

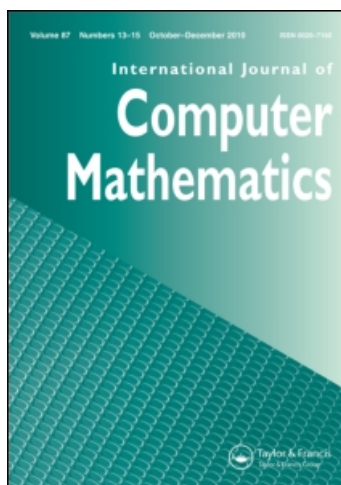
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Application of He's homotopy perturbation method to linear programming problems

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In this paper, we use He's homotopy perturbation method (HPM) for solving linear programming (LP) problems. By applying HPM for this class of problems, optimal solutions of a primal LP problem and its corresponding dual problem can be obtained at the same time. The efficiency of the method is shown by solving some examples.

Keywords: homotopy perturbation method; linear programming problem

2000 AMS Subject Classifications: 90C05; 55P99; 47A55

1. Introduction

Homotopy perturbation method (HPM) was presented first by He in 1998 [6,7] since then, different kinds of problems are solved by using this method, such as nonlinear partial differential equations [2–5,10,13], nonlinear systems of second-order boundary value problems [14], and also a system of linear equations [12].

Different methods such as the simplex method, the well-known and efficient method, are presented for solving linear programming (LP) problems. In this paper, we intend to use HPM to solve the following LP problem:

$$\begin{aligned} (P) \quad & \text{Min} \quad \mathbf{c}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{aligned} \quad (1)$$

where $\mathbf{c} = [c_j]$ is a n vector, $\mathbf{b} = [b_i]$ is an m vector, $\mathbf{A} = [a_{ij}]$ ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$) is an $m \times n$ matrix, and $\mathbf{x} = (x_1, x_2, \dots, x_n)^t$ is a n vector. A review of some methods for solving nonlinear programming problems is provided in some sections of [11], where Lagrange multiplier method and a method without using Lagrange multipliers are reviewed in order to solve nonlinear optimization problems. In Section 2, a brief review of HPM for solving a system of linear equations

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is presented and in Section (3), solving LP problems by using HPM is discussed, and finally in Section (4), some LP problems are solved by applying this method and the results are compared with the exact solutions obtained from the simplex method.

2. Homotopy perturbation method for solving a system of linear equations

Consider the following linear system:

$$\mathbf{Ax} = \mathbf{b}, \quad (2)$$

where

$$\mathbf{A} = [a_{ij}], \quad \mathbf{b} = [b_i], \quad \mathbf{x} = [x_j], \quad i = 0, \dots, m \quad j = 0, \dots, n.$$

Let $L(\mathbf{u}) = \mathbf{Au} - \mathbf{b}$ and $F(\mathbf{u}) = \mathbf{u} - \mathbf{w}_0$, where \mathbf{w}_0 is a known vector. Consider homotopy $H(\mathbf{u}, p)$ as follows:

$$H(\mathbf{u}, p) = (1 - p)F(\mathbf{u}) + pL(\mathbf{u}) = 0, \quad (3)$$

where $p \in [0, 1]$. It is clear that

$$H(\mathbf{u}, 0) = F(\mathbf{u}) \quad \text{and} \quad H(\mathbf{u}, 1) = L(\mathbf{u}).$$

HPM uses the parameter p as an expanding parameter [8,9] to obtain

$$\mathbf{u} = \mathbf{u}_0 + p\mathbf{u}_1 + p^2\mathbf{u}_2 + \dots \quad (4)$$

as p tends to 1, it gives an approximation to the solution of Equation (2) as follows:

$$\mathbf{x} = \lim_{p \rightarrow 1} \mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \dots$$

Putting Equation (4) into Equation (3), and comparing the coefficients of identical degrees of p on both sides, we find

$$\begin{aligned} p^0 : \mathbf{u}_0 &= \mathbf{w}_0, \\ p^1 : (\mathbf{A} - \mathbf{I})\mathbf{u}_0 + \mathbf{u}_1 - \mathbf{w}_0 - \mathbf{b} &= \mathbf{0}, \mathbf{u}_1 = \mathbf{b} - (\mathbf{A} - \mathbf{I})\mathbf{u}_0 + \mathbf{w}_0, \\ p^2 : (\mathbf{A} - \mathbf{I})\mathbf{u}_1 + \mathbf{u}_2 &= \mathbf{0}, \mathbf{u}_2 = -(\mathbf{A} - \mathbf{I})\mathbf{u}_1, \\ &\vdots \end{aligned}$$

consequently in general case

$$\mathbf{u}_{n+1} = -(\mathbf{A} - \mathbf{I})\mathbf{u}_n, \quad n = 1, 2, \dots$$

