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Variational iteration method for solving mth-order boundary value problems

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Abstract

In this paper, the variational iteration method is applied to solve mth-order boundary value problems. Using this method we only need to apply an iteration to obtain solutions of remarkable accuracy. By giving 3 examples and by comparing the obtained result with the exact solution, the efficiency of the method will be shown.

Keywords: Variational, iteration, boundary-value problems.

Introduction

In the recent years many different methods were proposed to solve boundary value problems (BVPs), such as homotopy perturbation method (HPM) (He Ji-Huan, 2006; Muhammad & Tauseef, 2007), variational iteration method (VIM) (Xu Lan, 2007; He Ji-Huan, 2007) and modified decomposition method (MDM) (Mestrovic, 2007). He Ji-Huan (2007) applied variational iteration method for solving eighth-order initial-boundary value problems and Xu Lan (2007) applied this method for solving fourth-order boundary value problems. In this paper, we apply the variational iteration method proposed by He Ji-Huan (1997, 1998, 1999 & 2000) to find approximate solutions for mth-order boundary value problems.

To illustrate the basic idea of VIM, we consider the following general nonlinear system:

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N is a nonlinear operator and g (x) is an inhomogeneous forcing term. According to the variational iteration method (He Ji-Huan, 1997, 1998, 1999 & 2000), we can construct a correction functional for the system as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\{Lu_n(s) + N\tilde{u}_n(s) - g(s)\}ds, \tag{2}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory (Inokuti *et al.,* 1978), the subscripts n in (2), denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n$ =0.

We consider the general boundary value problem of the following type:

$$y^{(m)}(x) + f(x; y, y', \ldots, y^{(m-1)}) = 0, \qquad a \le x \le b\,, \tag{3}$$

with suitable boundary conditions. We can construct a correction functional as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\{y_n^{(m)}(\xi) + f(\xi, \tilde{y}_n(\xi), \tilde{y}'_n(\xi), \ldots, \tilde{y}_n^{(m-1)}(\xi))\}d\xi. \tag{4}$$

By making the above functional stationary with respect to $y_n(x)$, we obtain the following stationary conditions:

$$\lambda(\xi)\Big|_{\xi=x} = 0, \quad \lambda'(\xi)\Big|_{\xi=x} = 0, \quad \lambda''(\xi)\Big|_{\xi=x} = 0, \ \cdots\ , \lambda^{(m-2)}(\xi)\Big|_{\xi=x} = 0,$$

$$1 + (-1)^{m-1}\lambda^{(m-1)}(\xi)\Big|_{\xi=x} = 0, \qquad \lambda^{(m)}(\xi) = 0.$$

We can readily identify the Lagrange multiplier as follows:

$$\lambda = (-1)^m \frac{(\xi - x)^{m-1}}{(m-1)!}. \tag{5}$$

Therefore, the following iteration formulation will be obtained:

$$y_{n+1}(x) = y_n(x) + (-1)^m \int_0^x \frac{(\xi - x)^{m-1}}{(m-1)!}\{y_n^{(m)}(\xi) + f(\xi, y_n(\xi), y'_n(\xi), \ldots, y_n^{(m-1)}(\xi))\}d\xi. \tag{6}$$

Having an initial approximation by using (6) we get the successive approximations.

Applications

To illustrate the method we consider a few examples of different order and then we will compare the obtained results with the exact solutions.

Example1. First, we consider the following BVP:

$$y^{(vi)}(x) + (5x+1)y(x) = (185x - 25x^2 + 10x^4) + (270 - 36x^2)\sin x, \qquad -1 \le x \le 1 \tag{7}$$

With the boundary conditions

$$\begin{aligned} &y(-1) = 4\cos 1, && y(1) = -2\cos 1, \\ &y'(-1) = \cos 1 + 4\sin 1, && y'(1) = \cos 1 + 2\sin 1, \\ &y''(-1) = -16\cos 1 + 2\sin 1, && y''(1) = 14\cos 1 - 2\sin 1, \end{aligned} \tag{8}$$

The exact solution of the problem given by

$$y_E(x) = (2x^3 - 5x + 1)\cos x,$$

which is taken from Siddiqi and Akram, (2008).

