A NEW APPROACH FOR SOLVING NONLINEAR NON-SMOOTH PROGRAMMING PROBLEM

A. V. KAMYAD N. HASSANZADEH

Abstract. In this paper, we introduce a new approach to solve nonlinear programming problems also when the objective function (functions) or constraint function (functions) are non-smooth ones. In this approach the nonlinear functions of the original problem is approximated by a piecewise linear functions. Then we find the global extremum of this approximated problem by solving a linear programming problem. Also, we will prove convergence of our approach. One of the main advantages of our approach is that the approach may be extended for problems where objective function or constraint functions or both are non-smooth functions by introducing a novel definition of global weak differentiation in the sense of ℓ_1 space. Also, other advantage of our approach is that we may obtain approximate solution by solving a corresponding a linear programming problem. Finally, numerical example are given to show the efficiency of the proposed approach to solve constraints nonlinear programming problems, especially when the objective function and constraint functions are non-smooth function.

key words : Nonlinear programming, linear programming, linearization, Taylor linear expansion, nonlinear non-smooth function.

Introduction

Kelley's cutting plane method [7] was introduced 1960 to solve nonlinear programming (NLP) problems by solving a sequence of linear programming (LP) problems. Although some other methods based on linear programming exist, such as the method of approximate programming [3, 6], LP techniques were quickly abandoned in favor of sequential quadratic programming (SQP) techniques. After Han proved local and global convergence of SQP methods in [4, 5], a large amount of research papers have been produced on SQP-based techniques. Indeed, many of the NLP solvers today use SQP techniques in one form or the other. There are some interesting recent papers on successive linear programming (SLP) techniques. In [1], a procedure is presented where linear programming and quadratic programming subproblems are successively solved to find the optimal solution. The linear programming problem provides an estimate of the active constraints within a trust region and a quadratic programming problem is constructed and solved using the active constraints at the optimal solution of the linear problem. However, the method in [1] utilizes linear programming problems mainly to estimate the active constraints in each iteration, and solves a quadratic, equality constrained, problem as well in each iteration. In [9] it is shown that LP techniques can be applied quite successfully to solve NLP problems efficiently, even without having to solve quadratic subproblems. But They supposed functions are continuously differentiable over $\mathbb{R}^n$. contrary to [8], the objective or constraint are not assumed to be convex and non-smooth. In [8] it is assumed that constraints include linear constraints defining a bounded region X. In this paper, we introduce to definition of weak differentiation for non-smooth functions([10, 11]). we may approximate our approach by AVK method[2]. Then we may solve nonlinear non-smooth programming by linearization of nonlinear non-smooth objective function and constraint functions.

Our approach solves problems of the form:

$$\begin{aligned} & max\ f(x) \\ & s.t. \\ & \quad g(x) \leq b \\ & \quad x \in \Omega = \prod_{i=1}^{n} [a_i, b_i] \subseteq \mathbb{R}^n \end{aligned}$$

where f and g are continuous functions (may be non-smooth) defined on Ω.

In this paper, without losing the generality, we assume that $a = 0$ and $b = 1$. Since, the function F defined below is a bijective function. So instead of $[a, b]$ we may use $[0, 1]$. That is:

$$F : [a, b] \rightarrow [0, 1]$$
$$x \rightarrow \frac{x - a}{b - a}$$

Our approach is based on parametric linearization which is a generalization of Taylor linear expansion of smooth function. The paper is outlined as follows:

First, we consider problem in one dimension and we assume that functions f and g are smooth functions. Second, we assume the problem is n dimension also f and g are smooth. Third, we assume the problem is n dimension and f and g are non-smooth functions and solve the problem approximately by defining weak differentiation on an interval in n-dimension space.

1. **proposed approach for one dimensional problems**

First, consider the following nonlinear optimization problem:

$$\begin{aligned} & \max\ f(x) \\ & s.t. \\ & \quad x \in [0, 1] \end{aligned} \tag{1.1}$$

Where $f : [0, 1] \rightarrow \mathbb{R}$ is a nonlinear smooth function. Here, we approximate the nonlinear function $f(x)$ by a piecewise linear function.

Now, we state the following definition for linearization of nonlinear function $f(x)$.

Definition 1.1. *We consider a partition of an interval* $[0, 1]$ *on* $\mathbb{R}$ *as follows:*

$$p_n[0, 1] = \{0 = x_0, x_1, \ldots, x_n = 1\}$$

Where $0 = x_0 < x_1 < \ldots < x_n = 1$.