If we take $\mathbf{u}_0 = \mathbf{w}_0 = \mathbf{0}$, then we have

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{b}, \\ \mathbf{u}_2 &= -(\mathbf{A} - \mathbf{I})\mathbf{u}_1 = -(\mathbf{A} - \mathbf{I})\mathbf{b}, \\ \mathbf{u}_3 &= -(\mathbf{A} - \mathbf{I})\mathbf{u}_2 = (\mathbf{A} - \mathbf{I})^2\mathbf{b}, \\ &\vdots \\ \mathbf{u}_{n+1} &= -(\mathbf{A} - \mathbf{I})\mathbf{u}_n = (-1)^n (\mathbf{A} - \mathbf{I})^n \mathbf{b}, \end{aligned}$$

and so the solution of Equation (2) is in the following form:

$$\mathbf{u} = [\mathbf{I} - (\mathbf{A} - \mathbf{I}) + (\mathbf{A} - \mathbf{I})^2 - \dots] \mathbf{b}. \quad (5)$$

The convergence of series (5) for diagonally dominant matrices \mathbf{A} is proved by Keramati in [12].

3. Analysis of the method

To solve the LP problem (1), first consider its corresponding dual problem

$$\begin{aligned} (D) \quad & \text{Max} \quad \mathbf{w}\mathbf{b} \\ & \text{s.t.} \quad \mathbf{w}\mathbf{A} \leq \mathbf{c}, \end{aligned} \quad (6)$$

where \mathbf{w} is an m vector and unrestricted. The Karush–Kuhn–Tucker (KKT) optimality conditions for the solution of problem (1) are as follows [1]:

- (i) $\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0},$
- (ii) $\mathbf{w}\mathbf{A} + \mathbf{v} = \mathbf{c}, \quad \mathbf{w} \text{ unrestricted}, \mathbf{v} \geq \mathbf{0},$
- (iii) $\mathbf{v}\mathbf{x} = \mathbf{0},$

where \mathbf{v} is the vector of slack variables for the dual problem (6). Using conditions (i) and (ii), condition (iii) can be rewritten in the following form:

$$(\mathbf{c} - \mathbf{w}\mathbf{A})\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{c}\mathbf{x} - \mathbf{w}\mathbf{A}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{c}\mathbf{x} = \mathbf{w}\mathbf{A}\mathbf{x} \Rightarrow \mathbf{c}\mathbf{x} = \mathbf{w}\mathbf{b}.$$

Hence the KKT conditions are as follows:

- (i) $\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}.$
- (ii) $\mathbf{w}\mathbf{A} + \mathbf{v} = \mathbf{c}, \quad \mathbf{w} \text{ unrestricted}, \mathbf{v} \geq \mathbf{0}.$
- (iii) $\mathbf{c}\mathbf{x} = \mathbf{w}\mathbf{b}.$

Clearly Equation (7) is a system of linear equations. The following theorem reveals the relationship between the optimal solutions of the primal and dual problems (1), (6), and KKT conditions (7).

THEOREM 3.1 *If \mathbf{x}^* is the optimal solution of Equation (1), and $(\mathbf{w}^*, \mathbf{v}^*)$ is the optimal solution of Equation (6), then $(\mathbf{x}^*, \mathbf{w}^*, \mathbf{v}^*)$ satisfies the KKT conditions (7), and also, if $(\mathbf{x}^*, \mathbf{w}^*, \mathbf{v}^*)$ satisfies Equation (7), then \mathbf{x}^* is the optimal solution of Equation (1), and $(\mathbf{w}^*, \mathbf{v}^*)$ is the optimal solution of Equation (6).*

Proof See [1].