According to (6), we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^5}{5!}\{y_n^{(vi)}(s) + F(y_n(s))\}ds, \tag{9}$$

where

$$F(y_n(x)) = (5x+1)y_n(x) - (185x - 25x^2 + 10x^4) - (270 - 36x^2)\sin x$$

We get start with the following initial approximate value:

$$y_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5, \tag{10}$$

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Fig. 1a. Comparison of the approximate solution with the exact solution.

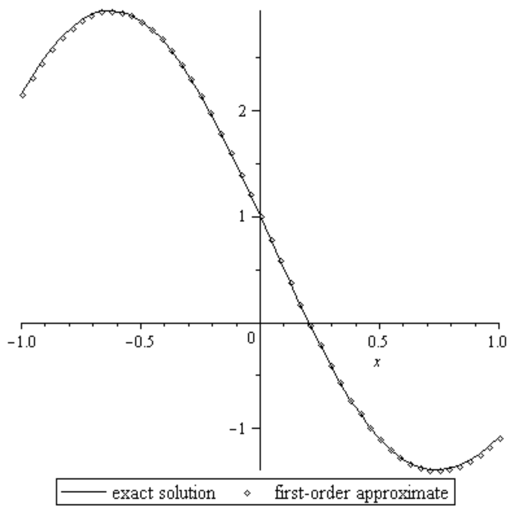


Fig. 1b. Absolute error $|y_E - y_1|$ for example 1.

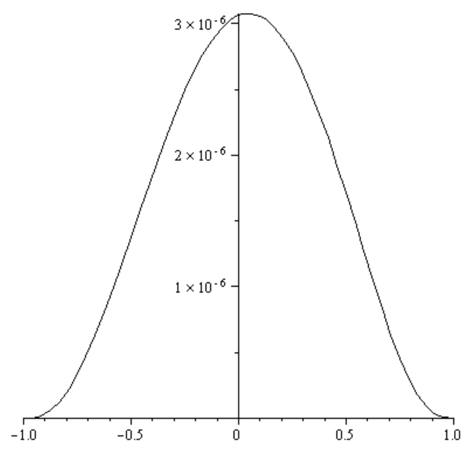


Fig. 2a. Comparison of the approximate solution with the exact solution.

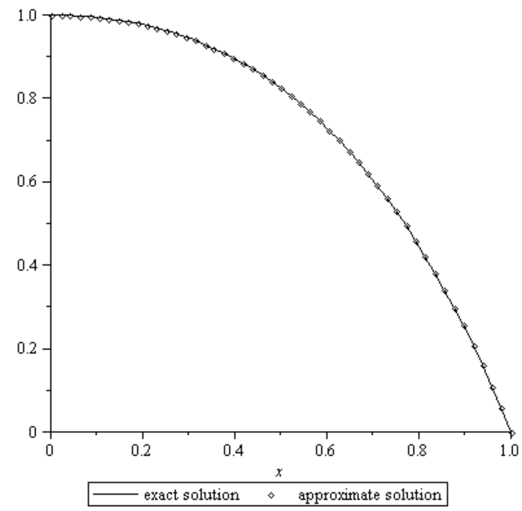


Fig. 2b. Absolute error $|y_E - y_1|$ for example 2.