We define the norm of partition as follows:

$$\|p_n\| = \max_{1 \leq i \leq n} \{|x_i - x_{i-}|\}$$

Note 1.2. *In this paper, we assume that* $x_i = \frac{i}{n}$ *for all* $i = 0, 1, \ldots, n$. *Therefore,* $\|p_n\| = \frac{1}{n}$.

Definition 1.3. *Let $f(x)$ be a nonlinear function on $[0,1]$. We define function $f_i(x)$ on $I_i = [x_{i-1}, x_i]$ as follows:*

$$f_i(x) \cong f'(a_i)x + f(a_i) - a_i f'(a_i) \quad i = 1, 2, \dots, n\ ,\ x \in I_i \tag{1.2}$$

Where $a_i \in I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$ is a mid point i.e. $a_i = \frac{2i-1}{n}$ for all $i = 1, 2, \dots, n$.

We define

$$m_i = f'(a_i) = f'(\frac{2i-1}{2n}), \qquad i = 1, 2, \dots, n$$
$$b_i = f(a_i) - a_i f'(a_i) = f(\frac{2i-1}{2n}) - \frac{2i-1}{2n} f'(\frac{2i-1}{2n})$$

If length of interval I_i be a small number, then $f_i(x) \cong m_i x + b_i$ for all $i = 1, 2, \dots, n$. Now, we define $G_n(x)$ as a piecewise linear approximation of $f(x)$ on $[0,1]$ as follows:

$$G_n(x) = \sum_{i=1}^{n} f_i(x) \times \chi_{[\frac{i-1}{n}, \frac{i}{n}]}(x) \tag{1.3}$$

where $\chi_{[\frac{i-1}{n}, \frac{i}{n}]}(x)$ is the characteristic function and defined as follows:

$$\chi_{[\frac{i-1}{n}, \frac{i}{n}]}(x) = \begin{cases} 1 & x \in [\frac{i-1}{n}, \frac{i}{n}] \\ 0 & x \notin [\frac{i-1}{n}, \frac{i}{n}] \end{cases}$$

Now, we prove the uniformly convergence theorem of our approach. this means where we choose n a big number closeness G_n to f is independent of $x \in [0,1]$.

Theorem 1.4. *Let f be a non-smooth function and G_n defined in (1.3). Now, we claim where if n tends to infinity, then G_n tends to f uniformly on $[0,1]$.*

Proof. Let $E_n(x) = G_n(x) - f(x)$ for all $x \in [0,1]$. There is $1 \leq i \leq n$ and $\eta_i \in (x_{i-1}, x_i) = (\frac{i-1}{n}, \frac{i}{n})$ such that

$$f(x) - f_i(x) = \frac{f''(\eta_i)}{2}\left(x - \frac{2i-1}{2n}\right)^2$$

Where $x \in [x_{i-1}, x_i]$. We know $\|p_n\| = \frac{1}{n}$. Thus, let $M_i = \sup_{x \in I_i} |f''(x)|$. Therefore,

$$|f(x) - f_i(x)| \leq \frac{M_i}{2}\left(x_i - \frac{2i-1}{2n}\right)^2 = \frac{M_i}{8n^2}$$

We set $M = \max\{M_1, M_2 \dots, M_n\}$. Consequently,

$$|E_n(x)| = |G_n(x) - f(x)| \leq \sum_{i=1}^{n} |f(x) - f_i(x)| \leq \sum_{i=1}^{n} \frac{M}{8n^2} = \frac{M}{8n}$$

Now, If n tends to infinity, then $E_n(x) \to 0$ or equivalently $G_n(x) \to f(x)$, for all $x \in [0,1]$. It is clear, that interval $[0,1]$ is compacted and $G_n(x)$ is continuous on $[0,1]$. Therefore, It is uniform continuous on $[0,1]$. According to definition of uniform continuous, given $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in [0,1]$, we have:

$$|x - y| < \delta \Rightarrow |G_n(x) - G_n(y)| < \frac{\varepsilon}{3}$$

Also, there is $N(y) > 0$ such that for all $n, m > N(y)$ we have $|G_n(y) - G_m(y)| < \frac{\varepsilon}{3}$. Therefore, $\sup_{y\in[0,1]} |G_n(y) - G_m(y)| \leq \frac{\varepsilon}{3}$. It is sufficient, we show that $\{G_n(x)\}$ is a cauchy uniformly sequence. There is $N > 0$ such that for all $n, m \geq N$, we have:

$$\begin{aligned}|G_n(x) - G_m(x)| &= |G_m(x) - G_n(y) + G_n(y) - G_m(y) + G_m(y) - G_m(x)| \\ &\leq |G_m(x) - G_n(y)| + |G_n(y) - G_m(y)| + |G_m(y) - G_m(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

