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On the probability of being a 2-Engel group

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Abstract

Let G be a finite group and $E_2(G)$ denote the probability that $[x, y, y] = 1$ for randomly chosen elements x, y of G. We will obtain lower and upper bounds for $E_2(G)$ in the case where the sets $E_G(x) = \{y \in G : [y, x, x] = 1\}$ are subgroups of G for all $x \in G$. Also the given examples illustrate that all the bounds are sharp.

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1 Introduction

For a given natural number n, the *n-Engel degree* of G, denoted by $E_n(G)$, is the probability that two randomly chosen elements x, y of G satisfy the n-Engel condition $[y,_n x] = 1$, that is

$$E_n(G) = \frac{|\{(x, y) \in G \times G : [y,_n x] = 1\}|}{|G|^2}.$$

The case $E_1(G)$, the commutativity degree of G is extensively studied in the literature. We intend to study $E_2(G)$ and give some lower and upper bound in terms of $E_2(G)$. Note that the situation is too complicated for computing $E_n(G)$, when $n > 2$.

2 Preliminary results

We begin with some elementary lemmas.

Lemma 2.1 *Let G be a group and $x \in G$. Then the followings statements are equivalent:*

(i) $E_G(x) \leq G$;

(ii) $[E_G(x), x, E_G(x), x] = 1$;

(iii) $[[E_G(x), x], [E_G(x), x]] = 1$ *that is* $[E_G(x), x]$ *is abelian.*

According to the above lemma we shall restrict ourselves to groups in variety $\mathcal{V}$ of all finite groups admitting the law $[[x, y], [x, z]] = 1$. This variety is studied by Macdonald [5] and Farrokhi and Moghaddam [2].

Lemma 2.2 *Let G be a finite group in variety $\mathcal{V}$ and $x \in G$. Then*

(i) $E_G(x) \leq G$;

(ii) $|E_G(x)| = |C_G(x)||G_G(x) \cap x^G|$;

(iii) $[C_G(x)x^G : C_G(x)] = [G : E_G(x)]$;

(iv) $|C_G(x)x^G| = [G : C_G(x) \cap x^G]$ *divides* $|G|$.

Lemma 2.3 *Let G be a finite group in variety $\mathcal{V}$ with an element x such that $C_G(x)x^G = G$. Then*

(i) $G = [x, G] \rtimes C_G(x)$;

(ii) *If* $x \in L(G)$, *then* $x \in Z(G)$.

3 Main theorems

Our main theorems give sharp lower and upper bounds for 2-Engel degree of a finite group.

Theorem 3.1 *Let G be a finite non 2-Engel group belonging to variety $\mathcal{V}$ and $p = \min \pi(G)$. Then*

$$E_2(G) \leq \frac{1}{p} + \left(1 - \frac{1}{p}\right) \frac{|L_2(G)|}{|G|}$$

and if $L_2(G) \leq G$, then

$$E_2(G) \leq \frac{2p - 1}{p^2}.$$

Moreover, both of the upper bounds are sharp at any prime p.

Theorem 3.2 *Let G be a finite non 2-Engel group belonging to variety $\mathcal{V}$ and $p = \min \pi(G)$. Then*

$$E_2(G) \geq E_1(G) - (p-1)\frac{|Z(G)|}{|G|} + (p-1)\frac{k_G(L(G))}{|G|}$$

and if either G is a p-group, or G' is a cyclic 2-group or a generalized quaternion 2-group, then

$$E_2(G) \geq pE_1(G) - (p-1)\frac{|Z(G)|}{|G|}.$$

Moreover, both of the lower bounds are sharp at any prime p.

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