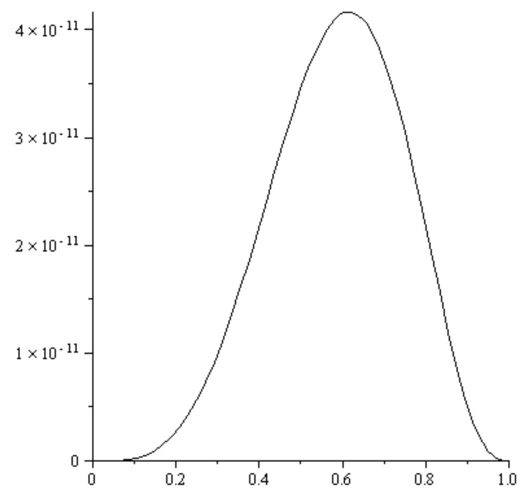


Fig. 3a. Comparison of the approximate solution with the exact solution

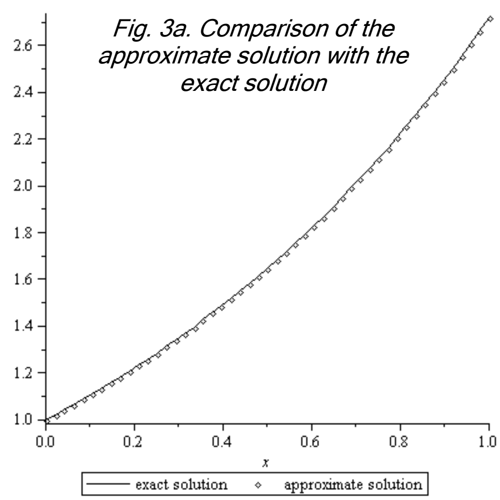
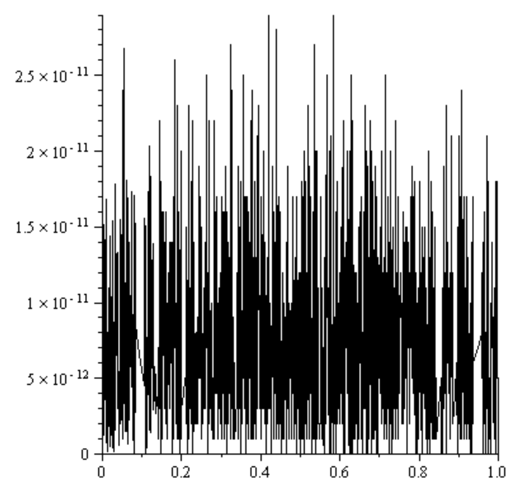


Fig. 3b. Absolute error $|y_E - y_1|$ for example 3



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where a_0, a_1, a_2, a_3, a_4 and a_5 are unknown constants to be further determined. By the iteration formula (9), we obtain the following first-order approximation:

$$y_1(x) = y_0(x) + \int_0^x \frac{(s-x)^5}{5!}\{y_0^{(vi)}(s) + F(y_0(s))\}ds$$
$$= 31290 + 425x + a_0 + a_1x + a_2x^2 + a_3x^3 - 4450x^2 - \frac{49}{2}x^3$$
$$+ a_4x^4 + \frac{225}{4}x^4 + a_5x^5 + \frac{157}{120}x^5 - \frac{a_0}{1008}x^7 - \frac{a_1}{4032}x^8$$
$$- \frac{a_2}{12096}x^9 - \frac{a_3}{30240}x^{10} - \frac{a_4}{66528}x^{11} - \frac{a_5}{133056}x^{12}$$
$$- \frac{a_0}{720}x^6 - \frac{a_1}{5040}x^7 - \frac{a_2}{20160}x^8 - \frac{a_3}{60480}x^9 - \frac{a_4}{151200}x^{10}$$
$$- \frac{a_5}{332640}x^{11} + 247x\cos x - 672\sin x - 31290\cos x$$
$$+ 2545x^2\cos x - 10x^4\cos x + 36x^2\sin x + 240x^3\sin x$$
$$-13740x\sin x. \tag{11}$$

Incorporating the boundary conditions from equation (8), into $y_1(x)$, we can determine the values of the unknown constants as follows

$$a_0 = 0.99999, \quad a_1 = -5.0000004, \quad a_2 = -0.499992,$$
$$a_3 = 4.5000017, \quad a_4 = 0.041662, \quad a_5 = -1.20833434, \tag{12}$$

Therefore, the first-order approximate solution can be obtained as follows:

$$y_1(x) = -13740\, x\sin x + 36x^2\sin x + 240x^3\sin x + 247x\cos x - 10x^4\cos x$$
$$+2545x^2\cos x - 31290\cos x - 672\sin x + 0.099999\, x^5 - 19.99999833x^3$$
$$+56.291662x^4 - 4450.499992x^2 + 0.0012648808x^8 + 10^{-8}x^7 - 0.000149085x^2$$
$$+0.000003x^{11} - 0.0013888x^6 - 0.00003306x^9 + 0.000009081x^{12} + 419.9999996x + 31290.999. \tag{13}$$

Comparison of the approximate solution, eqn. (13), with the exact one are shown in figs. 1a and 1b.