Let $N' = \max\{N, N(y)\}$. Thus for all $n, m \geq N'$, $G_n(x)$ is a cauchy uniformly sequence and uniformly continuous on $[0, 1]$. Then $G_n(x)$ is equicontinuity on $[0, 1]$. Consequently, it is uniformly convergent. $\square$

Now, if $f(x)$ in the problem (1.1) is replaced with piecewise linear function $G_n(x)$, we arrive to the following optimization problem.

$$\begin{aligned}&\max \sum_{i=1}^{n} f_i(x) \times \chi_{[\frac{i-1}{n},\frac{i}{n}]}(x) \\ &s.t. \\ &\qquad x \in [0, 1]\end{aligned} \tag{1.4}$$

We may solve the following problems equivalent to (1.4) for all $i = 1, 2, \ldots, n$:

$$\begin{aligned}&\max\ f_i(x) \\ &s.t. \\ &\qquad \frac{i-1}{n} \leq x \leq \frac{i}{n}\end{aligned} \tag{1.5}$$

There exits an index as $i_0 \in \{1, 2, \ldots, n\}$ such that optimal solution is in I_{i_0}. Because, nonlinear smooth function $f(x)$ is continuous and interval of $[0, 1]$ is compacted. Also, obtained the optimal solution by solving n linear programming problems on $I_i = [x_{i-1}, x_i]$ are distinct. Therefore, we may solve the following problem equivalent to (1.5):

$$\begin{aligned}&\max\ \{m_1x_1 + b_1, m_2x_2 + b_2, \ldots, m_nx_n + b_n\} \\ &s.t. \\ &\qquad \frac{i-1}{n} \leq x_i \leq \frac{i}{n} \qquad i = 1, 2, \ldots, n\end{aligned} \tag{1.6}$$

We may obtain the optimal solution of the problem (1.6) by solving n linear programming problems. Or, we must solve the following n linear programming problem for all $i = 1, 2, \ldots, n$:

$$\begin{aligned}&z_i = \max\ \{m_ix_i + b_i\} \\ &s.t. \\ &\qquad \frac{i-1}{n} \leq x_i \leq \frac{i}{n}\end{aligned} \tag{1.7}$$

We may say $z_k = \max\{m_kx_k + b_k\}$ or x_k is the optimal solution of our problem and z_k is

$$z_k = \max\{z_1^*, z_2^*, \ldots, z_n^*\}$$

It is clear that when n tends to infinity our solution x_k tends to the globally optimal solution of problem (1.1), Therefore, when n is very big natural number for obtaining a solution very near to global optimal solution of problem (1.1) we should solve n linear programming as (1.14). So in practice it taken a long time. So we introduce a single linear programming whose solution is the same as the solution of solving the above mentioned linear programming problems.

1.1. **Solving n linear programming by a unique linear programming problem.** Consider the problem (1.5). For linearization this problem, first we may solve $\min - \max$ problem instead of problem (1.5) as follows:

$$\begin{aligned} &\min \max_{1\le i\le n} \{m_i x_i + b_i\} \\ &s.t. \\ &\qquad \frac{i-1}{n} \le x_i \le \frac{i}{n} \qquad , i = 1, 2, \ldots, n \end{aligned}$$

For solving this problem, let

$$z = \max_{1\le i\le n} \{m_i x + b_i\}$$

Therefore, the problem transformed equivalently to the following linear programming problem:

$$\begin{aligned} &\min z \\ &s.t. \\ &\qquad m_i x_i + b_i \le z \qquad i = 1, 2, \ldots, n \\ &\qquad \frac{i-1}{n} \le x_i \le \frac{i}{n} \qquad i = 1, 2, \ldots, n \end{aligned} \tag{1.8}$$

By solving problem above, we obtain an approximation solution of nonlinear programming problem (1.1). In the theorem (1.4), we showed that if $n \to \infty$, then approximation optimal solution is convergent to main optimal solution.