We can rewrite Equation (7) as follows:

$$\begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{0}\mathbf{w}^t + \mathbf{0}\mathbf{v}^t &= \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \\ \mathbf{0}\mathbf{x} + \mathbf{A}^t\mathbf{w}^t + \mathbf{v}^t &= \mathbf{c}^t, \quad \mathbf{v} \geq \mathbf{0}, \\ \mathbf{c}\mathbf{x} - \mathbf{b}^t\mathbf{w}^t + \mathbf{0}\mathbf{v}^t &= 0, \end{aligned}$$

or

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^t & \mathbf{I} \\ \mathbf{c} & -\mathbf{b}^t & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w}^t \\ \mathbf{v}^t \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c}^t \\ 0 \end{bmatrix}, \quad \mathbf{x} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}, \quad (8)$$

which is a system of $(m + n + 1)$ equations and $(2n + m)$ variables. Now we add zero equations, that is, $(\mathbf{0}, \mathbf{0}, \mathbf{0}) \cdot (\mathbf{x}, \mathbf{w}^t, \mathbf{v}^t) = \mathbf{0}$, to Equation (8) to gain a $(2n + m) \times (2n + m)$ square matrix.

Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^t & \mathbf{I} \\ \mathbf{c} & -\mathbf{b}^t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{x} \\ \mathbf{w}^t \\ \mathbf{v}^t \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c}^t \\ \mathbf{0} \end{bmatrix}, \quad (9)$$

hence Equation (8) changes to the following system of linear equations:

$$\mathbf{BX} = \mathbf{d}. \quad (10)$$

Now Equation (10) can be solved by HPM, which is mentioned in previous section, and according to Theorem 3.1, the solution of this system is the optimal solution of problems (1) and (6). By solving Equation (10), optimal solution of primal problem (1) is found as well as its dual problem (6). ■

4. Numerical examples

We provide the following examples to illustrate the method and show the efficiency of this method.

Example 1 Consider the following LP problem (P1), and its corresponding dual problem (D1),

$$\begin{aligned} \text{(P1) Min} \quad & z = 2x_1 - 5x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 = 1, \\ & -x_1 + 8x_2 = 4, \quad x_1, x_2 \geq 0. \\ \text{(D1) Max} \quad & y = w_1 + 4w_2 \\ \text{s.t.} \quad & 3w_1 - w_2 + v_1 = 2, \\ & 2w_1 + 8w_2 + v_2 = -5, \\ & w_1, w_2 \text{ unrestricted}, \quad v_1, v_2 \geq 0. \end{aligned}$$

The optimal solution of (P1) is $(x_1^*, x_2^*) = (0, 0.5)$, the optimal objective value corresponding to it is $z^* = -2.5$, and the dual problem (D1) has its optimal solutions on a ray, but according to the duality theorems [1], we know that the optimal objective value of the dual problem is also $y^* = -2.5$. Now we use HPM to solve the above example. To have a square matrix, one zero equation is added to the linear system (8). After some manipulations, the following system is obtained:

$$\begin{bmatrix} 1 & 2/3 & 0 & 0 & 0 & 0 \\ -1/8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1/4 & 1 & 0 & 1/8 \\ 2 & -5 & -1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0.5 \\ 2/3 \\ -5/8 \\ 0 \\ 0 \end{bmatrix}.$$

Now by using eight terms of the series (5), the following values for the primal and dual variables are obtained: $(x_1, x_2, w_1, w_2, v_1, v_2) = (0, 0.5, 0.4231, -0.7308, 0, 0)$, which is equal to the exact optimal solutions of problems (P1) and (D1).

Example 2 Consider the following LP problem (P2) and its dual (D2):

$$\begin{aligned}
 \text{(P2) Max} \quad & z = x_1 + 2x_2 - x_3 + 4x_4 \\
 \text{s.t.} \quad & 5x_1 - x_2 + (2/3)x_3 - 3x_4 = 6, \\
 & x_1 - 3x_2 - x_3 = 3, \\
 & x_1 + 2x_2 + 4x_3 - x_4 = 2.5, \\
 & -x_1 - x_2 + 2x_4 = 7.2, \\
 & x_i \geq 0, \quad i = 1, 2, 3, 4, \\
 \text{(D2) Min} \quad & y = 6w_1 + 3w_2 + 2.5w_3 + 7.2w_4 \\
 \text{s.t.} \quad & 5w_1 + w_2 + w_3 - w_4 \geq 1, \\
 & -w_1 - 3w_2 + 2w_3 - w_4 \geq 2, \\
 & (2/3)w_1 - w_2 + 4w_3 \geq -1, \\
 & -3w_1 - w_3 + 2w_4 \geq 4, \\
 & w_i, \text{ unrestricted}, \quad i = 1, 2, 3, 4.
 \end{aligned}$$