Example 2. Now, we consider the following BVP:

$$y^{(vii)}(x) = y(x) - 7e^x, \quad 0 \le x \le 1 \tag{14}$$

with boundary conditions

$$y(0) = 1, y'(0) = 0, y''(0) = -1, y'''(0) = -2,$$
$$y(1) = 0, y'(1) = -e, y''(1) = -2e, \tag{15}$$

and the exact solution is $y_E(x) = (1-x)e^x$.

According to (6), we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(s-x)^6}{6!}\{y_n^{(vii)}(s) - y_n(s) + 7e^s\}ds. \tag{16}$$

Now, we assume that an initial approximation has the following form:

$$y_0(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6, \tag{17}$$

where $a_0, a_1, a_2, a_3, a_4, a_5$ and a_6 are unknown constants to be further determined. By the iteration formula (16), we obtain the following first-order approximation:

$$y_1(x) = y_0(x) - \int_0^x \frac{(s-x)^6}{6!}\{y_0^{(vii)}(s) - y_0(s) + 7e^s\}ds$$
$$= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + 7$$
$$+7x + \frac{7}{6}x^3 + \frac{7}{20}x^6 + \frac{7}{120}x^5 + \frac{7}{2}x^2 + \frac{7}{24}x^4$$
$$-7e^x + \frac{a_1}{40320}x^8 + \frac{a_4}{1663200}x^{11} + \frac{a_3}{40320604800}x^{10}$$
$$+ \frac{a_2}{181440}x^9 + \frac{a_6}{8648640}x^{13} + \frac{a_5}{3991680}x^{12} + \frac{a_0}{5040}x^7. \tag{18}$$

Incorporating the boundary conditions, Esq. (15), into $y_1(x)$, we can determine the values of the unknown constants

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{-1}{2}, \quad a_3 = \frac{-1}{3},$$
$$a_4 = -0.1250000, \quad a_5 = -0.0333333, \quad a_6 = -0.0069445. \tag{19}$$

Finally, we obtain the following first-order approximate solution:

$$y_1(x) = 8 + 7x + 3x^2 + 0.8333333333x^3 + 0.1666666667x^4 + 0.0250000333x^5$$
$$+0.0027777222\, x^6 + 0.1984126984\times10^{-3}x^7 - 7.515632515\times10^{-8}x^{11} - 7e^x$$
$$-0.2755731922\times10^{-5}x^9 - 5.511463845\times10^{-7}x^{10} - 8.350694446\times10^{-9}x^{12}$$
$$-8.029586153\times10^{-10}x^{13}, \tag{20}$$

Figs. 2a and 2b show the comparison between the exact solution and the first-order approximate solution. We easily observe that, the higher accuracy is obtained without any difficulty.

Example 3. As a last example we consider the following non- linear BVP:

$$y^{(ix)} = e^{-x}y^2(x), \quad 0 \le x \le 1, \tag{21}$$

with boundary conditions

$$y(0) = y'(0) = y''(0) = y'''(0) = y^{(iv)}(0) = 1,$$
$$y(1) = y'(1) = y''(1) = y'''(1) = e, \tag{22}$$

and the exact solution $y_E(x) = e^x$.

Employing (6) to the BVP, we have the following iteration formulation

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(s-x)^8}{8!}\{y_n^{(ix)}(s) - e^{-s}y_n^2(s)\}ds. \tag{23}$$

We start with the following initial approximation:

$$y_0(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8, \tag{24}$$

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where $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$ and a_8 are unknown constants to be further determined.