1.2. **Piecewise linearization of constrained nonlinear programming problem in $\mathbb{R}$.**

Consider constraint nonlinear programming problem as follows:

$$\begin{aligned} &\max f(x) \\ &s.t. \\ &\qquad g(x) \le b \\ &\qquad 0 \le x \le 1 \end{aligned} \tag{1.9}$$

Where $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are nonlinear smooth functions. We may obtain an optimal solution of problem (1.8) by transforming it to linearly form. For this propose, from (1.2), we have

$$\begin{aligned} f_i(x) &\cong f'(\frac{2i-1}{n})x + f(\frac{2i-1}{n}) - (\frac{2i-1}{n})\ g'(\frac{2i-1}{n}) \quad i = 1, 2, \ldots, n \\ g_i(x) &\cong g'(\frac{2i-1}{n})x + g(\frac{2i-1}{n}) - (\frac{2i-1}{n})\ g'(\frac{2i-1}{n}) \quad i = 1, 2, \ldots, n \end{aligned} \tag{1.10}$$

Now, we define $G_n(x)$ and $H_n(x)$ as a piecewise linear approximation of $f(x)$ and $g(x)$ an $[0,1]$ as follows:

$$G_n(x) = \sum_{i=1}^{n} f_i(x) \times \chi_{[\frac{i-1}{n},\frac{i}{n}]}(x) \tag{1.11}$$
$$H_n(x) = \sum_{i=1}^{n} g_i(x) \times \chi_{[\frac{i-1}{n},\frac{i}{n}]}(x)$$

respectively, where $\chi_{[\frac{i-1}{n},\frac{i}{n}]}(x)$ is the characteristic function. According to theorem (1.4), we know that if $n \to \infty$ then $G_n(x) \to f(x)$ and $H_n(x) \to g(x)$ for all $x \in [0,1]$. Now, if $f(x)$ and $g(x)$ in problem (2.4) is replaced with piecewise linear function $G_n(x)$ and $H_n(x)$, respectively. we arrive to the following optimization problem:

$$\begin{aligned} \max\ & \sum_{i=1}^{n} f_i(x) \times \chi_{[\frac{i-1}{n},\frac{i}{n}]}(x) \\ s.t.\ & \\ & \sum_{i=1}^{n} g_i(x) \times \chi_{[\frac{i-1}{n},\frac{i}{n}]}(x) \leq b \\ & x \in [0,1] \end{aligned} \tag{1.12}$$

We may solve the following n linear programming problem equivalent to (1.12) for all $i = 1, 2, \ldots, n$:

$$\begin{aligned} \max\ & f_i(x) \\ s.t.\ & \\ & g_i(x) \leq b \\ & \frac{i-1}{n} \leq x \leq \frac{i}{n} \end{aligned} \tag{1.13}$$

It is clear, obtained the optimal solution by solving n linear programming problems on $I_i = [x_{i-1}, x_i]$ are distinct. Therefore, we solve the following problem instead of (1.13)

$$\begin{aligned} \max\ & \{f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\} \\ s.t.\ & \\ & g_i(x_i) \leq b \qquad i = 1, 2, \ldots, n \\ & \frac{i-1}{n} \leq x_i \leq \frac{i}{n} \qquad i = 1, 2, \ldots, n \end{aligned} \tag{1.14}$$

We may obtain optimal solution of the problem (1.14) by solving n linear programming problems. For this propose, First, we solve the following problem for all $i = 1, 2, \ldots, n$:

$$\begin{aligned} z_i = \max\ & f_i(x_i) \\ s.t.\ & \\ & g_i(x_i) \leq b \\ & \frac{i-1}{n} \leq x_i \leq \frac{i}{n} \end{aligned} \tag{1.15}$$

Next, we choose optimal solution corresponding to z^* where

$$z^* = \max\{z_1^*, z_2^*, \ldots, z_n^*\}$$

We state our approach for linearization the problem (1.9), first we may solve $\min - \max$ problem instead of problem (1.14) as follows:

$$\begin{aligned} \min \ & \max_{1 \le i \le n} \{f_i(x_i)\} \\ s.t. & \\ & g_i(x_i) \le b \qquad , i = 1, 2, \ldots, n \\ & \frac{i-1}{n} \le x_i \le \frac{i}{n} \qquad , i = 1, 2, \ldots, n \end{aligned}$$

For solving this problem, let

$$z = \max_{1 \le i \le n} \{f_i(x_i)\}$$

Therefore, the problem transformed to the following linear programming problem:

$$\begin{aligned} \min \ & z \\ s.t. & \\ & f_i(x_i) \le z \qquad i = 1, 2, \ldots, n \\ & g_i(x_i) \le b \qquad i = 1, 2, \ldots, n \\ & \frac{i-1}{n} \le x_i \le \frac{i}{n} \qquad i = 1, 2, \ldots, n \end{aligned} \tag{1.16}$$

By solving problem above, we obtain an approximation solution of nonlinear programming problem (1.1).

