12540092483933185558 4641}+\frac{1103527685870509261824 0000}{12540092483933185558 4641}e,$$

$$a_5=\frac{621809780656953475456 2577}{169460709242340345384 65}-\frac{457543629986091323875200}{338921418484680690769 3}e,$$

$$a_6=-\frac{112493890519394664098718263}{752455490359911335078 46}-\frac{689704456585724854642 3680}{12540092483933185558 4641}e. \quad (12)$$

This gives us the approximate solution for (7) and (8).

The numerical results of this application compared with the exact solution are presented in Table 1. We observe the higher accuracy is obtained without any difficulty.

Table 1. Comparison of the first-order approximate solution with the exact solution.

x	y_E	y_1	Absolute error
0	1	1	0.00e-00
0.1	0.9946538262	0.9946538275	1.26e-09
0.2	0.9771222064	0.9771222018	4.60e-09
0.3	0.9449011656	0.944901164	2.00e-09
0.4	0.8950948188	0.895094819	1.00e-09
0.5	0.8243606355	0.824360631	4.00e-09
0.6	0.7288475200	0.728847543	2.30e-08
0.7	0.6041258121	0.60412578	3.00e-08
0.8	0.4451081856	0.44510815	4.00e-08
0.9	0.2459603111	0.24596027	4.00e-08
1	0	0	0.00e-00

2.2. Nonlinear integro-differential equation

Now, we consider the following nonlinear integro-differential equation

$$y^{(vii)} = 1 + \int_0^x e^{-x} y^2(t)dt, \quad 0 \le x \le 1. \tag{13}$$

With the boundary conditions

$$y(0) = y'(0) = y''(0) = y'''(0) = 1,$$
$$y(1) = y'(1) = y''(1) = e. \tag{14}$$

The exact solution of this problem is $y(x) = e^x$. According to (6), we have the following iteration formulation

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(s-x)^6}{6!} \left\{ y_n^{(vii)}(s) - 1 - \int_0^s e^{-t} {y_n}^2(t)dt \right\} ds. \tag{15}$$

We start with the following initial approximation

$$y_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6, \tag{16}$$

where a_0, a_1, a_2, a_3, a_4, a_5 and a_6 are unknown constants to be further determined.

After we apply the iteration formula (15) to the BVP, we get the following first-order approximation

$$y_1(x) = y_0(x) - \int_0^x \frac{(s-x)^6}{6!} \left\{ -1 - \int_0^s e^{-t} {y_0}^2(t)dt \right\} ds. \tag{17}$$

Incorporating the boundary conditions, equation (14), into $y_1(x)$, we obtain

$$a_0 = 1,\ a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{6},$$

$$a_4 = \frac{4816098}{115585067},\ a_5 = \frac{578307}{69403294},\ a_6 = \frac{218465}{157257279}. \tag{18}$$

Putting the values of a_i's, $i = 0, 1, \ldots, 6$ one can get the approximate solution.

Comparison of the first-order approximate solution with the exact one is shown in Table 2.

Table 2. Comparison of the first-order approximate solution with the exact solution.

x	y_E	y_1	Absolute error
0	1	1	0.00e-00
0.1	1.1051709181	1.1051710761	1.57e-07
0.2	1.2214027582	1.2214028349	7.67e-08
0.3	1.3498588076	1.3498589739	1.66e-07
0.4	1.4918246976	1.491824729	3.13e-08
0.5	1.6487212707	1.648721216	5.47e-08
0.6	1.8221188004	1.822118855	5.46e-08
0.7	2.0137527075	2.013752699	8.47e-09
0.8	2.2255409285	2.225540953	2.45e-08
0.9	2.4596031112	2.459603164	5.28e-08
1	2.7182818284	2.718281828	1.96e-08

3. Conclusion

In this paper, variational iteration method is employed to solve the linear and nonlinear BVPs for seventh order integro-differential equations. The method is applied in a direct way without using linearization, transformation and discretization. The numerical results in Tables 1-2 show that the presented method provides highly accurate numerical solutions for solving this type of the BVPs.

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