These problems are solved by Lingo software and the following values for $x_i, w_i (i = 1, 2, 3, 4)$ are obtained:

$$\begin{aligned}
 \mathbf{x}^* &= (x_1^*, x_2^*, x_3^*, x_4^*) = (4.934, 0.3908, 0.7617, 6.2624), \\
 (\mathbf{w}^*, \mathbf{v}^*) &= (2.234, -4.007, -1.6241, 4.539, 0, 0, 0, 0),
 \end{aligned} \tag{11}$$

and the optimal objective value is $z^* = y^* = 30.0035$. To use HPM, four zero equations should be added to system (8). By considering 35 terms of series (5), the following values are found:

$$(\mathbf{x}^*, \mathbf{w}^*, \mathbf{v}^*) = (4.9336, 0.3905, 0.7619, 6.262, 2.2337, -4.0066, -1.6239, 4.5386, 0, 0, 0, 0). \tag{12}$$

Comparing Equations (11) and (12), one can see that Equation (12) is a good approximation of the optimal solutions of problems (P2) and (D2).

Note 1: Consider the LP problem (1), with m constraints. By the theory of simplex method, it is clear that there are at most m nonzero variables at the optimal solution.

Note 2: Consider linear system (10), the terms in series (5) have the form $(\mathbf{B} - \mathbf{I})^k \mathbf{d}$; having in mind these terms, one can conclude that zero row-vectors in matrix \mathbf{B} makes a variable of vector \mathbf{X} equal to zero, and so it is important to add zero equations to Equation (8) in a correct place.

Example 3 Consider the following LP problem (P3), and its dual (D3), the exact solution of these problems are $(\mathbf{x}^*, \mathbf{s}^*) = (2.25, 1.5, 0, 0, 0)$, where \mathbf{s} is the vector of slack variables for (P3), and $(\mathbf{w}^*, \mathbf{v}^*) = (-0.5, -0.75, 0, 0, 0.75)$. The optimal objective value is $z^* = y^* = -5.25$. To solve this problem by HPM, according to the Note 1, we know that at the optimal solution there are at most two and three nonzero variables for problems (P3) and (D3), respectively. So we add zero equations in the following way. Note that in this problem, four zero equations

are needed.

$$\begin{aligned}
 \text{(P3)} \quad & \text{Min} \quad z = -x_1 - 2x_2 + x_3 \\
 & \text{s.t.} \quad 2x_1 + x_2 + x_3 \leq 6, \\
 & \quad \quad 2x_2 - x_3 \leq 3, \quad x_1, x_2, x_3 \geq 0. \\
 \text{(D3)} \quad & \text{Max} \quad y = 6w_1 + 3w_2 \\
 & \text{s.t.} \quad 2w_1 \leq -1, \\
 & \quad \quad w_1 + 2w_2 \leq -2, \\
 & \quad \quad w_1 - w_2 \leq 1, \\
 & \quad \quad w_1, w_2 \leq 0, \quad v_1, v_2 \geq 0.
 \end{aligned}$$

After some simple manipulations, the following system of linear equations is obtained:

$$\begin{bmatrix}
 1 & .5 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & -.5 & 0 & .5 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1/6 & 1/3 & -1/6 & 0 & 0 & 1 & .5 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & .5 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & .5 & 1 & 0 & .5 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1
 \end{bmatrix}
 \begin{bmatrix}
 \mathbf{x} \\
 \mathbf{s} \\
 \mathbf{w} \\
 \mathbf{v}
 \end{bmatrix}
 =
 \begin{bmatrix}
 3 \\
 1.5 \\
 0 \\
 0 \\
 0 \\
 -.5 \\
 -1 \\
 0 \\
 0 \\
 1
 \end{bmatrix},$$

where the fifth equation of this system is the third KKT condition, ($\mathbf{c}\mathbf{x} = \mathbf{w}\mathbf{b}$). By considering just two terms of series (5), the optimal solutions of problems (P3) and (D3) will be obtained as $(\mathbf{x}^*, \mathbf{s}^*, \mathbf{w}^*, \mathbf{v}^*) = (2.25, 1.5, 0, 0, 0, -0.5, -0.75, 0, 0, 0.75)$. Note that if the places for adding zero equations are not chosen correctly, the series (5) may not converge.

5. Conclusion

In this paper, we used He's HPM to solve an LP problem. We used KKT optimality conditions for LP problem (1), and changed the problem of solving an LP problem to solving a system of linear equations, and then used HPM to solve this system. If the matrix \mathbf{B} be strictly diagonally dominated, the rate of convergence will increase [12].

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