Corollary 1.5. *Let f and g be non-smooth functions in general and G_n and H_n defined in (1.11). Now, we claim where if n tends to infinity, then G_n and H_n tends to f and g, respectively. So our approximate solution optimal to our original nonlinear programming.*

Proof. By applying of the theorem (1.4), we get that if $n \to \infty$ then $G_n \to f$ and $H_n \to g$. Now, let $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ be a set contain the optimal solution by solving n linear programming problems as (1.15) is in $\mathcal{B}$. Therefore, the optimal solution problem (1.16) tends to the optimal solution problem (1.9). Also, the optimal value z^* tends to the optimal value of problem (1.9). □

2. Extension of the proposed approach for n dimensional problems

Consider the following nonlinear optimization problem:

$$\begin{aligned} \max \ & f(x) \\ s.t. & \\ & x \in \mathcal{A} \end{aligned} \tag{2.1}$$

Where $\mathcal{A} = \prod_{i=1}^{n} [0, 1] \subseteq \mathbb{R}^n$ and $f : \mathcal{A} \to \mathbb{R}$ is nonlinear smooth function. Here, we introduce a piecewise linear parametric approximation for $f(x)$ which is the extension of definition (1.1).

Definition 2.1. *Consider the nonlinear smooth function $f : \mathcal{A} \to \mathbb{R}$, where $\mathcal{A} = [0,1]^n$. Also, consider $p_{n_i}[0,1]$ as a partition of $[0,1]$ as follows:*

$$p_{n_i}([0,1]) = \{0 = x_i^0, \dots, x_i^{k_i}, \dots, x_i^{n_i} = 1\}$$

Where $x_i^0 < \dots < x_i^{k_i} < \dots < x_i^{n_i}$, $k_i = 0, 1, \dots, n_i$ and $i = 1, 2, \dots, n$. So $\mathcal{A}$ is partitioned to N cells where $N = n_1 \times n_2 \times \dots \times n_n$. Let shown the k^{th} cell by S_k for all $k = 1, 2, \dots, N$. let $s_k = (s_k^1, s_k^2, \dots, s_k^n)$ is used to shown mid point of S_k. Now, $f_k(x)$ is defined as a linear parametric approximate of $f(x)$ for $x \in S_k$ as follows:

$$f_k(x) \cong \nabla f(x)|_{x=s_k}.(x - s_k) + f(s_k) \tag{2.2}$$

where $x \in S_k$ for all $k = 1, 2, \dots, N$.
We define

$$\begin{array}{ll} m_k = \nabla f(x)|_{x=s_k} & k = 1, 2, \dots, N \\ b_k = f(s_k) - \nabla f(x)|_{x=s_k} \times s_k & k = 1, 2, \dots, N \end{array}$$

Now, $G_N(x)$ is defined as a piecewise linear approximate of $f(x)$ as follows:

$$G_N(x) = \sum_{k=1}^{N} f_k(x) \times \chi_{S_k}(x) \tag{2.3}$$

where $\chi_{S_k}(x)$ is the characteristic function and defined as follows:

$$\chi_{S_k}(x) = \begin{cases} 1 & x \in S_k \\ 0 & x \notin S_k \end{cases}$$

Theorem 2.2. *Let f be a non-smooth function and G_N defined in (1.3). Now, we claim where if N tends to infinity, then G_N tends to f uniformly.*

Proof. Let $E_k(x) = f(x) - f_k(x)$ for all $x \in [0,1]^n$. There is $1 \le k \le N$ and $\eta_k \in S_k$ such that

$$f(x) - f_k(x) = \frac{\nabla^2 f(\eta_k)}{2} \dot{(}x - s_k)^2$$

Where $x \in S_k$. We know $\|S_k\| = \frac{1}{n_1} \times \frac{1}{n_2} \times \dots \times \frac{1}{n_n} = \frac{1}{N}$. Thus, let $M_k = \sup_{x \in S_k} |\nabla^2 f(x)|$. Therefore,

$$|f(x) - f_k(x)| \le \frac{M_k}{2} |x_k - x_{k-1}|^2 \le \frac{M_k}{2} \|S_k\| \le \frac{M_k}{2} \frac{1}{N^2}$$

Suppose $M = \max\{M_1, M_2 \dots, M_N\}$. Consequently,

$$|G_N(x) - f(x)| \le \sum_{k=1}^{N} |f(x) - f_k(x)| \le N.\frac{M}{2} \frac{1}{N^2} = \frac{M}{2N}$$

Now, If N tends to infinity. Then for all $x \in [0,1]^n$:

$$\lim_{N \to \infty} G_N(x) = f(x)$$

As same case of one dimensional, we may show that $G_N(x)$ is uniformly convergent to $f(x)$ on $[0,1]^n$. $\square$