By the iteration formulation, equation (23), we can easily obtain $y_1(x)$. Incorporating the boundary conditions, equation (22), into $y_1(x)$, we can determine the values of the unknown constants

$$\begin{aligned} &a_0 = 1,\ \ a_1 = 1,\ \ a_2 = \frac{1}{2},\ \ a_3 = \frac{1}{6},\ \ a_4 = \frac{1}{24}, \\ &a_5 = \frac{1271735358}{152608243453}, a_6 = \frac{47580044437}{34257630333145}, \\ &a_7 = \frac{2010798606}{10134427926287}, a_8 = \frac{167102214}{6737556683771}. \end{aligned} \tag{25}$$

Therefore, we obtain the following first-order approximate value of the solution

$$\begin{aligned} y_1(x) = {} & 2.066193132\times10^{10} - 2.066193132\times10^{10}e^{-x} \\ & + (1.348410029\times10^{10}e^{-x} - 7.177831023\times10^{9})x \\ & + (1.14384398\times10^{9} - 4.296978615\times10^{9}e^{-x})x^2 \\ & - (1.095233493\times10^{8} + 8.890603385e^{-x})x^3 \\ & + (6.914519599\times10^{6} - 1.339217946\times10^{8}e^{-x})x^4 \\ & - (2.958897279\times10^{5} + 1.560408682\times10^{7}e^{-x})x^5 \\ & + (8221.333021 - 1.457933950\times10^{6}e^{-x})x^6 \\ & - (146.3618316 + 1.116898263\times10^{5}e^{-x})x^7 \\ & + (1.205853824 - 7108.752682e^{-x})x^8 \\ & - 378.2509240e^{-x}x^9 - 16.82396340e^{-x}x^{10} \\ & - 0.6211745796e^{-x}x^{11} - 0.01873996165e^{-x}x^{12} \\ & - 0.0004486336609e^{-x}x^{13} - 0.000008080209258e^{-x}x^{14} \\ & - 9.84191216\times10^{-8}e^{-x}x^{15} - 6.15119569\times10^{-10}e^{-x}x^{16}. \end{aligned} \tag{26}$$

Comparison of the first-order approximate solution with the exact one is shown in Fig. 3a and 3b.

Remark. The VIM algorithm is coded in the computer package Maple11. The Maple environment variable Digits contorolling the number of significant digits is set to 22 in calculations done in non-linear example 3.

Conclusion

In this paper, we have used the variational iteration method for finding the solution of mth-order linear and nonlinear boundary value problems. The method is applied in a direct way without using linearization, transformation, discretization. The numerical results given in the Figs. 1-3; show that the present method provides highly accurate numerical solutions for solving this type of the BVPs.

References

1. He Ji-Huan (1997) A new approach to nonlinear partial differential quations. *Commun. Non-linear Sci.* 2, 230-235.
2. He Ji-Huan (1998) Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Comp. Methods Appl. Mech. Engg*. 167(1-2), 57-68.
3. He Ji-Huan (1999) Variational iteration method-A kind of non-linear analytical technique: some eg. *Int. J. Non-linear Mech.* 34, 699-708.
4. He Ji-Huan (2000) Variational iteration method for autonomous ordinary differential systems. *Appl. Math. Comp*. 114, 115-123.
5. He Ji-Huan (2006) Homotopy perturbation method for solving boundary value problems. *Phy. Letts. A.* 350, 87-88.
6. He Ji-Huan (2007) The variational iteration method for eighth-order initial-boundary value problems. *Phy. Scripta*. 76, 680-682.
7. Inokuti *et al.* (1978) General use of the lagrange multiplier in non-linear mathematical physics. In: Variational method in the mechanics of solids. Nemat-Nasser S (Ed.), Pergamon press, Oxford. pp: 156-162.
8. Mestrovic M (2007) The modified decomposition method for eight-order boundary value problems. *Appl. Math. Comp*. 188, 1437-1444.
9. Muhammad AN and Tauseef MS (2007) An effecint algorithm for solving fifth- order boundary value problems. *Math. Comp. Model*. 45, 954-964.
10. Siddiqi SS and Akram G (2008) Septic spline solutions of sixth-order boundary value problems. *J. Comp. Appl. Math.* 215, 288-301.
11. Xu Lan (2007) The variational iteration method for fourth order boundary value problems. Chaos Solitons & Fractals. doi:10.1016/j.chaos. 6.013.