Now, if $f(x)$ in the problem (2.1) is replaced with piecewise linear function $G_N(x)$, we arrive to the following optimization problem.

$$\begin{aligned} &\max \sum_{k=1}^{N} f_k(x) \times \chi_{S_k}(x) \\ &s.t. \\ &\quad x \in [0,1]^n \end{aligned} \tag{2.4}$$

It is clear that optimal solution is situated in a cell. Therefore, we solve a parametric programming problem as follows:

$$\begin{aligned} &\max \sum_{k=1}^{N} f_k(x'_k) \times \chi_{S_k}(x'_k) \\ &s.t. \\ &\quad x'_k \in S_k \qquad k = 1, 2, \ldots. N \end{aligned} \tag{2.5}$$

where $N = n_1 \times n_2 \times \ldots n_n$. We may obtain optimal solution of the problem (2.5) by solving N linear programming problems. For this propose, First, we solve the following problem for all $k = 1, 2, \ldots, N$:

$$\begin{aligned} z_k = \max &\; \{m_k x'_k + b_k\} \\ s.t. & \\ & x_k \in S_k \end{aligned} \tag{2.6}$$

Next, we choose optimal solution corresponding to z^* where

$$z^* = \max\{z_1^*, z_2^*, \ldots, z_N^*\}$$

In continue, we will show a new approach for obtaining solution of problem.

2.1. **Solving N linear programming by a unique linear programming problem.** Consider the problem (2.5). For linearization this problem, first we may solve $\min - \max$ problem instead of problem (2.5) as follows:

$$\begin{aligned} &\min \max_{1 \le k \le N} \{m_k x'_k + b_k\} \\ &s.t. \\ &\quad x'_k \in S_k \end{aligned}$$

For solving this problem, let

$$z = \max_{1 \le k \le N} \{m_k x'_k + b_k\}$$

Therefore, the problem transformed to the following linear programming problem:

$$\begin{aligned} &\min z \\ &s.t. \\ &\quad m_k x'_k + b_k \le z \qquad k = 1, 2, \ldots, N \\ &\quad x'_k \in S_k \qquad k = 1, 2, \ldots, N \end{aligned} \tag{2.7}$$

By solving the problem above, we obtain a approximate solution of nonlinear programming problem (2.1).

2.2. **Piecewise linearization general constrained nonlinear programming problem.**
Consider constraint nonlinear programming problem as follows:

$$\begin{aligned} &\max\ f(x) \\ &\quad s.t. \\ &\qquad g(x) \leq b \\ &\qquad x \in [0,1]^n \end{aligned} \tag{2.8}$$

Where $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are nonlinear smooth functions. We may obtain an optimal solution of problem (2.8) by transforming it to linearly form. For this propose, from (2.2), we have

$$\begin{aligned} f_k(x) &\cong \nabla f(x)|_{x=s_k}.(x-s_k) + f(s_k) \quad k = 1,2,\ldots,N \\ g_k(x) &\cong \nabla g(x)|_{x=s_k}.(x-s_k) + g(s_k) \quad k = 1,2,\ldots,N \end{aligned} \tag{2.9}$$

As same case of one dimensional, in this case, we may transformed the problem (2.8) to linear programming problem as follows:

$$\begin{aligned} &\min\ z \\ &\quad s.t. \\ &\qquad f_k(x'_k) \leq z \qquad k = 1,2,\ldots,N \\ &\qquad g_k(x'_k) \leq b \qquad k = 1,2,\ldots,N \\ &\qquad x'_k \in S_k \qquad k = 1,2,\ldots,N \end{aligned} \tag{2.10}$$

3. **Extension to non-smooth nonlinear programming problems**

In general it is reasonable to assume that the objective function is a non-smooth ones. Therefore, we define type of generalized differentiation for non-smooth functions in coincide with usual differentiation for smooth function.

For linearization, we use from relation bellows:

$$f(x) \cong f(s) + (x-s)f'(s)$$

Suppose f' is not defined in s. we introduce weak differentiation for calculating differentiation of f in s we have:

$$f'(s) = \lim_{x \to s} \frac{f(x) - f(s)}{x - s}$$

Given $\varepsilon > 0$, for all $x \in (s-\varepsilon, s+\varepsilon)$, we define:

$$f'(s) \cong \frac{f(x) - f(s)}{x - s}, \qquad x \in (s-\varepsilon, s+\varepsilon)$$

For this propose, we refer to the following approach for calculating $f'(s)$:

$$\min \int_0^1 \left[\int_{s-\varepsilon}^{s+\varepsilon} |f(x) - f(s) - (x-s)f'(s)| dx \right] ds$$

We can extended above approach to n dimensional.

Theorem 3.1. *Consider the nonlinear smooth function $f : [0,1]^n \to \mathbb{R}$. Then the optimal solution of the following optimization problem is the function $f'(x)$.*

$$\min_{p(.)} \int_0^1 \int_0^1 \dots \int_0^1 |f(x) - [f(s) + p(s).(x-s)]| dx_1 dx_2 \dots dx_n \tag{3.1}$$

where $s = (s_1, s_2 \dots, s_n) \in [0,1]^n$ is an arbitrary point and $p(.)$ is a vector of the form $(p_1(.), p_2(.), \dots, p_n(.))$.

Proof. See [10]. □

Now based on theorem (3.1) the following definition may be stated for non-smooth functions.

Definition 3.2. *Let $f : [0,1]^n \to \mathbb{R}$ is a non-smooth function. The global weak differentiation with respect to x in the sense of ℓ_1 space is defined as the $p(.)$ the optimal solution of the minimization problem which is shown in (3.1).*

In the case that $n = 1$, we may obtain differentiation of $f(x)$ on $[0,1]$ by partitioning interval $[0,1]$ to n subinterval of $[x_{i-1}, x_i]$ such that $x_i = \frac{i}{n}$ for all $i = 1, 2, \dots, n$. Let s_i be a point in $\left[\frac{i-1}{n}, \frac{i}{n}\right]$. we show differentiation of $f(x)$ at $x = s$ by $p(s)$. We may obtain values of $p(s)$, by solving the following problem:

$$\min \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(x) - f(s_i) - (x-s).p(s_i)| dx \times \frac{1}{n}$$

where $x_i = \frac{i}{n}$ and $s_i \in (x_{i-1}, x_i)$ is an arbitrary point for all $i = 1, 2, \dots, n$. (note: we may define $s_i = \frac{2i-1}{2n}$ for all $i = 1, 2, \dots, n$)

4. **numerical examples**

In this section, we solve some constraint nonlinear smooth and non-smooth programming problems by transforming to a linear programming problem.

Example 4.1. *Consider nonlinear smooth programming problem as follows:*

$$\begin{aligned} &\max e^x \\ &\quad s.t. \\ &\qquad \sin(x) - x \le 1 \\ &\qquad 1 \le x \le 3 \end{aligned}$$

We may convert interval $[1,3]$ to $[0,1]$. For this propose, we may define bijective function $F(x)$ as follows:

$$\begin{aligned} F(x) : [1,3] &\to [0,1] \\ x &\to \frac{x-1}{2} \end{aligned}$$

Now, we have:

$$\begin{aligned} &\max e^{(2x+1)} \\ &\quad s.t. \\ &\qquad \sin(2x+1) - 2x \le 2 \\ &\qquad 0 \le x \le 1 \end{aligned}$$

the optimal solution and the optimal value are $x^* = 3$ *and* $z^* = 20.0855$*,respectively. We may partitioned interval of* $[0, 1]$ *to subinterval* $I_i = [x_{i-1}, x_i]$ *where* $x_i = \frac{i}{n}$ *for all* $i = 1, 2, \ldots, n$*. So, we may transform the above problem to the following equivalent linear programming problem:*

$$
\begin{aligned}
&\min\ z\\
&\quad s.t.\\
&\qquad 2e^{\left(1+\frac{2i-1}{n}\right)}\left(x-\frac{2i-1}{2n}\right)+e^{\left(1+\frac{2i-1}{n}\right)} \le z \quad i=1,2,\ldots,n\\
&\qquad \left[2\cos\left(1+\frac{2i-1}{n}\right)-2\right]\left(x-\frac{2i-1}{2n}\right)+\sin\left(1+\frac{2i-1}{n}\right)\frac{2i-1}{n} \le 2 \quad i=1,2,\ldots,n\\
&\qquad \frac{i-1}{n} \le x_i \le \frac{i}{n}
\end{aligned}
$$

Where $x_i = \frac{i}{n}$ *for all* $i = 1, 2, \ldots, n$*. In the following table showed optimal solutions for values of distinct* n*.*

n	*optimal solution*	*optimal value*
10	*2.9000*	*16.3567*
50	*2.9800*	*19.2941*
100	*2.9900*	*19.6868*
200	*2.9950*	*19.8854*
300	*2.9967*	*19.9520*
400	*2.9975*	*19.9853*
500	*2.9980*	*20.0053*
600	*2.9983*	*20.0187*
1000	*2.9990*	*20.0454*

Example 4.2. *Consider nonlinear smooth programming problem as follows:*

$$
\begin{aligned}
&\max\ \arctan(x+2y)\\
&\quad s.t.\\
&\qquad x-\sin(y) \le 1\\
&\qquad 0 \le x, y \le 1
\end{aligned}
$$

the Optimal solution and the optimal value are $x^* = y^* = 1$ *and* $z^* = 1.249046$*,respectively. We may partitioned interval of* $[0, 1] \times [0, 1]$ *to subinterval* $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ *where* $x_i = \frac{i}{n}$ *and* $y_j = \frac{j}{n}$ *for all* $i, j = 1, 2, \ldots, n$*. So, we may transform the above problem* *to the following equivalent linear programming problem:*

$$\begin{aligned}
&\min z\\
&\quad s.t.\\
&\quad \frac{1}{1+\left(\frac{2i-1}{2n}+\frac{2j-1}{n}\right)^2}\left[\left(x-\frac{2i-1}{2n}\right)+2\left(y-\frac{2j-1}{2n}\right)\right]+\arctan\left(\frac{2i-1}{2n}+\frac{2j-1}{n}\right)\le z\\
&\quad \left(x-\frac{2i-1}{2n}\right)-\cos\left(\frac{2j-1}{2n}\right)\left(y-\frac{2j-1}{2n}\right)+\left(\frac{2i-1}{2n}-\sin\left(\frac{2j-1}{2n}\right)\right)\le 1\\
&\quad \frac{i-1}{n}\le x_i\le\frac{i}{n} \qquad i=1,2,\dots,n\\
&\quad \frac{j-1}{n}\le y_j\le\frac{j}{n} \qquad j=1,2,\dots,n
\end{aligned}$$

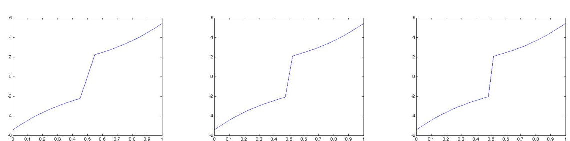
Where $x_i=\frac{i}{n}$ and $y_j=\frac{j}{n}$ for all $i,j=1,2,\dots,n$. In the following table showed optimal solutions for values of distinct n.

n	*optimal solution*	*optimal value*
10	*(0.9500,0.9500)*	*1.2169*
20	*(0.975,0.975)*	*1.2335*
30	*(0.9833,0.9833)*	*1.2388*
40	*(0.9875,0.9875)*	*1.2414*

Example 4.3. *Consider nonlinear non-smooth function as $f(x)=e^{|2x-1|}$ on $[0,1]$. For obtaining weak differentiation function of $f(x)$ i.e. $f'(x)$ on $[0,1]$, first, we partitioned the interval of $[0,1]$ to n subinterval. That is $I_i=[x_{i-1},x_i]=[\frac{i-1}{n},\frac{i}{n}]$ for all $i=1,2,\dots,n$. We may solve the following problem:*

$$\min\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}|f(x)-f(\frac{2i-1}{2n})-(x-\frac{2i-1}{2n}).p(\frac{2i-1}{2n})|dx\times\frac{1}{n}$$

Differentiation $f(x)$ for $n=10,20,30$ showed in the following figure, respectively:



1. Obtained differentiations for n=10,20,30 from left to right, respectively

Example 4.4. *Consider nonlinear non-smooth programming problem as follows:*

$$\begin{aligned}
&\max e^{|2x-1|}\\
&\quad s.t.\\
&\qquad \sin(x)\le 1\\
&\qquad 0\le x\le 1
\end{aligned}$$

We may solve our problem as previous examples. Therefore, obtained the optimal solutions as follows:

n	*optimal solution*	*optimal value*
10	*0.9373*	*2.3979*
20	*0.9760*	*2.5909*
30	*0.9855*	*2.6406*

The optimal solution is $x^* = 1$, $z^* = 2.7183$, *exactly.*

5. **conclusion**

In this paper we introduce a new approach to solve constrained nonlinear non-smooth programming problems. The main advantage of this approach is that an approximate of global solution is obtained. Also the approach can be extended for problem with non-smooth nonlinear programming problem by introducing a novel definition of global weak differentiation in the sense of ℓ_1 space. Also, we may obtain an approximate solution of a nonlinear non-smooth programming problems by solving a linear programming problem where the norm of problem is very small.

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Department of applied mathematics, ferdowsi university of mashhad, iran
E-mail address: *avkamyad@yahoo.com*
E-mail address: *na_ha87@yahoo.com*